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ULAM-HYERS STABILITY OF SINGULAR INTEGRAL EQUATIONS, VIA WEAKLY PICARD OPERATORS

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Abstract. In this paper we investigate the Ulam-Hyers stability of several integral equations with singularity. First we give some results concerning the Ulam-Hyers stability of integral equations with weak singularities. Our approach is also suitable for studying some fractional differential equations. In order to emphasize this aspect we prove that some conditions (5) in S. Abbas, M. Benchohra, *Ulam-Hyers stability for the Darboux problem for partial fractional differential and integro-differential equations via Picard operators* published in Results Math. **65**(2014), 67-79 (respectively condition (3.1) from S. Abbas, M. Benchohra, A. Petruşel, *Ulam stability for partial fractional differential inclusions via Picard operators theory*, Electron. J. Qual. Theory Differ. Equ., 2014, No. 51, 1-13) can be omitted without losing the validity of the obtained results. In the second part we establish some generalized Ulam-Hyers-Rassias stability results for the Bessel equation and related equations. Our approach is based on fixed point methods and the obtained results are more general than those established by Byungbae Kim and Soon-Mo Jung in *Bessel's differential equation and its Hyers-Ulam stability* appeared in J. Inequal. Appl., Volume 2007.
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1. Introduction

The history of Ulam-Hyers (and of many generalized Ulam-Hyers type) stability started in 1940, when S.M. Ulam posed the question of stability of group homeomorphisms ([28]):

Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$ does there exists a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality

$$d(h(xy), h(x)h(y)) < \delta, \ \forall x, y \in G_1,$$

then there exists a homomorphism $H : G_1 \to G_2$ with

$$d(h(x), H(x)) < \varepsilon, \ \forall x \in G_1?$$

One year later D.H. Hyers solved the problem under the assumption that G_1 and G_2 are Banach spaces ([17]). He proved that each solution of the inequality

$$||f(x+y) - f(x) - f(y)|| \le \varepsilon, \ \forall x, y \in G_1,$$

can be approximated by an exact solution, an additive function. In this case, the Cauchy additive functional equation, $f(x+y) = f(x) + f(y)$, is said to have the Ulam-Hyers stability.

Since then the initial question was generalized to a wide range of equations and several important results were obtained (for functional equations see [21], for operatorial equations see [24]). In the last decade the problem of Ulam-Hyers stability for various types of equations (such as functional, differential, operatorial, fractional differential, integral, partial differential equations, etc.) was studied in many recent papers (see for example [13], [25], [24], [2], [8], [9]). For differential and integral equations mainly three different approaches were used: direct calculations ([5]), power series method ([22], [19], [20]), fixed point method ([24], [25], [2], [9]). Each method has its advantages and disadvantages and in most cases the same results can be proved using several methods (see for example [13] and the references therein). The main purpose of this paper is to prove the Ulam-Hyers stability of weakly singular integral equations using fixed point methods (in terms of weakly Picard operator theory) and to study the stability of some singular integral equations which arose from the study of special functions. Due to the fact that several fractional differential equations can be reduced to weakly singular integral equations, our results have implications also in terms of fractional differential equations.

2. Ulam Hyers stability of weakly singular integral equations

We recall the following definitions:

Definition 2.1 ([24]) Let (X, d) be a metric space and $A : X \to X$ be an operator. By definition, the fixed point equation

$$x = A(x) \tag{2.1}$$

is said to be generalized Ulam-Hyers stable if there exists a $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ increasing, continuous in 0, with $\psi(0) = 0$, such that for each $\varepsilon > 0$ and for each solution y^* of the inequality

$$d(y, A(y)) \le \varepsilon,$$

there exists a solution x^* of the equation (2.1) such that

$$d(y^*, x^*) \leq \psi(\varepsilon).$$

If $\psi(t) = c_A \cdot t$, for each $t \in \mathbb{R}_+$, the equation (2.1) is called Ulam-Hyers stable.

Definition 2.2 (adapted from [24]) Let (K, d) be a metric space,

$$X = \{x : K \to K | x \text{ is continuous}\}$$

the set of continuous functions defined on K and $A : X \to X$ be an operator. The fixed point equation

$$x = A(x) \tag{2.2}$$

(which is equivalent to $x(t) = A(x)(t)$, for all $t \in K$) is said to be generalized Ulam-Hyers-Rassias stable if there exists a $c_A > 0$ such that for each solution y^* of the inequality

$$d(y(t), A(y)(t)) \leq \varphi(t), \quad \forall t \in K$$

there exists a solution x^* of the equation (2.1) such that

$$d(y^*(t), x^*(t)) \leq c_A \varphi(t), \quad \forall t \in K.$$

Definition 2.3 ([24]) $A : X \to X$ is weakly Picard operator if the sequence of successive approximations, $A^n(x)$, converges for all $x \in X$ and the limit (which may be depend on x) is a fixed point of T.

Definition 2.4 ([24]) A weakly Picard operator $A : X \to X$ is said to be ψ*-weakly Picard operator* if

$$d(x, A^\infty(x)) \leq \psi(d(x, A(x))), \text{for all } x \in X,$$

for a $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ increasing function which is continuous in 0 and $\psi(0) = 0$. In the case when $\psi(t) = ct$, for all $t \geq 0$ and a fixed $c > 0$, we say that A is c-weakly Picard operator.

Theorem 2.5 ([24]) *Let (X, d) be a metric space. If $A : X \to X$ is a ψ-weakly Picard operator, then the fixed point equation $u = Au$ is generalized Ulam-Hyers stable. Moreover if A is c-weakly Picard, then the fixed point equation $u = Au$ is Ulam-Hyers stable.*

Definition 2.6 The equation

$$u(x) = f(x) + \int_a^x K_1(x, s) g(s, u(s)) ds, \tag{2.3}$$

with $f \in C[a, b]$, $g \in C([a, b] \times \mathbb{R})$ is called weakly singular if there exist $H_1 \in C([a, b] \times [a, b])$ and $\alpha \in (0, 1)$ such that

$$K_1(x, s) = \frac{H_1(x, s)}{|x - s|^\alpha}, \ \forall x, s \in [a, b] \text{ with } x \neq s.$$

In this case we also say that the kernel K_1 is weakly singular.

Definition 2.7 The equation

$$u(x,y) = f(x,y) + \int_a^x \int_a^y K_2(x,y,s,t)g(s,t,u(s,t))dsdt, \tag{2.4}$$

with $f \in C([a,b] \times [a,b])$, $g \in C([a,b] \times [a,b] \times \mathbb{R})$ is called weakly singular if there exist $H_2 \in C\left([a,b]^4\right)$ and $\alpha_1, \alpha_2 \in (0,1)$ such that

$$K_2(x,y,s,t) = \frac{H_2(x,y,s,t)}{|x-s|^{\alpha_1}|y-t|^{\alpha_2}}, \ \forall x,y,s,t \in [a,b] \text{ with } x \neq s \text{ and } y \neq t.$$

In this case we also say that the kernel K_2 is weakly singular.

The connection between Caputo type fractional differential equations and weakly singular integral equations was used by several authors ([4], [12], [2], [3]) to obtain existence, uniqueness and Ulam-Hyers type stability, but most of the above cited authors use an extra condition to guarantee the validity of their results (condition (10) in [4], condition 3.3 in [12], condition (5) in [2], condition (3.1) in [3]). In what follows we would like to emphasize that these conditions can be omitted without losing the validity of the established results. First we will prove the Ulam-Hyers stability of linear weakly singular integral equations by using the properties of the iterated kernels (Theorem 2.8 and Theorem 2.10). After this we will prove an Ulam-Hyers type stability result for the nonlinear case (Theorem 2.12) and at the end of this section we will prove a similar result for weakly singular Fredholm-Volterra equations (Theorem 2.15).

Theorem 2.8 *If $f \in C[a,b]$ and the equation*

$$u(x) = f(x) + \int_a^x K(x,s)u(s)ds \tag{2.5}$$

is weakly singular, then it is also Ulam-Hyers stable.

Proof. For the operator $A : C([a,b]) \to C([a,b])$,

$$A(u)(x) = f(x) + \int_a^x K(x,s)u(s)ds$$

there exists $n \in \mathbb{N} \setminus \{0\}$ such that $\widehat{A} = A^n$ is a contraction relative to a Bielecki norm (see [7]). Theorem 2.5 implies that the equation $u = \widehat{A}(u)$ is Ulam-Hyers stable. The contractivity of $\widehat{A}$ also implies that both of the equations $u = \widehat{A}(u)$ and $u = A(u)$ have a unique solution in $C[a,b]$ and moreover the above equations have a common solution.

Consider $\varepsilon > 0$ a fixed number. If $\tilde{u}$ is an approximate solution for the equation $u = Au$, with $\tilde{u}(t) = \eta(t) + A(\tilde{u})(t)$, where $|\eta(t)| \leq \varepsilon$ for each $t \in [a,b]$, then it is also an approximate solution for the equation $u = \widehat{A}(u)$. This can be proved by a straightforward calculation. Indeed $\tilde{u}(t) = \eta(t) + A(\tilde{u})(t)$ implies

$$\tilde{u}(t) = \eta(t) + \sum_{j=1}^{n-1} \int_a^t K_1^{(j)}(t,s)\eta(s)ds + A^n(\tilde{u})(t),$$

where $K_1^{(j)}$, $0 \leq j \leq n$ are the iterated kernels defined by

$$K_1^{(0)}(x,s) \equiv 1,\ K_1^{(1)}(x,s) = K_1(x,s)$$

and

$$K_1^{(j+1)}(x,s) = \int_s^x K_1^{(j)}(x,t)K_1(t,s)dt \text{ for } x,s \in [a,b] \text{ and } j \geq 1.$$

Hence

$$|\tilde{u}(t) - A^n(\tilde{u})(t)| \leq \varepsilon \sum_{j=0}^{n-1} \int_a^t K_1^{(j)}(t,s)ds = c_1 \cdot \varepsilon,$$

where $c_1 \in \mathbb{R}$. This inequality implies that

$$d_\tau(\tilde{u}, \widehat{A}(\tilde{u})) = \max_{t\in[a,b]} e^{-\tau(t-a)}|\tilde{u}(t) - A^n(\tilde{u})(t)| \leq c_1 \cdot \varepsilon$$

and from the Ulam-Hyers stability of the equation $u = \widehat{A}(u)$ there exist a constant $c_2 > 0$ such that $d_\tau(\tilde{u}, u) \leq c_2 \cdot c_1 \cdot \varepsilon$, where u is the unique solution of the equation $u = A(u)$. This concludes the proof.

Remark 2.9 *For $a = 0$ the previous theorem implies the existence, uniqueness of the solution and the Ulam-Hyers stability of the fractional differential equation*

$$\begin{cases} D^q(y - T_{m-1}[y])(x) = y(x) & \\ y^{(k)}(0) = y_0^{(k)}, & 0 \leq k \leq m-1 \end{cases}$$

where T_{m-1} is the Taylor polynomial of order $(m-1)$ centered at the origin. The existence and uniqueness of the solution for this equation (and also for a nonlinear version of it) was studied in [14] *by using the Mittag-Leffler function.*

Theorem 2.10 *If $f \in C([a,b] \times [a,b])$, and the equation*

$$u(x,y) = f(x,y) + \int_a^x \int_a^y K_2(x,y,s,t)u(s,t)dsdt \tag{2.6}$$

is weakly singular, then it is also Ulam-Hyers stable.

Proof. We define the iterated kernels in a similar manner: $K_2^{(0)} \equiv 1, K_2^{(1)} = K_2$ and

$$K_2^{(j+1)}(x,y,s,t) = \int_s^x \int_t^y K_2^{(j)}(x,y,q,r)K_2(q,r,s,t)dqdr \text{ for } j \geq 1.$$

By a straightforward computation we can prove the following properties of the iterated kernels: If

$$|K_2(x,y,s,t)| \leq \frac{P_2}{|x-s|^{\alpha_1}|y-t|^{\alpha_2}}, \ \ \forall x,y,s,t \in [a,b] \text{ with } x \neq s \text{ and } y \neq t.$$

and

$$|K_2^{(j)}(x,y,s,t)| \leq \frac{P_2}{|x-s|^{\alpha_1'}|y-t|^{\alpha_2'}}, \ \ \forall x,y,s,t \in [a,b] \text{ with } x \neq s \text{ and } y \neq t.$$

then

- for $\alpha_1+\alpha_1' > 1$ and $\alpha_2+\alpha_2' > 1$ $K_2^{(j+1)}$ is also weakly singular with

$$|K_2^{(j+1)}(x,y,s,t)| \leq \frac{P_3}{|x-s|^{\alpha_1+\alpha_1'-1}|y-t|^{\alpha_2+\alpha_2'-1}}, \ \ \forall x,y,s,t \in [a,b], x \neq s, y \neq t;$$

- for $\alpha_1+\alpha_1' \leq 1$ and $\alpha_2+\alpha_2' \leq 1$ $K_2^{(j+1)}$ is continuous and satisfies

$$|K_2^{(j+1)}(x,y,s,t)| \leq P_4(x-s)^{1-\alpha_1-\alpha_1'}(y-t)^{1-\alpha_2-\alpha_2'}, \forall x,y,s,t \in [a,b];$$

- for $\alpha_1+\alpha_1' > 1$ and $\alpha_2+\alpha_2' \leq 1$ $K_2^{(j+1)}$ is continuous in y,t, has a weak singularity in x,s and satisfies the inequality

$$|K_2^{(j+1)}(x,y,s,t)| \leq \frac{P_5(y-t)^{1-\alpha_2-\alpha_2'}}{(x-s)^{\alpha_1+\alpha_1'-1}}, \forall x,y,s,t \in [a,b], x \neq s;$$

- for $\alpha_1+\alpha_1' \leq 1$ and $\alpha_2+\alpha_2' > 1$ $K_2^{(j+1)}$ is continuous in x,s, has a weak singularity in y,t and satisfies the inequality

$$|K_2^{(j+1)}(x,y,s,t)| \leq \frac{P_6(x-s)^{1-\alpha_1-\alpha_1'}}{(y-t)^{\alpha_2+\alpha_2'-1}}, \forall x,y,s,t \in [a,b], y \neq t;$$

where P_3, P_4, P_5, P_6 are some real numbers. These properties and the fact that $\alpha_1, \alpha_2 \in (0,1)$ guarantees the existence of a number n such that the n^{th} iterate of the operator $A_2 : C([a,b]\times[a,b]) \to C([a,b]\times[a,b])$ defined by

$$A_2(u)(x,y) = f(x,y) + \int_a^x \int_a^y K_2(x,y,s,t)u(s,t)dsdt$$

is a contraction with respect to a Bielecki norm (being a regular Volterra operator). Moreover the sum

$$\sum_{j=0}^{n-1} \int_a^x \int_a^y K_2^{(j)}(x,y,s,t)dsdt$$

is bounded (because each integral that appears in this sum is bounded), hence the previous proof can be repeated step by step for the operator A_2.

Remark 2.11 *This theorem implies the Ulam-Hyers stability of the following Darboux type problem for (Caputo type) fractional differential equation*

$$\begin{cases} {}^C D_\theta^r u(x,y) = u(x,y), & x,y \in [a,b] \\ u(x,0) = \varphi(x) & x \in [a,b] \\ u(0,y) = \psi(y) & y \in [a,b] \\ \varphi(0) = \psi(0) \end{cases}$$

where $r = (r_1, r_2) \in (0,1)\times(0,1)$, $\varphi, \psi : [a,b] \to \mathbb{R}$ *are absolutely continuous functions. A nonlinear version of this problem was studied in* [2]. *Our approach suggests that the Ulam-Hyers stability (such as the existence and uniqueness of the solution) can be obtained without additional assumptions on the Lipschitz constants. This aspect will be completely clarified in the following theorem.*

Theorem 2.12 *If* $f \in C([a,b]\times[a,b])$, $g \in C([a,b]\times[a,b]\times\mathbb{R})$ *has the Lipschitz property in the last variable and equation (2.4) is weakly singular, then it is also Ulam-Hyers stable.*

Proof. If we denote by L the Lipschitz constant of the function g, then for the integral operator $A_3 : C([a,b]\times[a,b]) \to C([a,b]\times[a,b])$ defined by

$$A_3(u)(x,y) = f(x,y) + \int_a^x \int_a^y K_2(x,y,s,t) g(s,t,u(s,t)) dsdt$$

we have

$$|A_3(u)(x,y) - A_3(v)(x,y)| \le L \int_a^x \int_a^y |K_2(x,y,s,t)||u(s,t) - v(s,t)| dsdst$$

$$\le L \cdot H^* \int_a^x \int_a^y \frac{e^{\tau(s-a)} e^{\tau(t-a)}}{|x-s|^{\alpha_1}|y-t|^{\alpha_2}} e^{-\tau(s-a)} e^{-\tau(t-a)} |u(s,t) - v(s,t)| dsdst$$

$$\le L \cdot H^* d_\tau(u,v) \int_a^x \int_a^y \frac{e^{\tau(s-a)} e^{\tau(t-a)}}{|x-s|^{\alpha_1}|y-t|^{\alpha_2}} dsdst$$

$$\le L \cdot H^* d_\tau(u,v) \left(\int_a^x \frac{e^{\tau(s-a)}}{|x-s|^{\alpha_1}} ds \right) \left(\int_a^y \frac{e^{\tau(t-a)}}{|y-t|^{\alpha_2}} dt \right),$$

where H^* is the maximum of $|H_1(x,y,s,t)|$ for $x,y,s,t \in [a,b]$. If we apply Hölder's inequality for the last two integrals we obtain:

$$|A_3(u)(x,y) - A_3(v)(x,y)| \le$$

$$\le L \cdot H^* d_\tau(u,v) \left(\frac{(b-a)^{2-p\alpha_1-p\alpha_2}}{(1-p\alpha_1)(1-p\alpha_2)} \right)^{1/p} \frac{e^{\tau(x-a)} e^{\tau(y-a)}}{(\tau q)^{\frac{2}{q}}},$$

where $1 < p < \min\left\{\frac{1}{\alpha_1}, \frac{1}{\alpha_2}\right\}$ and $\frac{1}{p} + \frac{1}{q} = 1$. This inequality implies that there exist $\tau > 0$ such that A_3 is a contraction with respect to the Bielecki metric d_τ. Due to Theorem 2.5 the fixed point equation $u = A_3(u)$ has a unique solution in $C([a,b] \times [a,b])$ and is Ulam-Hyers stable. This property implies the Ulam-Hyers stability in the usual (pointwise) sense. If $\varepsilon > 0$ is a fixed constant and $\tilde{u}$ is an ε-solution for $u = A_3(u)$, then $|\tilde{u}(x,y) - A_3(\tilde{u})(x,y)| \le \varepsilon$ for all $x, y \in [a,b]$. This inequality implies that $d_\tau(\tilde{u}, A_3(\tilde{u})) \le \varepsilon$. From the Ulam-Hyers stability of the fixed point equation in $(C([a,b] \times [a,b]), d_\tau)$ we deduce the existence of a constant $c > 0$ (independent of ε) such that $d_\tau(u, \tilde{u}) < c \cdot \varepsilon$, where u is the unique solution of the equation $u = A_3(u)$. But this inequality implies $|u(x,y) - \tilde{u}(x,y)| \le e^{2\tau(b-a)} \cdot c \cdot \varepsilon$, so we have the Ulam-Hyers stability of the integral equation in the usual (pointwise) sense.

Remark 2.13 *Theorem 2.12 implies the Ulam-Hyers stability for the nonlinear fractional differential Darboux problem*

$$\begin{cases} {}^C D_\theta^r u(x,y) = g(x,y,u(x,y)), & x,y \in [a,b] \\ u(x,0) = \varphi(x) & x \in [a,b] \\ u(0,y) = \psi(y) & y \in [a,b] \\ \varphi(0) = \psi(0) & \end{cases}$$

where $r = (r_1, r_2) \in (0,1) \times (0,1)$, $\varphi, \psi : [a,b] \to \mathbb{R}$ are absolutely continuous functions and g has the Lipschitz property with respect to the last variable. This result shows that condition (5) from Theorem 3.1. in [2] *can be omitted without losing the Ulam-Hyers stability of the equation. The same observation applies to condition 3.3 in* [12], *condition (10) in* [4], *condition (3.1) in* [3].

Definition 2.14 The equation

$$\begin{aligned} u(x,y) &= f(x,y) + \int_a^x \int_a^y K_1(x,y,s,t) g_1(x,y,s,t,u(s,t)) ds dt + \\ &+ \int_a^b \int_a^b K_2(x,y,s,t) g_2(x,y,s,t,u(s,t)) ds dt, \end{aligned} \tag{2.7}$$

with $f \in C([a,b] \times [a,b])$ is called weakly singular if one of the kernels K_1 and K_2 is weakly singular and the other is continuous or if both kernels are weakly singular. Suppose

$$K_i(x,y,s,t) = \frac{H_i(x,y,s,t)}{|x-s|^{\alpha_{1i}} |y-t|^{\alpha_{2i}}}, \ \forall x,y,s,t \in [a,b] \text{ with } x \ne s \text{ and } y \ne t$$

for $i \in \{1,2\}$ and H_i^* is the maximum of $H_i(x,y,s,t)$ when $x,y,s,t \in [a,b]$.

Theorem 2.15 *If $f \in C([a,b] \times [a,b])$, $g_1, g_2 \in C\left([a,b]^4 \times \mathbb{R}\right)$ are Lipschitz functions with respect to the last variable (L_1 and L_2 are the corresponding Lipschitz constants) and the condition $(*)$ is satisfied, then equation (2.7) is Ulam-Hyers stable.*

Condition $(*)$*: There exist* $\tau > 0$ *such that the matrix*

$$\begin{bmatrix} A & B \\ Ae^{2\tau(b-a)} & B \end{bmatrix}$$

is convergent to zero, where

$$A = L_1 \cdot H_1^* \cdot \left(\frac{(b-a)^{2-p\alpha_{11}-p\alpha_{21}}}{(1-p\alpha_{11})(1-p\alpha_{21})} \right)^{1/p} \frac{1}{(\tau q)^{\frac{2}{q}}}$$

$$B = L_2 \cdot H_2^* \cdot \frac{2^{\alpha_{12}+\alpha_{22}}(b-a)^{2-\alpha_{12}-\alpha_{22}}}{(1-\alpha_{12})(1-\alpha_{22})}.$$

Condition $(*)$ is fulfilled if $H_2^* = 0$, so for the Volterra case we do not need any additional condition. If $H_1^* = 0$ we obtain the usual additional condition on the Lipschitz constant for Fredholm equations.

Proof. Using the same technique as in the proof of Theorem 2.12 for the operator $A_4 : C([a,b] \times [a,b]) \to C([a,b] \times [a,b])$ defined by

$$\begin{aligned} A_4(u)(x,y) = & f(x,y) + \int_a^x \int_a^y K_1(x,y,s,t) g_1(x,y,s,t,u(s,t)) ds dt + \\ & + \int_a^b \int_a^b K_2(x,y,s,t) g_2(x,y,s,t,u(s,t)) ds dt \end{aligned} \tag{2.8}$$

we obtain

$$d_\tau(A_4 u, A_4 v) \le A \cdot d_\tau(u,v) + B \cdot d(u,v) \text{ and}$$
$$d(u,v) \le Ae^{2\tau(b-a)} \cdot d_\tau(u,v) + B \cdot d(u,v),$$

where d denotes the Chebyshev metric. By considering the vector valued metric $\mathbf{d} : C([a,b] \times [a,b]) \times C([a,b] \times [a,b]) \to \mathbb{R}^2$, $\mathbf{d}(u,v) = (d_\tau(u,v), d(u,v))^t$ we can apply Perov's fixed point theorem in the space $X = (C([a,b] \times [a,b]), \mathbf{d})$ for the operator A_4. Theorem 2.5 is valid also for generalized metric spaces (with vector valued metrics), hence the considered equation is Ulam-Hyers stable.

The study of weakly singular Fredholm-Volterra mixed equations is crucial in the study of fractional differential equations with nonlocal conditions. Our approach can be used also to obtain a part of the results from [16].

3. Ulam-Hyers stability of the Bessel equation

In this section we study the generalized Ulam-Hyers-Rassias stability of the modified Bessel equation on compact intervals of the form $[0,T]$, where $T \in \mathbb{R}$. This approach can be used also for studying the Ulam-Hyers-Rassias stability of the Bessel, spherical Bessel, modified spherical, generalized Bessel or generalized and normalized Bessel equation. The main idea is to study the Ulam-Hyers stability of an equivalent singular integral equation in a well chosen weighted Banach space.

For the sake of completeness we recall the above mentioned equations and we give the equivalent integral equation we work with.

Definition 3.1 ([10]) If $p, b, c, K \in \mathbb{R}$ the equation

$$x^2y'' + xy' + (x^2 - p^2)y = 0 \text{ is called Bessel equation;} \tag{3.1}$$

$$x^2y'' + xy' - (x^2 + p^2)y = 0 \text{ is called modified Bessel equation;} \tag{3.2}$$

$$x^2y'' + 2xy' + (x^2 - p(p+1))y = 0 \text{ is called spherical Bessel equation;} \tag{3.3}$$

$$x^2y'' + 2xy' - (x^2 + p(p+1))y = 0 \text{ is called modified spherical Bessel equation;} \tag{3.4}$$

$$x^2y'' + bxy' + (cx^2 - p^2 + (1-b)p)y = 0 \text{ is called generalized Bessel equation;} \tag{3.5}$$

$$4x^2y'' + 4Kxy' + cxy = 0 \text{ is called generalized and normalized Bessel equation.} \tag{3.6}$$

In [22] the authors proved the following theorem:

Theorem 3.2 ([22]) *Let $y : (-\rho, \rho) \to \mathbb{C}$ be a given analytic function which can be represented by a power-series expansion centered at $x = 0$. Suppose there exists a constant $\varepsilon > 0$ such that*

$$|x^2y''(x) + xy'(x) + (x^2 - p^2)y(x)| \leq \varepsilon$$

for all $x \in (-\rho, \rho)$ and for some positive nonintegral number p. Let $\rho_1 = \min\{1, \rho\}$. Suppose, further, that

$$x^2y''(x) + xy'(x) + (x^2 - p^2)y(x) = \sum_{m=0}^{\infty} a_m x^m$$

satisfies

$$\sum_{m=0}^{\infty} |a_m x^m| \leq K \left| \sum_{m=0}^{\infty} a_m x^m \right|$$

for all $x \in (-\rho, \rho)$ and for some constant K. Then there exists a Bessel function $y_h : (-\rho, \rho) \to \mathbb{C}$ such that $|y(x) - y_h(x)| \leq C\varepsilon$, $x \in (-\rho, \rho)$, where $\rho_0 < \rho_1$ is any positive number and C is some constant which depends on ρ_0.

This result gives a partial answer to the Ulam-Hyers stability problem of the Bessel equation under highly restrictive conditions on the right hand side. Our aim is to prove the generalized Ulam-Hyers-Rassias stability using less restrictive conditions. For this we need the equivalent integral equations and some Ulam-Hyers type stability properties of these equations.

Lemma 3.3 *The above mentioned differential equations are equivalent to the following integral equations:*

- $y(x) = y(0) - \int\limits_0^x \ln\frac{x}{s}\frac{s^2-p^2}{s}y(s)ds$ *for equation* (3.1)
- $y(x) = y(0) + \int\limits_0^x \ln\frac{x}{s}\frac{s^2+p^2}{s}y(s)ds$ *for equation* (3.2)
- $y(x) = y(0) + \int\limits_0^x \left(\frac{1}{x}-\frac{1}{s}\right)(s^2-p(p+1))y(s)ds$ *for equation* (3.3)
- $y(x) = y(0) - \int\limits_0^x \left(\frac{1}{x}-\frac{1}{s}\right)(s^2+p(p+1))y(s)ds$ *for equation* (3.4)
- $y(x) = y(0) - \int\limits_0^x \frac{x^{1-b}-s^{1-b}}{1-b}\cdot\frac{cs^2-p^2+(1-b)p}{s^{2-b}}y(s)ds$ *for equation* (3.5) *and* $b \neq 1$ *and* $y(x) = y(0) - \int\limits_0^x \ln\frac{x}{s}\frac{cs^2-p^2}{s}y(s)ds$ *if* $b = 1$;
- $y(x) = y(0) - \frac{c}{4}\int\limits_0^x \frac{x^{1-K}-s^{1-K}}{1-K}\cdot s^{K-1}y(s)ds$ *for equation* (3.6) *and* $K \neq 1$ *and* $y(x) = y(0) - \frac{c}{4}\int\limits_0^x \ln\frac{x}{s}y(s)ds$ *if* $K = 1$.

For the sake of simplicity we perform the calculations for the modified Bessel equation and at the end we give a more general result for all the equations (3.1)-(3.6).

For a fixed constant $\alpha > 1$ consider the set

$$X_\alpha = \{u \in C([0,T]) | \exists M > 0 \text{ such that } |u(x) - u(0)| \leq M \cdot x^\alpha\}.$$

Lemma 3.4 *On the set*

$$H_a := \{u \in X_\alpha | u(0) = a\}$$

the functional $d_\alpha : H_a \times H_a \to \mathbb{R}$ *defined by*

$$d_\alpha(u,v) = \min\{M \in \mathbb{R} | |u(x) - v(x)| \leq Mx^\alpha,\ \forall x \in [0,T]\}$$

is a metric.

Proof. Observe that the set

$$\mathcal{M} = \{M \in \mathbb{R} | |u(x) - v(x)| \leq Mx^\alpha,\ \forall x \in [0,T]\}$$

is nonempty because $u, v \in H_a$ and if $M \in \mathcal{M}$ for some $M \in \mathbb{R}$, then $[M, \infty) \subset \mathcal{M}$. Morever due to the continuity of $u, v \in H_a$ the set $\mathcal{M}$ is closed, hence d_α is well defined. It is clear that $d_\alpha(u,v) \geq 0$ for all $u, v \in H_a$ and $d_\alpha(u,v) = 0$ implies $u = v$. Moreover if $M_1 = d_\alpha(u,v)$ and $M_2 = d_\alpha(v,w)$ then from the inequalities

$$|u(x) - v(x)| \leq M_1 \cdot x^\alpha \text{ and}$$

$$|v(x) - w(x)| \leq M_2 \cdot x^\alpha$$

we obtain

$$|u(x) - w(x)| \leq |u(x) - v(x)| + |v(x) - w(x)| \leq (M_1 + M_2)x^\alpha.$$

This inequality implies that $d_\alpha(u, w) \leq M_1 + M_2$, hence d_α is a metric.

Lemma 3.5 *The space (H_a, d_α) is a complete metric space and if a sequence $(u_n)_{n\geq 0}$ converges in H_a, then it converges uniformly in $C[0, T]$.*

Proof. Observe that $d_\alpha(u, v) < \varepsilon$ implies $|u(x) - v(x)| < \varepsilon \cdot T^\alpha$, hence the second assertion is true. Due to this property if $(u_n)_{n\geq 0}$ is a Cauchy sequence in (H_a, d_α), then it is also a Cauchy sequence in $(C([0, T]), d)$, so there exist a function $u^* \in C([0, T])$ such that $u_n \to u^*$ uniformly on $[0, T]$. Since $u_n(0) = a$, for all $n \geq 0$, we obtain $u^*(0) = a$. $(u_n)_{n\geq 0}$ being a Cauchy sequence in (H_a, d_α) for all $\varepsilon > 0$ there exists $n(\varepsilon) \in \mathbb{N}$ such that

$$|u_{n+p}(x) - u_n(x)| \leq \varepsilon x^\alpha, \text{ for all } x \in [0, T] \text{ and } n \geq n(\varepsilon), p \in \mathbb{N}.$$

If $p \to \infty$ we obtain

$$|u^*(x) - u_n(x)| \leq \varepsilon x^\alpha, \text{ for all } x \in [0, T] \text{ and } n \geq n(\varepsilon), \tag{3.7}$$

so for a sufficiently large, but fixed n we have

$$|u^*(x) - u^*(0)| \leq |u^*(x) - u_n(x)| + |u_n(x) - a| \leq \varepsilon x^\alpha + M_n \cdot x^\alpha.$$

This inequality implies $u^* \in H_a$ while (3.7) implies $u_n \to u^*$ in H_a, so (H_a, d_α) is a complete metric space.

Theorem 3.6 *If a function $\tilde{y} \in X_\alpha$, with $\alpha \geq \max\{1, \sqrt{T^2 + p^2}\}$ satisfies the inequality*

$$|x^2\tilde{y}'' + x\tilde{y}' - (x^2 + p^2)\tilde{y}| \leq c_2 x^\alpha, \text{ for all } x \in [0, T], \tag{3.8}$$

where $T > 0$, and $p \in \mathbb{R}$, then there exist a constant c and a solution y of the modified Bessel equation (3.2) *with the property*

$$|\tilde{y}(x) - y(x)| \leq \frac{c_2}{\alpha^2 - T^2 - p^2} x^\alpha, \forall x \in [0, T].$$

Proof. If $x^2\tilde{y}'' + x\tilde{y}' - (x^2 + p^2)\tilde{y} = h(x)$, then $\tilde{y}$ is a solution of the integral equation

$$y(x) = y(0) + \int_0^x \ln \frac{x}{s} \frac{s^2 + p^2}{s} y(s) ds + \int_0^x \ln \frac{x}{s} \frac{h(s)}{s} ds.$$

Consider the integral operator $A : X_\alpha \to X_\alpha$

$$A(y)(x) = y(0) + \int_0^x \ln \frac{x}{s} \frac{s^2 + p^2}{s} y(s) ds.$$

For this operator we have

$$|A(u)(x) - A(v)(x)| \leq (T^2 + p^2) \int_0^x \ln \frac{x}{s} \frac{|u(s) - v(s)|}{s} ds \leq M(T^2 + p^2) \int_0^x \ln \frac{x}{s} s^{\alpha-1} ds,$$

where $M = d_\alpha(u, v)$. But $\int_0^x \ln \frac{x}{s} s^{\alpha-1} ds = \frac{x^\alpha}{\alpha^2}$, so we obtain the inequality

$$d_\alpha(A(u), A(v)) \leq \frac{T^2 + p^2}{\alpha^2} d_\alpha(u, v).$$

Due to the Banach contraction principle and to the assumptions on α, the operator A is a contraction on X_α. This guarantees the existence and uniqueness of a solution $y \in X_\alpha$ whith $y(0) = \tilde{y}(0)$. Moreover if z is the difference between the solutions y and the approximate solution $\tilde{y}$, then z satisfies the integral equation

$$z(x) = \int_0^x \ln \frac{x}{s} \frac{s^2 + p^2}{s} z(s) ds + \int_0^x \ln \frac{x}{s} \frac{h(s)}{s} ds.$$

Condition (3.8) guarantees that $h \in H_0$, so the operator $B : H_0 \to H_0$,

$$B(u)(x) = \int_0^x \ln \frac{x}{s} \frac{s^2 + p^2}{s} u(s) ds + \int_0^x \ln \frac{x}{s} \frac{h(s)}{s} ds$$

is well defined and it is a contraction, so there exist a unique function $z^* \in H_0$ with $z^* = Bz^*$. This unique function z^* is in fact $\tilde{y} - y$, so we have

$$d_\alpha(y, \tilde{y}) \leq \frac{1}{1 - \frac{T^2 + p^2}{\alpha^2}} d_\alpha(\tilde{y}, A(\tilde{y})).$$

This completes the proof.

The fixed point equation associated with the operator A is Ulam-Hyers stable in (H_a, d_α) and this property implies the generalized Ulam-Hyers-Rassias stability in the usual sense. Using the same technique the following result can be proved:

Theorem 3.7 *The Bessel type equations* ((3.1)-(3.6)) *are generalized Ulam-Hyers-Rassias stable on any compact interval* $[0, T]$ *in the following sense:*

If for a function $\tilde{y} \in H_a$ *we have*

$$|L(\tilde{y})(x)| < c_1 x^\alpha,$$

for some $c_1 \in \mathbb{R}$, *all* $x \in [0, T]$, *and* $\alpha > 1$ *satisfying the following condition:*

- $\alpha^2 > T^2 + p^2$ *for equation* (3.1) *and* (3.2)*;*
- $\alpha(\alpha + 1) > T^2 + p(p + 1)$ *for equation* (3.3) *and* (3.4)*;*
- $\alpha(\alpha - 1 + b) > cT^2 + p^2 + |1 - b|p$ *for* (3.5)*;*
- $(\alpha + 1)(\alpha + k) > cT/4$ *for* (3.6)*,*

then there exist a solution $y \in H_a$ of the corresponding integral equation $L(y)(x) = 0$ such that $|\tilde{y}(x) - y(x)| \le C_L \cdot c_2 \cdot x^\alpha$ for all $x \in [0, T]$, where L denotes the differential operator associated to one of the equations (3.1)-(3.6) *and the constant C_L depends only on L.*

A direct approach based on Gronwall type inequality is also possible if we can prove separately the existence of the solution in the set H_0 for the studied equation. For (3.2) if $\tilde{y}$ is an approximated solution and y is a solution with the same initial condition, the function $z = |\tilde{y} - y|$ satisfies

$$|z(x)| \le (T^2 + p^2) \int_0^x \ln \frac{x}{s} \cdot \frac{|z(s)|}{s} ds + \int_0^x \ln \frac{x}{s} \cdot \frac{|h(s)|}{s} ds.$$

Now if $|h(x)| \le c_2 x^\alpha$, for all $x \in [0, T]$ and $z \in H_0$, then from the initial estimation $|z(x)| \le c_1 x^\alpha$, by successive iteration we obtain

$$|z(x)| \le c_n x^\alpha,$$

where

$$c_{n+1} = c_n \left(\frac{p^2}{\alpha^2} + \frac{T^2}{(\alpha + 1)^2} \right) + \frac{c}{\alpha^2}.$$

If α is sufficiently large ($\frac{p^2}{\alpha^2} + \frac{T^2}{(\alpha+1)^2} < 1$), then we obtain

$$|z(x)| \le \frac{c_2 x^\alpha}{\alpha^2 \left(1 - \frac{p^2}{\alpha^2} - \frac{T^2}{(\alpha+1)^2}\right)}.$$

The same Ulam-Hyers constant can be obtained by using the reasoning from the proof of Theorem 3.7 (if we do not use a majorant for the nonsingular part inside the kernel of the integral). In this equality we can have also equality, so this Ulam-Hyers constant is the best possible constant (for the modified Bessel equation and the majorizing function x^α). The best Ulam-Hyers constant can also be calculated for the rest of the Bessel type equations.

From the inequality

$$u(x) \le c_1^2 \int_0^x \ln \frac{x}{s} \frac{u(s)}{s} ds + \frac{c_2 x^\alpha}{\alpha^2}, \tag{3.9}$$

where $c_1, c_2, \alpha \in (0, \infty)$ and $\alpha > \max\{1, c_1\}$, we can not prove

$$u(x) \le \frac{c_2 x^\alpha}{\alpha^2 - c_1^2} \quad \text{for } x \in [0, T]$$

in the general case of $u \in C([0, T])$. By iterating the given inequality we can obtain

$$u(x) \le \frac{c_1^{2n}}{(2n-1)!} \int_0^x \left(\ln \frac{x}{s}\right)^{2n-1} \frac{u(s)}{s} ds + e_n x^\alpha, \tag{3.10}$$

where $e_n = \frac{c_2}{\alpha^2 - c_1^2}\left(1 - \left(\frac{c_1^2}{\alpha^2}\right)^n\right)$, but an apriori bound for u is still required in order to obtain an upper bound for u.

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References

[1] S. Abbas, M. Benchohra, *Darboux problem for perturbed partial differential equations of fractional order with finite delay*, Nonlinear Anal. Hybrid Syst., **3**(2009), Issue 4, 597-604.

[2] S. Abbas, M. Benchohra, *Ulam-Hyers stability for the Darboux problem for partial fractional differential and integro-differential equations via Picard operators*, Results Math., **65**(2014), 67-79, published online September 28, 2013.

[3] S. Abbas, M. Benchohra, A. Petruşel, *Ulam stability for partial fractional differential inclusions via Picard operators theory*, Electron. J. Qual. Theory Differ. Equ., 2014, No. 51, 1-13, http://www.math.u-szeged.hu/ejqtde/

[4] S. Abbas, *Darboux problem for partial functional differential equations with infinite delay and Caputo's fractional derivative*, Adv. Dyn. Syst. Appl., **5**(2010), no. 1, 1-19.

[5] D.R. Anderson, J.M. Otto, *Hyers-Ulam stability of certain linear differential equations with singularities* (to appear).

[6] S. András, I.A. Rus, *Iterates of Cesaro operators, via fixed point principle*, Fixed Point Theory, **11**(2010), no. 2, 171–178.

[7] S. András, *Weakly singular Volterra and Fredholm-Volterra integral equations*, Stud. Univ. Babeş-Bolyai Math., **48**(2003), no. 3, 147–155.

[8] S. András, A. Mészáros, *Ulam-Hyers stability of dynamic equations on time scales via Picard operators*, Appl. Math. Comput., **219**(2013), no. 9, 4853-4864.

[9] S. András, A. Mészáros, *Ulam-Hyers stability of elliptic partial differential equations in Sobolev spaces*, Appl. Math. Comput., **229**(2014), 131-138.

[10] Á. Baricz, *Generalized Bessel functions of the first kind*, Lecture notes in Mathematics, Springer, 2010.

[11] L.C. Becker, *Resolvents and solutions of weakly singular linear Volterra integral equations*, Nonlinear Anal., **74**(2011), 1892-1912.

[12] L.C. Becker, *Resolvents for weakly singular kernels and fractional differential equations*, Nonlinear Anal., **75**(2012), no. 13, 4839-4861.

[13] L. Cădariu, V. Radu, *Fixed points and the stability of Jensen's functional equation*, J. Inequal. Appl., **4**(2003), no. 1, article 3.

[14] K. Diethelm, *Analysis of fractional differential equations*, J. Math. Anal. Appl., **265**(2002), 229-248.

[15] P. Găvruţa, L. Găvruţa, *A new method for the generalized Hyers-Ulam-Rassias stability,* Int. J. Nonlinear Anal. Appl., **1**(2010), No. 2, 11-18.

[16] E. Hernández, D. O'Regan, K. Balachandran, *Existence results for abstract fractional differential equations with nonlocal conditions via resolvent operators*, Indag. Math. (N.S.), **24**(2013), no. 1, 68-82.

[17] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. USA, **27**(1941), 222–224.

[18] R.W. Ibrahim, *Ulam-Hyers stability for Cauchy fractional differential equation in the unit disk*, Abstr. Appl. Anal., Volume 2012, Article ID 613270.

[19] S.M. Jung, *Legendre's differential equation and its Hyers-Ulam stability*, Abstr. Appl. Anal., Volume 2007, Article ID 56419.

[20] S.M. Jung, H. Şevli, *Power series method and approximate linear differential equations of second order*, Adv. Difference Equ., 2013, 2013:76.

[21] S.M. Jung, *Hyers-Ulam-Rassias stability of functional equations in nonlinear analysis*, Springer, 2011.

[22] B. Kim, S.M. Jung, *Bessel's differential equation and its Hyers-Ulam stability*, J. Inequal. Appl., Volume 2007, Article ID 21640.

[23] I.A. Rus, *Picard operators and applications*, Sci. Math. Jpn., **58**(2003), 191–219.

[24] I.A. Rus, *Remarks on Ulam stability of the operatorial equations,* Fixed Point Theory, **10**(2009), 305–320.

[25] I.A. Rus, *Ulam stability of ordinary differential equations*, Stud. Univ. Babeş-Bolyai, Math., **54**(2009), No. 4, 125–133.

[26] I.A. Rus, *Gronwall lemma approach to the Hyers-Ulam-Rassias stability of an integral equation,* 147-152, in: Nonlinear Analysis and Variational Problems (P.M. Pardalos, Th. M. Rassias and A.A. Khan (Eds.)), Springer, 2010.

[27] I.A. Rus, *Results and problems in Ulam stability of operatorial equations and inclusions,* 323-352, in: Handbook of Functional Equations: Stability Theory (Th. M, Rassias (Ed.)), Springer, 2014.

[28] S.M. Ulam, *A collection of mathematical problems*, Interscience, New York, 1960.

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