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CRITICAL KRASNOSELSKII-SCHAEFER TYPE FIXED POINT THEOREMS FOR WEAKLY SEQUENTIALLY CONTINUOUS MAPPINGS AND APPLICATION TO A NONLINEAR INTEGRAL EQUATION

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Abstract. In this paper, we first state some new fixed point theorems for operators of the form $A + B$ on a bounded closed convex set of a Banach space, where A is a weakly compact and weakly sequentially continuous mapping and B is either a weakly sequentially continuous nonlinear contraction or a weakly sequentially continuous separate contraction mapping. Second, we study the fixed point property for a larger class of weakly sequentially continuous mappings under weaker assumptions and we explore this kind of generalization by looking for the multivalued mapping $(I - B)^{-1}A$, when $I - B$ may not be injective. To attain this goal, we extend H. Schaefer's theorem to multivalued mappings having weakly sequentially closed graph. Our results generalize many known ones in the literature, in particular those obtained by C. Avramescu (2004, Electron. J. Qual. Theory Differ. Equ., $17, 1 - 10$), C. S. Barroso (2003, Nonlinear Anal., $55, 25 - 31$), T.A. Burton (1998, Appl. Math. Lett., $11, 85 - 88$), Y. Liu and Z. Li (2006, J. Math. Anal. Appl., $316, 237 - 255$ and 2008, Proc. Amer. Math. Soc., $316, 1213 - 1220$), H. Schaefer (1955, Math. Ann., $129, 415 - 416$) and M.A. Taoudi (2010, Nonlinear Anal., 72 $(1), 478 - 482$). Finally, we use our abstract results to derive an existence theory for an integral equation in a reflexive Banach space.
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1. Introduction

Many problems arising from the most diverse areas of natural science, when modeled under the mathematical point of view, involve the study of solutions of nonlinear equations of the form

$$Ax + Bx = x, \ x \in \Omega, \tag{1.1}$$

where Ω is a nonempty closed and convex subset of a Banach space E. In particular, many problems in integral equations can be formulated in terms of Eq. (1.1). A useful prototype for solving equations of the type (1.1) is the celebrated fixed point theorem due to Krasnoselskii in 1958 (see for example Theorem 11.B in [26]) which asserts that, if Ω is a nonempty closed, bounded and convex subset of a Banach space E; A and B are two maps from Ω into E so that (i) A is compact, (ii) B is a contraction mapping and (iii) $A\Omega + B\Omega \subset \Omega$, then $A + B$ has at least one fixed point in Ω. Since then, a wide class of problems, for instance in integral equations and stability theory, have been contemplated by the Krasnoselskii fixed point approach. However, there are various problems arising in mathematical physics and population dynamics which may be written in the form (1.1) but, in general, A and B do not satisfy the assumptions (i)-(iii); (see for example [3]-[8], [11], [13], [22] and [25]). So, many generalizations of the Krasnoselskii's theorem have been estabished in the litterature in the course of time by modifying the assumptions (i), (ii) or (iii). In [11], Burton improved the Krasnoselskii principle by requiring, instead of (iii), the more general condition $[x = Bx + Ay, y \in \Omega] \Longrightarrow x \in \Omega$. His result applies to stability theory, integral equations and cover cases where Krasnoselskii's theorem can not apply; (see [11] and [12]). Subsequently, Burton and Kirk extended Krasnoselskii's approach by combining a result of Schaefer on fixed point from a priori bound with Banach's theorem. Using Schaefer's theorem (see [24]), the authors obtained the following Krasnoselskii-Schaefer type fixed point theorem: if $A, B : E \longrightarrow E$ are two mappings such that B is a contraction and A is continuous and maps bounded sets into compact sets, then either

(i) (i) $x = \lambda B(\frac{x}{\lambda}) + \lambda Ax$ has a solution in E for $\lambda = 1$, or

(ii) (ii) the set of all such solutions,$0 < \lambda < 1$, is unbounded.

Since infinite dimentional Banach spaces are not locally compact, the authors have suggested a locally convex topology approach which is a weak topology of Banach spaces and proposed valuable attempts to prove the analogue of Krasnoselskii's fixed point theorem for the weak topology. In [3], Barroso established a new version of Krasnoselskii's theorem using the weak topology of a Banach space. His result requires the weak continuity and the weak compactness of A while B must be a bounded linear operator satisfying the condition $\|B^p\| \leq 1$ for some integer $p \geq 1$. Recently, Barroso and Texeira [4] established a fixed point theorem for the sum $A + B$ of a weakly sequentially continuous mapping A and a weakly sequentially continuous contraction B. The result requires the weak compactness of $(I - B)^{-1}A$. In [25], Taoudi relaxed the last condition and proved an analogue of Krasnoselskii's theorem for the weak topology by assuming the weak compactness of A. His analysis was based on the technique of measures of weak noncompactness and Arino et al.'s fixed point theorem ([1]). Motivated by the problem of existing of solution for some transport equation in L_1-spaces and due to the lack of compactness in that spaces, Ben Amar et al. established new valuable generalizations of Krasnoselskii's theorem in the setting of weak topology; (see [5] and [6]). In [5], the authors extended Krasnoselskii's theorem to the class of weakly compact operators in Dunford-Pettis spaces in order to prove some existence result for a source problem with general boundary conditions in L_1-spaces. Subsequently, in [6], Ben Amar et al. gave new variants of Krasnoselskii's theorem in general Banach spaces. Notice that the arguments of the authors was based on the notion of weak sequential continuity since it is not always possible to show that a given operator between Banach spaces is weakly continuous, quite often its weak sequential continuity presents no problem. The obtained results are then used to prove the existence of solution on L_1-spaces to a nonlinear boundary value problem originally proposed by Rotenberg ([23]) to model the growth of cell population.

We shall emphasize on the fact that, in all the papers mentioned above, the strategy of the authors consists in giving suffisent conditions assuring the invertibility of the operator $I-B$ in order to prove the fixed point property of the operator $(I-B)^{-1}A$ and to deduce the existence of solution for Eq. (1.1). So, it would be interesting to study the problem in the case when $I-B$ is not injective. It should be noticed that this case was considered in [21] by Liu and Li in the setting of strong topology.

The aim of the abstract part of the present paper is two folds. First, we extend a number of previously known generalizations of Krasnoselskii's theorem to a larger class of contraction mappings in the setting of weak topology. Namely, we are interested by nonlinear contraction ([9]) and separate contraction mappings ([20]); (see Theorems 3.1, 3.2, 3.3 and 3.4). Our analysis is based on the notion of the De Blasi measure of weak noncompactness (see [14]) which allows as to cover earlier results in the litterature; (see [3], [4], [6], [18] and [25]).

Second, we concentrate on the study of the existence of fixed point for the sum of two mappings, A and B in the critical case; i.e. the case when $I-B$ may not be injective and we investigate this kind of generalization by looking for the multivalued mapping $(I-B)^{-1}A$. More precisely, we prove that if A and B are weakly sequentially continuous mappings from a nonempty closed and convex subset Ω of a Banach space E into E such that

(i) $A(\Omega) \subset (I-B)(\Omega))$;
(ii) $A(\Omega)$ is a relatively weakly compact subset of E;
(iii) if $(I-B)x_n \rightharpoonup y$, then there exists a weakly convergent subsequence of $(x_n)_n$;
(iv) for every y in the range of $I-B$, $D_y = \{x \in \Omega$ such that $(I-B)x = y\}$ is a convex set;
then there exists $x \in \Omega$ such that $x = Ax + Bx$.

Moreover, we establish the analogue of Burton and Kirk's theorem ([13]) in the critical case for the weak topology: Given $A, B : E \longrightarrow E$ two weakly sequentially continuous mappings, then under the hypotheses:

(i) $A(E) \subset (I-B)(E)$;
(ii) there exists a closed convex, balanced and absorbing weak neighborhood U of θ (the zero of E) such that the set $A(nU)$ is relatively weakly compact for all $n \in \mathbb{N}$,
(iii) if $(I-B)x_n \rightharpoonup y$, then there exists a weakly convergent subsequence $(x_{n_k})_k$ of $(x_n)_n$;
(iv) for every y in the range of $I-B, D_y = \{x \in E : (I-B)x = y\}$ is convex;

we show that either for any $\lambda \in [0,1]$ there exists an $x \in E$ such that $x = \lambda B(\frac{x}{\lambda}) + \lambda Ax$ or the set $\{x \in E : \exists\ \lambda \in]0,1[,\ x = \lambda B(\frac{x}{\lambda}) + \lambda Ax\}$ is unbounded.

Notice here that B is allowed to be non injective and the fact that B is a contraction mapping (as in [13]) turns out to be a special case of the assumptions (iii) and (iv); (see Corollary 3.1). Hence, the critical case considered in our approach, improves and encompass a multitude of existing generalizations of the Krasnoselskii principle dealing with the sum of two weakly sequentially continuous mappings; (see [3]-[6], [13], [15], [22] and [25]). Furthermore, the two mentioned critical results extend the two main theorems of Liu and Li in [21] to the setting of weak topology.

To illustrate the applicability of the theories, we prove the existence of a solution for the following nonlinear integral equation

$$x(t) = f(t, x(t)) + \lambda \int_0^t g(s, x(s))ds, \; x \in \mathcal{C}(J, E), \tag{1.2}$$

where $J = [0, T]$, $\lambda \in (\frac{1}{2}, 1)$, $(E, \|.\|)$ is a reflexive Banach space, $\mathcal{C}(J, E)$ is the Banach space of all continuous functions from J to E endowed with the sup-norm $\|.\|_\infty$, defined by $\|x\|_\infty = \sup \{\|x(t)\|;\; t \in J\}$, for each $x \in \mathcal{C}(J, E)$ and f and g satisfy some conditions (see Section 4).

Our paper will be divided into three main parts. In Section 2, we recall some basic definitions and we give known results for future use. Section 3 is devoted to establish some new Krasnoselskii-Schaefer type fixed point theorems for some classes of weakly sequentially continuous mappings under weaker assumptions on mappings and domains as it is known. An existence theory for the problem (1.2) is the topic of Section 4.

2. Preliminaries

Let Ω_E be the set of bounded subsets of a Banach space E and let $\mathcal{K}^w$ be the family of all weakly compact subsets of E. In addition, let B_r denote the closed ball centred at θ, the zero of E, with radius $r > 0$. The De Blasi [14] measure of weak non-compactness is the map $\beta : \Omega_E \longrightarrow [0, +\infty)$ defined by

$$\beta(X) = \inf\{r > 0 : \text{ there exists } Y \in \mathcal{K}^w \text{ such that } X \subset Y + B_r\}, \text{ here } X \in \Omega_E.$$

For convenience we recall some properties of β :
Let $X_1, X_2 \in \Omega_E$. Then:
(i) $X_1 \subset X_2$ implies $\beta(X_1) \leq \beta(X_2)$.
(ii) $\beta(X_1) = 0$ iff $\overline{X_1^w} \in \mathcal{K}^w$, here $\overline{X_1^w}$ is the weak closure of X_1 in E.
(iii) $\beta(X_1) = \beta(\overline{X_1^w})$.
(iv) $\beta(X_1 \cup X_2) = \max\{\beta(X_1), \beta(X_2)\}$.
(v) $\beta(\lambda X_1) = \lambda\beta(X_1)$ for all $\lambda > 0$.
(vi) $\beta(conv(X_1)) = \beta(X_1)$.
(vii) $\beta(X_1 + X_2) \leq \beta(X_1) + \beta(X_2)$.

Definition 2.1. *Let Ω be a nonempty subset of a Banach space E. If F maps Ω into E, we say that:*

(i) *F is β-condensing if F is bounded and $\beta(F(D)) < \beta(D)$ for all bounded subsets D of Ω with $\beta(D) > 0$;*
(ii) *F is weakly compact if F is bounded and $F(D)$ is relatively weakly compact for every bounded subset $D \subset \Omega$.*
(iii) *F is weakly sequentially continuous if for every sequence $(x_n)_n \subset \Omega$ with $x_n \rightharpoonup x \in \Omega$, we have $Fx_n \rightharpoonup Fx$; here $\rightharpoonup$ denotes weak convergence.*

Remark 2.1. *Notice that a weakly compact mapping is β-condensing.*

Definition 2.2. [20, Definition 1.2] *Let (E, d) be a metric space and $F : E \longrightarrow E$. Then, F is said to be separate contraction if there exist two functions $\varphi, \psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ satisfying*
(i) *$\psi(0) = 0$, ψ is strictly increasing.*
(ii) *$d(F(x), F(y)) \leq \varphi(d(x, y))$.*
(iii) *$\psi(r) \leq r - \varphi(r)$ for $r > 0$.*

Remark 2.2. *The mapping $B : E \longrightarrow E$ defined by $(Bx)(t) = f(t, x(t))$, where $E = \mathcal{C}(\mathbb{R}, [0, 1])$ and $f : \mathbb{R} \times [0, 1] \longrightarrow [0, 1]$ with $f(t, x) = x - \frac{x^4}{4} + \frac{sin^2(t)}{4}$ is a separate contraction (see* [20, Example 4.1]*).*

Definition 2.3. [9] *Let $(E, \|.\|)$ be a Banach space. We say that $F : E \longrightarrow E$ is a nonlinear contraction mapping if there exists a continuous nondecreasing function $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ satisfying $\varphi(z) < z$ for $z > 0$, such that*

$$\|F(x) - F(y)\| \leq \varphi(\|x - y\|), \ \forall x, y \in E.$$

Remark 2.3. (i) *Note that every contraction is a nonlinear contraction (resp. a separate contraction) but the converse is not true.*
(ii) *We point out that the notions of separate contraction mapping and nonlinear contraction mapping are different and nonlinear contraction mappings do not generate separate contraction mappings.*

Remark 2.4. *One of the advantages of the weak topology of a Banach space E is the fact that if a set Ω is weakly compact, then every weakly sequentially continuous mapping $F : \Omega \longrightarrow E$ is weakly continuous. This is an immediate consequence of the Eberlein-Šmulian's theorem (see* [17, Theorem 8.12.4, p. 549]*).*

Theorem 2.1. [6, Theorem 2.5] *Let Ω be a nonempty, convex and closed set in a Banach space E. Assume $F : \Omega \longrightarrow \Omega$ is a weakly sequentially continuous mapping. If $F(\Omega)$ is relatively weakly compact, then F has a fixed point in Ω.*

Theorem 2.2. [7, Theorem 3.2] *Let Ω be a nonempty, convex and closed set in a Banach space E. Assume $F : \Omega \longrightarrow \Omega$ is a weakly sequentially continuous and β-condensing mapping. In addition, suppose that $F(\Omega)$ is bounded. Then, F has a fixed point.*

Remark 2.5. *Theorem 2.2 extends and improves Theorem 2.1.*

Let us now introduce the concept of multivalued mappings.

Definition 2.4. *Let Ω be a nonempty subset of a Banach space E. We call mutivalued mapping (or multi-function) defined on Ω every application $H : \Omega \longrightarrow \mathcal{P}(E)$, where $\mathcal{P}(E) := \{M$ such that $M \subset E$ and $M \neq \emptyset\}$.*

Remark 2.6. *(i) Every single valued mapping (or mapping) $F : \Omega \longrightarrow E$ can be identified with a multivalued mapping $H : \Omega \longrightarrow \mathcal{P}(E)$, (see* [26, p. 447]*) by setting*

$$H(x) = \{F(x)\} \text{ for all } x \in \Omega.$$

(ii) We denote $Gr(H) = \{(x, y) \in \Omega \times E : x \in \Omega$ and $y \in H(x)\}$ the graph of H.

Definition 2.5. *Let Ω be a nonempty subset of a Banach space E and $H : \Omega \longrightarrow \mathcal{P}(E)$ a mutivalued mapping. H is said to have a weakly sequentially closed graph if $Gr(H)$ is weakly sequentially closed, i.e., if for every sequence $(x_n)_n \subset \Omega$ with $x_n \rightharpoonup x$ and for every sequence $(y_n)_n$ with $y_n \in H(x_n), \forall n \in \mathbb{N}, y_n \rightharpoonup y$ implies $y \in H(x)$; here $\rightharpoonup$ denotes weak convergence.*

Theorem 2.3. [8, Theorem 2.2] *Let Ω be a nonempty closed and convex subset of a Banach space E and $H : \Omega \longrightarrow \mathcal{P}(\Omega)$ a multivalued mapping. Suppose that:*

(i) $H(\Omega)$ is relatively weakly compact.
(ii) H has a weakly sequentially closed graph.
(iii) The set $H(x)$ is not empty closed and convex for all $x \in \Omega$.

Then, there exists $x \in \Omega$ such that $x \in H(x)$.

Remark 2.7. *Since every weakly sequentially continuous single valued mapping can be identified with a multivalued mapping having a weakly sequentially closed graph, Theorem 2.3 is the multivalued analogue of Theorem 2.1.*

3. Fixed point theory

At the beginning of this section, we will state some new Krasnoselskii type fixed point theorems for different classes of weakly sequentially continuous mappings. The first result is formulated as

Theorem 3.1. *Let Ω be a nonempty closed bounded and convex subset of a Banach space E. In addition, let $A : \Omega \longrightarrow E$ be a weakly sequentially continuous mapping and $B : E \longrightarrow E$ satisfying:*

(i) $A(\Omega)$ is relatively weakly compact.
(ii) B is linear, bounded and there exists $p \in \mathbb{N}^*$ such that B^p is a separate contraction.
(iii) $[x = Bx + Ay,\ y \in \Omega] \Longrightarrow x \in \Omega$.

Then, there exists $x \in \Omega$ such that $x = Ax + Bx$.

Proof. Since B is linear, bounded and B^p is a separate contraction, then $(I - B^p)^{-1}$ exists on E, (see [20, Lemma 1.2]). Hence

$$(I - B)^{-1} = (I - B^p)^{-1} \sum_{k=0}^{p-1} B^k. \tag{3.1}$$

By Eq. (3.1), we have $(I-B)^{-1} \in \mathcal{L}(E)$, so $(I-B)^{-1}$ is weakly continuous. Define the mapping $F := (I-B)^{-1}A$. Since $A : \Omega \longrightarrow E$, then from assumption (iii) it follows that $F : \Omega \longrightarrow \Omega$. Since $(I-B)^{-1}$ is weakly continuous (see [10, p. 39]) and A is weakly sequentially continuous, so F is weakly sequentially continuous. Moreover, we have A maps bounded sets into relatively weakly compact sets and $(I-B)^{-1}$ is weakly continuous, then F maps bounded sets into relatively weakly compact sets. Hence, F fulfills the conditions of Theorem 2.1. □

Theorem 3.1 remains true if we suppose that there exists $p \in \mathbb{N}^*$ such that B^p is a nonlinear contraction.

Theorem 3.2. *Let Ω be a closed, bounded and convex subset of a Banach space E. In addition, let $A : \Omega \longrightarrow E$ be a weakly sequentially continuous mapping and $B : E \longrightarrow E$ satisfying:*

(i) $A(\Omega)$ is relatively weakly compact.
(ii) B is linear, bounded and there exists $p \in \mathbb{N}^*$ such that B^p is a nonlinear contraction.
(iii) $[x = Bx + Ay,\ y \in \Omega] \Longrightarrow x \in \Omega$.

Then, there exists $x \in \Omega$ such that $x = Ax + Bx$.

Proof. Since B is linear, bounded and B^p is a nonlinear contraction, then $(I-B^p)^{-1}$ exists on E, (see [9]). Now, reasoning as in the proof of Theorem 3.1, we get the desired result. □

Remark 3.1. (i) *Theorem 3.2 extends Theorem* 1.5 *in* [15] *to the case of weakly sequentially continuous mappings in the setting of weak topology.*
(ii) *Since nonlinear contraction mappings do not generate separate contraction mappings, so Theorems 3.1 and 3.2 are two different new generalizations of Krasnoselskii's fixed point theorem.*

The next result asserts:

Theorem 3.3. *Let Ω be a nonempty bounded closed and convex subset of a Banach space $(E, \|.\|)$. Suppose that $A : \Omega \longrightarrow E$ and $B : E \longrightarrow E$ are two weakly sequentially continuous mappings such that:*

(i) A is weakly compact.
(ii) B is a nonlinear contraction.
(iii) $(A+B)(\Omega) \subset \Omega$.

Then, there exists $x \in \Omega$ such that $x = Ax + Bx$.

Proof. First, we claim that B is β-condensing. Indeed, let D be a bounded subset of E such that $\beta(D) = d > 0$. Let $\varepsilon > 0$, then there exists a weakly compact set K of E satisfying $D \subseteq K + B_{d+\varepsilon}$. So, for $x \in D$ there exists $y \in K$ and $z \in B_{d+\varepsilon}$ such that $x = y + z$ and so

$$\|Bx - By\| \leq \varphi(\|x-y\|) \leq \varphi(\varepsilon + d).$$

It follows immediately, that

$$B(D) \subseteq B(K) + B_{\varphi(\varepsilon+d)}.$$

Moreover, since B is a weakly sequentially continuous mapping and K is weakly compact, then $B(K)$ is weakly compact (see Remark 2.4). Therefore, $\beta(B(D)) \leq \varphi(d+\varepsilon)$. Since $\varepsilon > 0$ is arbitrary, then $\beta(B(D)) \leq \varphi(d) < d = \beta(D)$. Hence, B is β-condensing, which ends the proof of the claim. On the other hand, it is easy to see that $A+B$ is weakly sequentially continuous. Thanks to Theorem 2.2, it suffices to show that $A+B$ is β-condensing. To see this, let D be a bounded subset of Ω. Taking into account the fact that $A(D)$ is relatively weakly compact and using the subadditivity of the De Blasi measure of weak noncompactness we get

$$\beta((A+B)(D)) \leq \beta(A(D)+B(D)) \leq \beta(A(D)) + \beta(B(D)) \leq \beta(B(D)).$$

So, if $\beta(B(D) \neq 0$ then

$$\beta((A+B)(D)) < \beta(D),$$

and hence $A+B$ is β-condensing, which ends the proof of the theorem. □

We point out that theorem 3.3 remains valid if we replace the assumption $(A+B)(\Omega) \subset \Omega$ by the following one $[x = Bx + Ay,\ y \in \Omega] \Longrightarrow x \in \Omega$, introduced in [11].

Theorem 3.4. *Let Ω be a nonempty bounded closed and convex subset of a Banach space E. Suppose that $A : \Omega \longrightarrow E$ and $B : E \longrightarrow E$ are two weakly sequentially continuous mappings such that:*

(i) A is weakly compact.
(ii) B is a nonlinear contraction.
(iii) $[x = Bx + Ay,\ y \in \Omega] \Longrightarrow x \in \Omega$.

Then, there exists $x \in \Omega$ such that $x = Ax + Bx$.

Proof. Let y be fixed in Ω. The map which assigns to each $x \in \Omega$ the value $Bx + Ay$ defines a nonlinear contraction from Ω into Ω. So, using Theorem 1 in [9] together with assumption (iii), the equation $x = Bx + Ay$ has a unique solution $x = (I-B)^{-1}Ay \in \Omega$. Therefore

$$(I-B)^{-1}A(\Omega) \subset \Omega. \tag{3.2}$$

Now, define the mapping $F : \Omega \longrightarrow \Omega$ by $F(x) := (I-B)^{-1}Ax$. Let $K = \overline{conv}(F(\Omega))$ be the closed convex hull of $F(\Omega)$. Clearly, K is closed convex bounded and $F(K) \subset K \subset \Omega$. We claim that K is weakly compact. If it is not the case, then $\beta(K) > 0$. Since $F(\Omega) \subseteq A(\Omega) + BF(\Omega)$, we obtain

$$\beta(K) = \beta(F(\Omega)) \leq \beta(A(\Omega) + BF(\Omega)) \leq \beta(A(\Omega)) + \beta(BF(\Omega)).$$

Taking into account the fact that A is weakly compact and B is β-condensing, we obtain

$$\beta(K) = \beta(F(\Omega)) \leq \beta(B(F(\Omega))) < \beta(F(\Omega)),$$

which is absurd. Hence, K is weakly compact. In view of Theorem 2.1, it remains to show that $F : K \longrightarrow K$ is weakly sequentially continuous. In fact, let $(x_n)_n \subset K$ such that $x_n \rightharpoonup x$. Because $F(K)$ is relatively weakly compact, it follows by the

Eberlein-Šmulian's theorem that there exists a subsequence $(x_{n_k})_k$ of $(x_n)_n$ such that $F(x_{n_k}) \rightharpoonup y$. The weakly sequentially continuity of B leads to $BF(x_{n_k}) \rightharpoonup By$. Also, from the equality $BF = -A + F$, it results that

$$-A(x_{n_k}) + F(x_{n_k}) \rightharpoonup -A(x) + y.$$

So, $y = F(x)$. We claim that $F(x_n) \rightharpoonup F(x)$. Suppose that this is not the case, then there exists a subsequence $(x_{\varphi_1(n)})_n$ and a weak neighborhood V^w of $(I-B)^{-1}Ax$ such that $(I-B)^{-1}Ax_{\varphi_1(n)} \notin V^w$, for all $n \in \mathbb{N}$. On the other hand, we have $x_{\varphi_1(n)} \rightharpoonup x$, then arguing as before, we find a subsequence $(x_{\varphi_1(\varphi_2(n))})_n$ such that $(I-B)^{-1}Ax_{\varphi_1(\varphi_2(n))}$ converges weakly to $(I-B)^{-1}Ax$, which is a contradiction and hence F is weakly sequentially continuous. □

Remark 3.2. *Since every contraction is a nonlinear contraction but the converse is not true, Theorem 3.4 extends Theorem* 2.1 *in* [25] *to a larger class of contraction mappings* (*i.e. nonlinear contractions*).

Notice that in Theorems 3.1, 3.2 and 3.4 our arguments are based on the invertibility of the mapping $I - B$ and our strategy consists in proving the fixed point property of the mapping $(I-B)^{-1}A$. Hence, it would be interesting to investigate the case when $I - B$ may not be injective.

In that line, the following result presents a critical type of Krasnoselskii's fixed point theorem.

Theorem 3.5. *Let Ω be a nonempty closed and convex subset of a Banach space E. Suppose that A and B are weakly sequentially continuous mappings from Ω into E such that:*

(i) $A(\Omega) \subset (I-B)(E)$ and $[x = Bx + Ay, y \in \Omega] \Longrightarrow x \in \Omega$ (or $A(\Omega) \subset (I-B)(\Omega)$).
(ii) $A(\Omega)$ is a relatively weakly compact subset of E.
(iii) If $(I-B)x_n \rightharpoonup y$, then there exists a weakly convergent subsequence of $(x_n)_n$.
(iv) For every y in the range of $I - B$, $D_y = \{x \in \Omega$ such that $(I-B)x = y\}$ is a convex set.

Then there exists $x \in \Omega$ such that $x = Ax + Bx$.

Proof. First, we assume that $I - B$ is invertible. For any given $y \in \Omega$, define $F : \Omega \longrightarrow \Omega$ by $Fy := (I-B)^{-1}Ay$. F is well defined by assumption (i).

Step 1: $F(\Omega)$ is relatively weakly compact. For any $(y_n)_n \subset F(\Omega)$, we choose $(x_n)_n \subset \Omega$ such that $y_n = F(x_n)$. Taking into account assumption (ii), together with the Eberlein-Šmulian's theorem (see [17, theorem 8 .12.4, p. 549]), we get a subsequence $(y_{\varphi_1(n)})_n$ of $(y_n)_n$ such that $(I-B)y_{\varphi_1(n)} \rightharpoonup z$, for some $z \in \Omega$. Thus, by assumption (iii), there exists a subsequence $y_{\varphi_1(\varphi_2(n))}$ converging weakly to $y_0 \in \Omega$.

Step 2 : F is weakly sequentially continuous. The result can be checked in the same way as in Theorem 3.4.
Consequently, using Theorem 2.1, we get the desired result.

Second, if $I-B$ is not invertible, $(I-B)^{-1}$ could be seen as a multivalued mapping. For any given $y \in \Omega$, define $H : \Omega \longrightarrow P(\Omega)$ by $Hy := (I-B)^{-1}Ay$. H is well defined by assumption (i). We should prove that H fulfills the hypotheses of Theorem 2.3.

Step 1 : $H(x)$ is a convex set for each $x \in \Omega$. This is an immediate consequence of assumption (iv).

Step 2 : H has a weakly sequentially closed graph. Let $x \in \Omega$ and $(x_n)_n \subset \Omega$ such that $x_n \rightharpoonup x$ and $y_n \in H(x_n)$ such that $y_n \rightharpoonup y$. By the definition of H, we have $(I-B)y_n = Ax_n$. Since A and $I-B$ are weakly sequentially continuous, we obtain $(I-B)y = Ax$. Thus $y \in (I-B)^{-1}Ax$.

Step 3 : $H(x)$ is closed for each $x \in \Omega$. This assertion follows from Steps 1 and 2 by setting $(x_n)_n \equiv x$.

Step 4 : $H(\Omega)$ is relatively weakly compact. This assertion is proved by using the same reasoning as the one in Step 1 of the first part of the proof.
In view of Theorem 2.3 we get $x \in H(x)$, for some $x \in \Omega$. Thus, there exists $x \in \Omega$ such that $x = Ax + Bx$. □

Remark 3.3. (i) *Theorem 3.5 improves a multitude of previously known generalizations of Krasnoselskii's fixed point theorem, since we study the fixed point property for a larger class of weakly sequentially continuous mappings under weaker assumptions as it is known; (see* [3], [4], [6], [18] *and* [25]*). Moreover, Theorem 3.5 extends* [21, Theorem 2.1] *to the setting of weak topology and shows that the closed convex subset* Ω *may not be bounded.*

(ii) *We shall emphasize on the fact that if* $B : E \longrightarrow E$ *is a* β*-condensing weakly sequentially continuous mapping so that* $B(E)$ *is a bounded subset of* E *and* $I-B$ *is invertible, then the assumptions* (iii) *and* (iv) *of Theorem 3.5 are satisfied. Indeed, suppose that* $(I-B)x_n \rightharpoonup y$*, for some* $(x_n)_n \subset \Omega$ *and* $y \in \Omega$*. Writing* x_n *as* $x_n = (I-B)x_n + Bx_n$ *and using the subadditivity of the De Blasi measure of weak noncompactness, we get*

$$\beta(\{x_n\}) \leq \beta(\{(I-B)x_n\}) + \beta(\{Bx_n\}).$$

Since $\overline{\{(I-B)x_n\}}^w$ *is weakly compact, we obtain* $\beta(\{x_n\}) \leq \beta(\{Bx_n\})$*. Now, we show that* $\beta(\{x_n\}) = 0$*. If we suppose the contrary, then using the fact that* B *is* β*-condensing, we obtain*

$$\beta(\{x_n\}) \leq \beta(\{Bx_n\}) < \beta(\{x_n\}),$$

which is absurd. So, $\beta(\{x_n\}) = 0$*. Consequently,* $\overline{\{x_n\}}^w$ *is weakly compact and then by the Eberlein-Šmulian's theorem, there exists a weakly convergent subsequence of* $(x_n)_n$*. Hence, the assumption* (iii) *is satisfied. On the other hand, since* $I-B$ *is invertible, we have for every* y *in the range of* $I-B$*, the set* D_y *is reduced to* $\{(I-B)^{-1}y\}$*, which is convex.*

In particular, if $B : E \longrightarrow E$ is a weakly sequentially continuous nonlinear contraction so that $B(E)$ is bounded, we get the following corollary:

Corollary 3.1. *Let Ω be a nonempty closed and convex subset of a Banach space E. Suppose that $A : \Omega \longrightarrow E$ and $B : E \longrightarrow E$ are two weakly sequentially continuous mappings such that:*

(i) *$A(\Omega)$ is relatively weakly compact.*
(ii) *B is a nonlinear contraction such that $B(E)$ is bounded.*
(iii) $[x = Bx + Ay, y \in \Omega] \Longrightarrow x \in \Omega$.

Then there exists $x \in E$ such that $x = Ax + Bx$.

Proof. Since B is a nonlinear contraction, so $I - B$ is invertible and $(I - B)(E) = E$ (see [9]), hence the first part of assumption (i) of Theorem 3.5 is fulfilled. Moreover, we have already proved that every weakly sequentially continuous nonlinear contraction is β-condensing (see proof of Theorem 3.2). Hence, in view of Remark 3.3 (ii), we deduce that B satisfies the assumptions (iii) and (iv) of Theorem 3.5. □

Remark 3.4. *Corollary 3.1 improves Theorem 3.4 since it shows that the condition that Ω is bounded can be removed.*

Using the technique used in the proof of Theorem 3.5, we have the following result.

Theorem 3.6. *Let Ω be a bounded, closed and convex nonempty subset of a Banach space E. Suppose that A and B are weakly sequentially continuous and map Ω into E such that:*

(i) $(I - A)(\Omega) \subset B(\Omega)$.
(ii) *$(I - A)(\Omega)$ is contained in a weakly compact subset of E.*
(iii) *If $Bx_n \rightharpoonup y$, then there exists a weakly convergent subsequence of $(x_n)_n$.*
(iv) *For every y in the range of B, $D_y = \{x \in \Omega$ such that $Bx = y\}$ is a convex set.*

Then there exists $y \in \Omega$ such that $y = Ay + By$.

Remark 3.5. *Theorem 3.6 extends Theorem* 2.4 *in* [21] *to weakly sequentially continuous mappings.*

The next theorem extends a result of H. Schaefer [24] to the case of multivalued mappings in the context of weak topology, dealing with the method of a priori estimate in the Leray-Schauder theory.

Theorem 3.7. *Let E be a Banach space and $H : E \longrightarrow \mathcal{P}(E)$ a multivalued mapping. Suppose that :*

(i) *H has a weakly sequentially closed graph.*
(ii) *There exists a closed convex, balanced and absorbing weak neighborhood U of θ such that the set $H(mU)$ is relatively weakly compact for all $m \in \mathbb{N}$.*
(iii) *The set $H(x)$ is closed, convex and not empty for all $x \in E$.*

Then, either for any $\lambda \in [0, 1]$ there exists an x such that

$$x \in \lambda H(x) \tag{3.3}$$

or the set $\{x \in E : \exists \lambda \in]0, 1[\ ,\ x \in \lambda H(x)\}$ is unbounded.

Proof. Denote by p the Minkowski functional of the set U. Since E endowed with its weak topology is locally convex, we get p is a weakly continuous seminorm and $U = \{x \in E;\ p(x) \leq 1\}$. Clearly, θ is the unique solution of Eq. (3.3) for $\lambda = 0$. If for $\lambda_0 \in (0, 1]$, there is no solution of Eq. (3.3) for $\lambda = \lambda_0$, we consider the weakly sequentially closed multivalued mapping G defined by $G(x) = \lambda_0 H(x)$, for all $x \in E$ and we shall show that for any natural m there exists $y_m \in \eta_m H(y_m)$ with $0 < \eta_m < 1$ and $p(y_m) = n$. To do this, let m be a natural number and define a weakly continuous retraction $r_m : E \longrightarrow mU$ by $r_m(x) = x$ for all $x \in mU$ and $r_m(x) = \frac{mx}{p(x)}$ for all x such that $p(x) > m$. Consider the composition $H_m = G \circ r_m$. In the following, we will prove that H_m satisfies the conditions of Theorem 2.3.

Step 1 : H_m is weakly sequentially closed. Let $x \in E$, $(x_n)_n \in E$ such that $x_n \rightharpoonup x$ and $y_n \in H_m(x_n)$ such that $y_n \rightharpoonup y$. Since the rectraction r_m is weakly continuous, we get $r_m(x_n) \rightharpoonup r_m(x)$. On the other hand, we have $y_n \in G(r_m(x_n))$, $y_n \rightharpoonup y$ and G is weakly sequentially closed. So, $y \in G(r_m(x))$, i.e., $y \in H_m(x)$. Consequently, H_m has a weakly sequentially closed graph.

Step 2 : $H_m(E)$ is relatively weakly compact. The assumption follows from the fact that $H_m(E) = G(mU)$ and the hypothesis (ii).

Step 3 : $H_m(x)$ is closed, convex and not empty for all $x \in E$. This is an immediate consequence of (iii).

Consequently, by Theorem 2.3 H_m has a fixed point x_m in E, i.e., there exists an x_m such that $x_m \in \lambda_0 H(r_m(x_m))$. Notice that the case $p(x_m) \leq m$ can't occur, otherwise we get $x_m = \lambda_0 H(x_m)$ which contradicts our assumption. Hence, $p(x_m) > m$ and thus

$$\frac{r_m(x_m)p(x_m)}{n} = \lambda_0 H(r_m(x_m)).$$

This gives that $y_m = \eta_m H(y_m)$ with

$$y_m = r_m(x_m),\ \eta_m = \frac{n\lambda_0}{p(x_m)} < 1 \text{ and } p(y_m) = n.$$ □

Remark 3.6. *Theorem 3.7 extends Theorem* 3.1 *in* [2] *to the case of multivalued mappings in the setting of weak topology.*

Since every weakly sequentially continuous single valued mapping can be identified with a multivalued mapping having a weakly sequentially closed graph, we obtain the following corollary.

Corollary 3.2. *Let E be a Banach space and $F : E \longrightarrow E$ be a weakly sequentially continuous mapping. Assume that there exists a closed convex, balanced and absorbing weak neighborhood U of θ such that the set $F(mU)$ is relatively weakly compact for all $m \in \mathbb{N}$.*
Then, either for any $\lambda \in [0, 1]$ there exists an x such that

$$x = \lambda F(x)$$

or the set $\{x \in E : \exists\lambda \in]0, 1[\ ,\ x = \lambda F(x)\}$ is unbounded.

Remark 3.7. *Corollary 3.2 is an alternative of Schaefer type fixed point theorem* (*see* [24]) *in the setting of weak topology.*

Having obtained these results, we are ready to state the following Krasnoseskii-Schaefer type fixed point theorem in the setting of weak topology.

Theorem 3.8. *Let E be a Banach space and $A, B : E \longrightarrow E$ two weakly sequentially continuous mappings satisfying:*

(i) $A(E) \subset (I - B)(E)$.
(ii) *There exists a closed convex, balanced and absorbing weak neighborhood U of θ such that the set $A(nU)$ is relatively weakly compact for all $n \in \mathbb{N}$.*
(iii) *If $(I - B)x_n \rightharpoonup y$, then there exists a weakly convergent subsequence $(x_{n_k})_k$ of $(x_n)_n$.*
(iv) *For every y in the range of $I - B, D_y = \{x \in E : (I - B)x = y\}$ is convex.*

Then, either for any $\lambda \in [0, 1]$ there exists an $x \in E$ such that $x = \lambda B(\frac{x}{\lambda}) + \lambda Ax$ or the set $\{x \in E : \exists \lambda \in]0, 1[,\ x = \lambda B(\frac{x}{\lambda}) + \lambda Ax\}$ is unbounded.

Proof. First, we assume that $I - B$ is invertible. For any given $y \in \Omega$, define $F : E \longrightarrow E$ by $Fy := (I - B)^{-1}Ay$. F is well defined by assumption (i).

Step 1: F is weakly sequentially continuous. Let $(y_n)_n = ((I - B)x_n)_n$ be a sequence in $(I - B)(E)$ such that $y_n \rightharpoonup y$. By item (iii), there exists a subsequence $(x_{\varphi(n)})_n$ converging weakly to $x' \in E$. The weakly sequentially continuity of $I - B$ leads to $(I - B)x_{\varphi(n)} \rightharpoonup (I - B)x'$. So, $y = (I - B)x'$ and then $x' = (I - B)^{-1}y$. Using the same reasoning as the one used in the proof of Theorem 3.4, we get $x_n \rightharpoonup (I-B)^{-1}y$. Then, $(I - B)^{-1}$ is weakly sequentially continuous. Since A is weakly sequentially continuous, then it is so for F.

Step 2 : $F(nU)$ is relatively weakly compact. The result can be seen by the same way as in Step 1 of the first part of the proof of Theorem 3.5.
Consequently, using Corollary 3.2, we get the desired result.

Second, if $I - B$ is not invertible, $(I - B)^{-1}$ could be seen as a multivalued mapping. For any given $y \in \Omega$, define $H : E \longrightarrow P(E)$ by $Hy := (I - B)^{-1}Ay$. H is well defined by assumption (i). Now, arguing as in the proof of the second part of Theorem 3.5, we prove that H satisfies the hypotheses of Theorem 3.7. So, using this theorem, we get the desired result. □

Remark 3.8. (i) *Theorem 3.8 is the analogue of Burton and Kirk's theorem* ([13]) *in the critical case for the weak topology.*
(ii) *Theorem 3.8 extends* [21, Theorem 2.2] *to the setting of weak topology.*

Using Theorem 3.8 and Remark 3.3, we obtain the following corollary:

Corollary 3.3. *Let E be a Banach space and $A, B : E \longrightarrow E$ two weakly sequentially continuous mappings satisfying:*

(i) $A(E) \subset (I - B)(E)$.
(ii) There exists a closed convex, balanced and absorbing weak neighborhood U *of* θ *such that the set* $A(nU)$ *is relatively weakly compact for all* $n \in \mathbb{N}$.
(iii) B *is a nonlinear contraction so that* $B(E)$ *is bounded.*

Then, either for any $\lambda \in [0, 1]$ *there exists an* $x \in E$ *such that* $x = \lambda B(\frac{x}{\lambda}) + \lambda Ax$ *or the set* $\{x \in E : \exists \lambda \in]0, 1[, \ x = \lambda B(\frac{x}{\lambda}) + \lambda Ax\}$ *is unbounded.*

Proof. The result follows immediately from Theorem 3.8 and Remark 3.3 (i). □

4. Application to nonlinear integral equation

The objective of this section is to prove the existence of solution for the following nonlinear integral equation

$$x(t) = f(t, x(t)) + \lambda \int_0^t g(s, x(s))ds, \ x \in \mathcal{C}(J, E), \tag{4.1}$$

where $J = [0, T]$, $\lambda \in (\frac{1}{2}, 1)$, $(E, \|.\|)$ is a reflexive Banach space and $\mathcal{C}(J, E)$ is the Banach space of all continuous functions from J to E endowed with the sup-norm $\|.\|_\infty$, defined by $\|x\|_\infty = \sup \{\|x(t)\|; \ t \in J\}$, for each $x \in \mathcal{C}(J, E)$.

Suppose that the functions involved in Eq. (4.1) satisfy the following conditions:

$(H1)$ the mapping $f : J \times E \longrightarrow E$ is such that:
(i) f is a nonlinear contraction with respect to the second variable, i.e.: there exists a continuous nondecreasing function $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ satisfying $\varphi(r) < r$, for $r > 0$ and

$$\|f(t, x) - f(t, y)\| \leq \varphi(\|x - y\|) \text{ for all } x, y \in E \text{ and } t \in J;$$

(ii) $\varphi(r) < (1 - \lambda)r$, for all $r > 0$;
(iii) $f(t, 0) = f(0, 0)$, for all $t \in J$;

$(H2)$ the mapping $x \longmapsto f(., x(.))$ is weakly sequentially continuous on $\mathcal{C}(J, E)$;

$(H3)$ for every $t \in J$, the mapping $g_t = g(t, .) : E \longrightarrow E$ is weakly sequentially continuous;

$(H4)$ for all $x \in \mathcal{C}(J, E)$, $g(., x(.))$ is Pettis integrable on J;

$(H5)$ there exist $\alpha \in L^1([0, T])$ and a nondecreasing continuous function ϕ from $[0, +\infty)$ to $(0, +\infty)$ such that $\|g(t, x)\| \leq \alpha(t)\phi(|\|x\| - \|f(0, 0)\||)$ for a.e. $t \in [0, T]$ and all $x \in E$. Further, assume that

$$\int_0^T \alpha(s)ds < \int_{\|f(0,0)\|}^{+\infty} \frac{dr}{\phi(r)}.$$

The existence result for Eq. (4.1) is given by:

Theorem 4.1. *Assume that the assumptions* $(H1)$*-*$(H5)$ *hold. Then, Eq.* (4.1) *has at least one solution* $x \in \mathcal{C}(J, E)$.

Proof. We set

$$\beta(t) = \int_{\|f(0,0)\|}^{t} \frac{dr}{\phi(r)} \text{ and } b(t) = \beta^{-1}\left(\int_0^t \alpha(s)ds\right).$$

Then,

$$\int_{\|f(0,0)\|}^{b(t)} \frac{dr}{\phi(r)} = \int_0^t \alpha(s)ds. \tag{4.2}$$

Now, we define the set

$$\Omega = \{x \in \mathcal{C}(J,E) \text{ such that } \|x(t)\| \le b(t) + \|f(0,0)\| \text{ for all } t \in J\}.$$

Clearly, Ω is a closed convex and bounded subset of $\mathcal{C}(J,E)$. Let us consider the nonlinear mappings $A, B : \mathcal{C}(J,E) \longrightarrow \mathcal{C}(J,E)$ defined as

$$(Ax)(t) = f(0,0) + \lambda \int_0^t g(s,x(s))ds$$

and

$$(Bx)(t) = f(t,x(t)) - f(0,0).$$

In the following, we will prove that A and B satisfy the assumptions of Theorem 3.4.

Step 1 : $A(\Omega) \subset \Omega$, $A(\Omega)$ is weakly equicontinuous and $A(\Omega)$ is relatively weakly compact.

(i) let $x \in \Omega$ be an arbitrary point. We shall prove that $Ax \in \Omega$. Let $t \in J$. Without loss of generality, we may assume that $(Ax)(t) \neq 0$. By the Hahn-Banach theorem, there exists $x_t \in E^*$ such that $\|x_t\| = 1$ and $\|(Ax)(t)\| = x_t((Ax)(t))$. So, using $(H5)$ and Eq. (4.2), we get

$$\begin{aligned} \|(Ax)(t)\| &= x_t\left(f(0,0) + \lambda \int_0^t g(s,x(s))ds\right) \\ &\le \|f(0,0)\| + \lambda \int_0^t \alpha(s)\phi(\|x(s)\|)ds \\ &\le \|f(0,0)\| + \lambda \int_0^t \alpha(s)\phi(b(s))ds \\ &= \|f(0,0)\| + \lambda(b(t) - \|f(0,0)\|) \\ &= \lambda b(t) + (1-\lambda)\|f(0,0)\| \qquad (4.3) \\ &< b(t) + \|f(0,0)\|. \end{aligned}$$

Hence, $Ax \in \Omega$.

(ii) Let $\varepsilon > 0$; $x \in \Omega$; $x^* \in E^*$; $t, t' \in J$ such that $t \le t'$ and $t' - t \le \varepsilon$. Let us show that $\|x^*\left((Ax)(t) - (Ax)(t')\right)\| \le \varepsilon$. We have

$$\|(Ax)(t) - (Ax)(t')\| = \lambda \left\| \int_0^t g(s,x(s))ds - \int_0^{t'} g(s,x(s))ds \right\| \le \int_t^{t'} \|g(s,x(s))\|ds.$$

Using $(H5)$, we obtain

$$\|(Ax)(t)-(Ax)(t')\| \leq \int_t^{t'} \alpha(s)\phi(b(s))ds \leq |b(t)-b(t')|.$$

(iii) The reflexiveness of E implies that for all $t \in J$, the subset $A(\Omega)(t) = \{(Ax)(t);\ x \in \Omega\}$ is relatively weakly compact. Since $A(\Omega)$ is weakly equicontinuous, so, by the Ascoli-Arzela theorem, we have $A(\Omega)$ is relatively weakly compact.

Step 2 : $A : \Omega \longrightarrow \Omega$ is weakly sequentially continuous.
Let $(x_n)_n \subset \Omega$ such that $x_n \rightharpoonup x \in \Omega$. Taking into account Dobrakov's theorem (see [16, p. 36]) and the fact that $(x_n)_n$ is bounded, we obtain $x_n(t) \rightharpoonup x(t)$ in E, for all $t \in J$. From assumption $(H3)$, we get $g(t, x_n(t)) \rightharpoonup g(t, x(t))$ in E, for all $t \in J$. So, by $(H5)$ and the dominated convergence theorem, we have $(Ax_n)(t) \rightharpoonup (Ax)(t)$ in E. Since $(Ax_n)_n$ is bounded $(A(\Omega) \subset \Omega)$, then $Ax_n \rightharpoonup Ax$ in $\mathcal{C}(J, E)$. Consequently, A is weakly sequentially continuous.

Step 3 : B is a nonlinear contraction.
Let $x, y \in \mathcal{C}(J, E)$. Using the assumption $(H1)(i)$, we infer that

$$\begin{aligned}\|(Bx)(t)-(By)(t)\| &= \|f(t,x(t))-f(t,y(t))\| \\ &\leq \varphi\left(\|x(t)-y(t)\|\right) \\ &\leq \varphi\left(\|x-y\|_\infty\right),\end{aligned}$$

for all $t \in J$. So,

$$\|Bx - By\|_\infty \leq \varphi(\|x-y\|_\infty).$$

Step 4 : $[x = Bx + Ay,\ y \in \Omega] \Longrightarrow x \in \Omega$.
First, we claim that $\|(I-B)x(t)\| \geq \|x(t)\| - \varphi(\|x(t)\|)$ for every $x \in \mathcal{C}(J, E)$ and $t \in J$. Indeed, taking into account hypothesis $(H1)(i)$, we get

$$\begin{aligned}\|(I-B)x(t)-(I-B)y(t)\| &\geq \|x(t)-y(t)\| - \|Bx(t)-By(t)\| \\ &\geq \|x(t)-y(t)\| - \varphi\left(\|x(t)-y(t)\|\right),\end{aligned}$$

for every $x, y \in \mathcal{C}(J, E)$ and $t \in J$. In particular, for $y = 0$, assumption $(H1)(iii)$ implies that

$$\|(I-B)x(t)\| \geq \|x(t)\| - \varphi(\|x(t)\|) \text{ for every } x \in \mathcal{C}(J,E) \text{ and } t \in J, \tag{4.4}$$

as claimed. Now, let $x \in \mathcal{C}(J, E)$ and $y \in \Omega$ such that $x = Bx+Ay$ and let us show that $\|x(t)\| \leq \|f(0,0)\|+b(t)$, for all $t \in J$. Without loss of generality, we may suppose that $x(t) \neq 0$. So, Eq. (4.4) and assumption $(H1)(ii)$ imply that $\|(I-B)x(t)\| \geq \lambda\|x(t)\|$, for all $t \in J$. Consequently, we have

$$\begin{aligned}\|x(t)\| &\leq \frac{1}{\lambda}\ \|((I-B)x)(t)\| \\ &\leq \frac{1}{\lambda}\left\|(Ay)(t)\right\|,\end{aligned}$$

for all $t \in J$. Hence, from Eq. (4.3), we deduce that

$$\begin{aligned} \|x(t)\| &\leq b(t) + \frac{1-\lambda}{\lambda}\|f(0,0)\| \\ &< b(t) + \|f(0,0)\|, \end{aligned}$$

for all $t \in J$. □

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