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THE n-DIMENSIONAL U-CONVEXITY AND GEOMETRY OF BANACH SPACES

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Abstract. In this paper, we introduce the modulus of n-dimensional U-convexity which simultaneously generalizes modulus of n-dimensional uniform convexity due to Kirk [14] and modulus of U-convexity due to Gao [9]. The properties of the modulus are investigated and the relationships between this modulus and other geometric properties of Banach spaces are studied. Some results on fixed point theory for nonexpansive mappings and normal structure in Banach spaces are improved.
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1. Introduction

Let X be a real Banach space with the dual space X^*. Denote by B_X and S_X the closed unit ball and the unit sphere of X, respectively. For two sets of vectors $\{x_1, x_2, \dots, x_{n+1}\} \subset X$ and $\{f_1, f_2, \dots, f_n\} \subset X^*$ where $n \in \mathbb{N}$, the following matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \langle x_1, f_1\rangle & \langle x_2, f_1\rangle & \cdots & \langle x_{n+1}, f_1\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle x_1, f_n\rangle & \langle x_2, f_n\rangle & \cdots & \langle x_{n+1}, f_n\rangle \end{bmatrix}$$

is denoted by $M(x_1, x_2, \dots, x_{n+1}; f_1, f_2, \dots, f_n)$.

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In 1951, Silverman [20] introduced the concept of volume of the convex hull of $x_1, x_2, \ldots, x_{n+1}$ in X by

$$V(x_1, x_2, \ldots, x_{n+1}) := \sup\{\det M(x_1, x_2, \ldots, x_{n+1}; f_1, f_2, \ldots, f_n) : f_1, f_2, \ldots, f_n \in S_{X^*}\}.$$

In 1979, by using this concept, Sullivan introduced the concept of n-dimensional uniform convexity:

Definition 1.1. [22] A Banach space X is *n-dimensional uniformly convex* (n-UR) if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\frac{1}{n+1}\|x_1 + x_2 + \cdots + x_{n+1}\| \leq 1 - \delta$$

whenever $x_1, x_2, \ldots, x_{n+1} \in S_X$ and $V(x_1, x_2, \ldots, x_{n+1}) \geq \varepsilon$.

It is obvious that $V(x_1, x_2) = \|x_1 - x_2\|$. For $n = 1$, this definition coincides with the classical uniform convexity of Clarkson [4].

Theorem 1.2. [22] *If a Banach space X is n-UR, then X is superreflexive.*

In 1988, Kirk introduced the modulus of n-dimensional uniform convexity as follows [14]:

Definition 1.3. Let X be a Banach space. Then

$$\delta_X^n(\varepsilon) := \inf\left\{1 - \frac{1}{n+1}\|x_1 + x_2 + \cdots + x_{n+1}\| : \begin{array}{c} x_1, x_2, \ldots x_{n+1} \in S_X, \\ V(x_1, x_2, \ldots, x_{n+1}) \geq \varepsilon \end{array}\right\},$$

where $0 \leq \varepsilon \leq 2$ is called the modulus of n-dimensional uniform convexity of X. Furthermore, $\varepsilon_0^n = \sup\{\varepsilon > 0 : \delta_X^n(\varepsilon) = 0\}$ is called the characteristic of n-dimensional uniform convexity.

He also proved the following sufficient condition for a Banach space to have normal structure in term of $\delta_X^n(\cdot)$:

Theorem 1.4. *If X is a Banach space with $\varepsilon_0^n < (\frac{1}{2})^{n-1}$, then X is reflexive and has normal structure.*

In particular, we have the following result:

Corollary 1.5. *Every n-UR space has normal structure.*

In 1989, Bae and Park extended the results for $n = 1, 2, 3$ and proved:

Theorem 1.6. [1] *If X is a Banach space with $\varepsilon_0^n < 1$ for $n = 1, 2, 3$, then X is reflexive and has normal structure.*

In [9], Gao introduced the modulus of U-convexity defined by

$$U_X(\varepsilon) = \inf\{1 - \frac{\|x+y\|}{2} : x, y \in S_X, \langle x - y, f\rangle \geq \varepsilon \text{ for some } f \in \nabla_x\},$$

where $0 \leq \varepsilon \leq 2$, and ∇_x denotes the set of norm one supporting functionals of $x \in S_X$.

For more information in this direction, see references, e.g., [2], [8], [10], [11], [13], [16], [17], [18], and [19].

In this paper, we first introduce the modulus of n-dimensional U-convexity which simultaneously generalizes modulus of n-dimensional uniform convexity due to Kirk [14] and modulus of U-convexity due to Gao [9]. Then the properties of this modulus are investigated and the relationships between this modulus and normal structure, reflexivity, and other geometric properties of Banach spaces are studied. Some results for nonexpansive mappings and normal structure in Banach spaces are improved.

2. Main results

Recall that for $x \in S_X$, $\nabla_x \subset S_{X^*}$ denotes the set of norm 1 supporting functionals of $x \in S_X$. We first introduce the following matrix: For two sets of vectors $\{x_1, x_2, \dots, x_{n+1}\} \subseteq X$ and $\{f_2 \in \nabla_{x_2}, f_3 \in \nabla_{x_3}, \dots, f_{n+1} \in \nabla_{x_{n+1}}\} \subseteq X^*$ where $n \in \mathbb{N}$, the following matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \langle x_1, f_2\rangle & \langle x_2, f_2\rangle & \cdots & \langle x_{n+1}, f_2\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle x_1, f_{n+1}\rangle & \langle x_2, f_{n+1}\rangle & \cdots & \langle x_{n+1}, f_{n+1}\rangle \end{bmatrix}$$

is denoted by $m(x_1, x_2, \dots, x_{n+1}; f_2, f_3, \dots, f_{n+1})$.

Similar to Silverman [20], we introduce another concept of volume by the convex hull of $x_1, x_2, \dots, x_{n+1}$ in X by

$$v(x_1, x_2, \dots, x_{n+1}) := \sup\{\det m(x_1, x_2, \dots, x_{n+1}; f_2, f_3, \dots, f_{n+1}) : f_2 \in \nabla_{x_2}, f_3 \in \nabla_{x_3}, \dots, f_{n+1} \in \nabla_{x_{n+1}}\}.$$

It is clear from the definition that:

Proposition 2.1. $v(x_1, x_2, \dots, x_{n+1}) \le V(x_1, x_2, \dots, x_{n+1})$.

The following example shows that the inequality in Proposition 2.1 can be strict.

Example 2.2. Consider the Hilbert space $H = l^2$, and let

$$e_1 = (1, 0, 0, 0, \dots),\ e_2 = (0, 1, 0, 0, \dots),\ e_3 = (0, 0, 1, 0, \dots),$$

and $f_2 = (0, 1, 0, 0, \dots) \in \nabla_{e_2}$, $f_3 = (0, 0, 1, 0, \dots) \in \nabla_{e_3}$. From the smoothness of H, we have $\{f_2\} = \nabla_{x_2}$ and $\{f_3\} = \nabla_{x_3}$. It is clear that

$$v(e_1, e_2, e_3) = \det m(e_1, e_2, e_3; f_2, f_3) = 1.$$

But let $f_2' = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, 0, \dots) \in S_H$, $f_3' = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, 0, \dots) \in S_H$, we have $V(e_1, e_2, e_3) \ge \frac{4}{\sqrt{6}}$.

We give an example to show some possible values of $v(x_1, x_2, \dots, x_{n+1})$ in some Banach spaces.

Example 2.3. Let $X = l_\infty$ be the space of bounded real sequences equipped with the supremum norm. For each $n \in \mathbb{N}$ there exist vectors with $x_1, x_2, \ldots, x_{n+1} \in S_X$ such that $v(x_1, x_2, \ldots, x_{n+1}) = 2^n$. Let

$$x_1 = (1,1,1,\ldots,1,1,1,0,\ldots),\ x_2 = (-1,1,1,\ldots,1,1,1,0,\ldots),$$
$$x_3 = (1,-1,1,\ldots,1,1,1,0,\ldots),\ldots,x_n = (1,1,1,\ldots,1,-1,1,0,\ldots),$$
$$x_{n+1} = (1,1,1,\ldots,1,1,-1,0,\ldots),$$

and

$$f_2 = (0,1,0,\ldots 0,0,0,0,\ldots) \in \nabla_{x_2},\ f_3 = (0,0,1,\ldots,0,0,0,0,\ldots) \in \nabla_{x_3},\ldots,$$
$$f_{n-1} = (0,0,0,\ldots,0,1,0,0,\ldots) \in \nabla_{x_{n-1}},\ f_n = (0,0,0,\ldots,0,0,1,0,\ldots) \in \nabla_{x_n},$$
$$f_{n+1} = (1,0,0,\ldots,0,0,0,0,\ldots) \in \nabla_{x_{n+1}}.$$

We have

$$\det m(x_1,x_2,\ldots,x_{n+1}; f_2,f_3,\ldots,f_{n+1})$$
$$= \det \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ \langle x_1,f_2\rangle & \langle x_2,f_2\rangle & \langle x_3,f_2\rangle & \cdots & \langle x_n,f_2\rangle & \langle x_{n+1},f_2\rangle \\ \langle x_1,f_3\rangle & \langle x_2,f_3\rangle & \langle x_3,f_3\rangle & \cdots & \langle x_n,f_3\rangle & \langle x_{n+1},f_3\rangle \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \langle x_1,f_n\rangle & \langle x_2,f_n\rangle & \langle x_3,f_n\rangle & \cdots & \langle x_n,f_n\rangle & \langle x_{n+1},f_n\rangle \\ \langle x_1,f_{n+1}\rangle & \langle x_2,f_{n+1}\rangle & \langle x_3,f_{n+1}\rangle & \cdots & \langle x_n,f_{n+1}\rangle & \langle x_{n+1},f_{n+1}\rangle \end{bmatrix}$$
$$= \det \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & -1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & -1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & -1 \\ 1 & -1 & 1 & \cdots & 1 & 1 \end{bmatrix} = -2^n.$$

If we let

$$g_2 = (-1,0,0,\ldots,0,0,0,0,\ldots) \in \nabla_{x_2},\ g_3 = (0,-1,0,\ldots,0,0,0,0,\ldots) \in \nabla_{x_3},\ldots,$$
$$g_{n-1} = (0,0,0,\ldots,-1,0,0,0,\ldots) \in \nabla_{x_{n-1}},\ g_n = (0,0,0,\ldots,0,-1,0,0,\ldots) \in \nabla_{x_n},$$
$$g_{n+1} = (0,0,0,\ldots,0,0,-1,0,\ldots) \in \nabla_{x_{n+1}},$$

we have

$$\det m(x_1,x_2,\ldots,x_{n+1}; g_2,g_3,\ldots,g_{n+1})$$
$$= \det \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ \langle x_1,g_2\rangle & \langle x_2,g_2\rangle & \langle x_3,g_2\rangle & \cdots & \langle x_n,g_2\rangle & \langle x_{n+1},g_2\rangle \\ \langle x_1,g_3\rangle & \langle x_2,g_3\rangle & \langle x_3,g_3\rangle & \cdots & \langle x_n,g_3\rangle & \langle x_{n+1},g_3\rangle \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \langle x_1,g_n\rangle & \langle x_2,g_n\rangle & \langle x_3,g_n\rangle & \cdots & \langle x_n,g_n\rangle & \langle x_{n+1},g_n\rangle \\ \langle x_1,g_{n+1}\rangle & \langle x_2,g_{n+1}\rangle & \langle x_3,g_{n+1}\rangle & \cdots & \langle x_n,g_{n+1}\rangle & \langle x_{n+1},g_{n+1}\rangle \end{bmatrix}$$

$$= \det \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 1 & -1 & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & -1 & -1 \\ -1 & -1 & -1 & \cdots & 1 & -1 \\ -1 & -1 & -1 & \cdots & -1 & 1 \end{bmatrix} = 2^n.$$

Similar to Sullivan [22], we can extend U-convexity [9] to the n-dimension and introduce a more generalized concept of n-dimensional uniform convexity as follows:

Definition 2.4. A Banach space X is *n-dimensional U-convex* (n-U) if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\frac{1}{n+1}\|x_1 + x_2 + \cdots + x_{n+1}\| \le 1 - \delta$$

whenever $x_1, x_2, \ldots, x_{n+1} \in S_X$ and $v(x_1, x_2, \ldots, x_{n+1}) \ge \varepsilon$.

Definition 2.5. Let $\nu_X^n = \sup\{v(x_1, x_2, \ldots, x_{n+1}) : x_1, x_2, \ldots x_{n+1} \in S_X\}$ be the upper bound of all n-dimensional volume in X.

Proposition 2.6. *For a Banach space X with $\dim(X) > n$, $\nu_X^n \ge 2$.*

Proof. We proceed by induction on n. For $n = 1$, consider x_1, and $x_2 = -x_1$, we have $v(x_1, x_2) = 2$. Let us now assume that the result is true for an integer $n \ge 1$, and $x_1, x_2, \ldots, x_{n+1} \in S_X$ such that $v(x_1, x_2, \ldots, x_{n+1}) \ge 2 - \varepsilon$. By using the Hahn-Banach theorem, we can take an $x_{n+2} \in S_X$ and an $f_{n+2} \in \nabla_{x_{n+2}}$, such that $\langle x_i, f_{n+2}\rangle = 0$ for $i = 1, 2, \ldots, n+1$ and $\langle x_{n+2}, f_{n+2}\rangle = 1$. We therefore have $v(x_1, x_2, \ldots, x_{n+2}) \ge 2 - \varepsilon$. Since ε is arbitrarily, the proof is complete. $\square$

Similar to Kirk [14], we can extend modulus of U-convexity [9] to n-dimension and introduce a more generalized concept of modulus of n-dimensional U-convexity as follows:

Definition 2.7. Let X be a Banach space. Then

$$U_X^n(\varepsilon) := \inf \left\{ 1 - \frac{1}{n+1}\|x_1 + x_2 + \cdots + x_{n+1}\| : \begin{array}{c} x_1, x_2, \ldots x_{n+1} \in S_X, \\ v(x_1, x_2, \ldots, x_{n+1}) \ge \varepsilon \end{array} \right\},$$

where $0 \le \varepsilon \le \nu_X^n$ is called the modulus of n-dimensional U-convexity of X.

Proposition 2.8. *For a Banach space X with $\dim(X) > n$, if $0 \le \varepsilon \le 2$, then $\delta_X^n(\varepsilon) \le U_X^n(\varepsilon)$.*

Lemma 2.9. *$U_X^n(\varepsilon)$ is a continuous function in $[0, \nu_X^n)$.*

Proof. The proof of this lemma is exactly same as the proof of Corollary 5 of [15]. $\square$

Lemma 2.10 (Bishop-Phelps-Bollobás [3]). *Let X be a Banach space, and let $0 < \varepsilon < 1$. Given $z \in B_X$ and $h \in S_{X^*}$ with $1 - \langle z, h\rangle < \frac{\varepsilon^2}{4}$, then there exist $y \in S_X$ and $g \in \nabla_y$ such that $\|y - z\| < \varepsilon$ and $\|g - h\| < \varepsilon$.*

Remark 2.11. It is easy to know that the condition of Theorem 2.10 can be extended to $1 - \langle z, h\rangle \le \frac{\varepsilon^2}{4}$ for given $z \in B_X$ and $h \in S_{X^*}$.

The following result was proved by James.

Theorem 2.12 ([12])**.** *Let X be a Banach space. Then X is not reflexive if and only if for any $0 < \varepsilon < 1$ there are a sequence $\{x_n\} \subseteq S_X$ and a sequence $\{f_n\} \subseteq S_{X^*}$ such that*

(a) $\langle x_m, f_n\rangle = \varepsilon$ *whenever* $n \le m$*; and*
(b) $\langle x_m, f_n\rangle = 0$ *whenever* $n > m$*.*

Theorem 2.13. *If X is a Banach space with $U_X^n(1) > 0$ where $n \in \mathbb{N}$, then X is reflexive.*

Proof. For the following $(n+1)\times(n+1)$ matrix,

$$\det \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & \cdots & 1-\frac{\varepsilon^2}{4} & 1-\frac{\varepsilon^2}{4} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} = 1.$$

If X is not reflexive, for $0<\varepsilon<1$, let $\{x_i\} \subseteq S_X$ and $\{f_i\} \subseteq S_{X^*}$, $i = 1,2,...,n+1$ satisfy the two conditions for $1-\frac{\varepsilon^2}{4}$ in Theorem 2.12. We have $\langle x_i, f_j\rangle = 1-\frac{\varepsilon^2}{4}$ whenever $j \le i$; and $\langle x_i, f_j\rangle = 0$ whenever $j > i$.

From the remark of Lemma 2.10, for $0<\varepsilon<1$, there are $\{y_i\} \subseteq S_X$ and $\{g_i\} \subseteq S_{X^*}$, $i = 1,2,\ldots,n+1$ such that $g_i \in \nabla_{y_i}, \|x_i - y_i\| < \varepsilon$ and $\|f_i - g_i\| < \varepsilon$ for $i = 1,2,\ldots,n+1$.

Since $\langle y_i, g_j\rangle = \langle x_i, f_j\rangle - \langle x_i - y_i, f_j\rangle + \langle y_i, g_j - f_j\rangle$ for $1 \le i \le n+1$ and $1 \le j \le n+1$, we have $1-\frac{\varepsilon^2}{4} - 2\varepsilon \le \langle y_i, g_j\rangle \le 1 - \frac{\varepsilon^2}{4} + 2\varepsilon$ whenever $j \le i$; and $-2\varepsilon \le \langle y_i, g_j\rangle \le 2\varepsilon$ whenever $j > i$.

Evaluating the determinant of $(n+1)\times(n+1)$ matrix as the sum of $(n+1)!$ product of entries, we have

$$\det \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ \langle y_1, g_2\rangle & \langle y_2, g_2\rangle & \cdots & \langle y_n, g_2\rangle & \langle y_{n+1}, g_2\rangle \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \langle y_1, g_{n+1}\rangle & \langle y_2, g_{n+1}\rangle & \cdots & \langle y_n, g_{n+1}\rangle & \langle y_{n+1}, g_{n+1}\rangle \end{bmatrix} \ge 1 - c\varepsilon,$$

where c is a constant.

So,

$$v(y_1, y_2, \ldots, y_{n+1}) := \sup\{\det m(y_1, y_2, \ldots, y_{n+1}; g_2, g_3, \ldots, g_{n+1}) : \\ g_2 \in \nabla_{y_2}, g_3 \in \nabla_{y_3}, \ldots, g_{n+1} \in \nabla_{y_{n+1}}\} \ge 1 - c\varepsilon.$$

On the other hand, since

$$\frac{\|x_1 + x_2 + \cdots + x_{n+1}\|}{n+1} \ge \langle \frac{x_1 + x_2 + \cdots + x_{n+1}}{n+1}, f_1\rangle = 1 - \frac{\varepsilon^2}{4},$$

we have

$$\frac{\|y_1 + y_2 + \cdots + y_{n+1}\|}{n+1} \ge 1 - \frac{\varepsilon^2}{4} - \varepsilon.$$

From the definition of $U_X^n(\varepsilon)$, we have $U_X^n(1 - c\varepsilon) < \frac{\varepsilon^2}{4} + \varepsilon$.

Since ε can be arbitrarily small, we have $U_X^n(1) = 0$. □

Definition 2.14. [6, 7] Let X and Y be Banach spaces. We say that Y is *finitely representable in* X if for any $\varepsilon > 0$ and any finite dimensional subspace $N \subseteq Y$ there is an isomorphism $T : N \to X$ such that for any $y \in N$, $(1-\varepsilon)\|y\| \le \|Ty\| \le (1+\varepsilon)\|y\|$.

The Banach space X is called *superreflexive* if any space Y which is finitely representable in X is reflexive.

Theorem 2.15. *If X is a Banach space with $U_X^n(1) > 0$ where $n \in \mathbb{N}$, then X is superreflexive.*

Proof. If X is not superreflexive, then there exists a Banach space Y such that Y can be finitely representable in X, but Y is not reflexive. From Theorem 2.13, if $0 < \varepsilon < 1$ is small enough, then $U_Y^n(1-\varepsilon) < \varepsilon$.

Therefore, there exist $\{y_i\} \subseteq S_Y$ for $i = 1, 2, 3, \ldots, n+1$ and $\{g_i\} \in \nabla_{y_i} \subseteq S_{Y^*}$ for $i = 2, \ldots, n+1$ such that

$$\det \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ \langle y_1, g_2 \rangle & \langle y_2, g_2 \rangle & \cdots & \langle y_n, g_2 \rangle & \langle y_{n+1}, g_2 \rangle \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \langle y_1, g_{n+1} \rangle & \langle y_2, g_{n+1} \rangle & \cdots & \langle y_n, g_{n+1} \rangle & \langle y_{n+1}, g_{n+1} \rangle \end{bmatrix} \ge 1 - \varepsilon,$$

but $1 - \frac{\|y_1 + y_2 + \ldots + y_{n+1}\|}{n+1} < \varepsilon$.

Let $N = \operatorname{span}\{y_1, y_2, ..., y_{n+1}\}$, and $T : N \to M \subseteq X$ be an isomorphism with range M.

Consider the conjugate mapping T^* of T. We have $\langle Ty_j, (T^*)^{-1} g_i \rangle = \langle y_j, g_i \rangle$ for $1 \le i, j \le n+1$. By Hahn-Banach theorem, $(T^*)^{-1} g_i$ can be extended to the whole space of X. We have $1 - \varepsilon \le \|T\| \le 1 + \varepsilon$, $1 - \varepsilon \le \|T^*\| \le 1 + \varepsilon$, and $1 - \varepsilon \le \|(T^*)^{-1}\| \le 1 + \varepsilon$.

Let $x_i = Ty_i$ and $f_i = (T^*)^{-1} g_i$ for $1 \le i \le n+1$, we have

$$\langle x_j, f_i \rangle = \langle Ty_j, (T^*)^{-1} g_i \rangle = \langle y_j, g_i \rangle.$$

If $i = j$, then $\langle x_i, f_i \rangle = \langle y_i, g_i \rangle = 1$, so $f_i \in \nabla_{x_i}$ and we have

$$\det \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ \langle x_1, f_2 \rangle & \langle x_2, f_2 \rangle & \cdots & \langle x_n, f_2 \rangle & \langle x_{n+1}, f_2 \rangle \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \langle x_1, f_{n+1} \rangle & \langle x_2, f_{n+1} \rangle & \cdots & \langle x_n, f_{n+1} \rangle & \langle x_{n+1}, f_{n+1} \rangle \end{bmatrix}$$
$$= \det \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ \langle y_1, g_2 \rangle & \langle y_2, g_2 \rangle & \cdots & \langle y_n, g_2 \rangle & \langle y_{n+1}, g_2 \rangle \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \langle y_1, g_{n+1} \rangle & \langle y_2, g_{n+1} \rangle & \cdots & \langle y_n, g_{n+1} \rangle & \langle y_{n+1}, g_{n+1} \rangle \end{bmatrix} \ge 1 - \varepsilon.$$

On the other hand,

$$\frac{\|x_1 + x_2 + \cdots + x_{n+1}\|}{n+1} = \frac{\|T(y_1 + y_2 + \cdots + y_{n+1})\|}{n+1}$$
$$\geq (1-\varepsilon)\frac{\|y_1 + y_2 + \cdots + y_{n+1}\|}{n+1}$$
$$\geq (1-\varepsilon)^2 > 1 - 2\varepsilon.$$

Since ε can be arbitrarily small, we have $U_X^n(1) = 0$. □

In 2008, Saejung proved the following result:

Lemma 2.16. [19] *If X is a Banach space with B_{X^*} is weak* sequentially compact and it fails to have weak normal structure, then for any $\varepsilon > 0$ and $n \in \mathbb{N}$ there are $\{x_1, x_2, \ldots, x_n\} \subseteq S_X$ and $\{f_1, f_2, \ldots, f_n\} \subseteq S_{X^*}$ such that*

(a) $|\|x_i - x_j\| - 1| < \varepsilon$, *for all $i \neq j$;*
(b) $\langle x_i, f_i\rangle = 1$, *for all $1 \leq i \leq n$; and*
(c) $|\langle x_j, f_i\rangle| < \varepsilon$, *for all $i \neq j$.*

Remark 2.17. From the proof of Lemma 2.16, we can choose $\{x_1, x_2, \ldots, x_{n+2}\} \subseteq S_X$ such that $\|\frac{x_1+x_2+\cdots+x_{n+1}}{n+1} - x_{n+2}\| > 1 - \varepsilon$.

Theorem 2.18. *If X is a Banach space with $U_X^n(1) > 0$ where $n \in \mathbb{N}$, then X has normal structure.*

Proof. From Theorem 2.13, X is reflexive. So, B_{X^*} is weak* sequentially compact, and normal structure and weak normal structure coincide.

Suppose X does not have weak normal structure, for $0 < \varepsilon < 1$ let $\{x_i\} \subseteq S_X$ and $\{f_i\} \subseteq S_{X^*}, i = 1, 2, ..., n+2$ satisfy the three conditions for ε in Lemma 2.16. We have $\langle x_i, f_i\rangle = 1$; and $|\langle x_i, f_j\rangle| < \varepsilon$ whenever $i \neq j$ and $1 \leq i, j \leq n+2$.

Consider $\{x_i - x_{n+2}\} \subseteq (1+\varepsilon)B_X$ and $\{f_i\} \subseteq S_{X^*}, i = 1, 2, \ldots, n+1$. We have $1 - \varepsilon < \langle x_i - x_{n+2}, f_i\rangle < 1 + \varepsilon$; and $-2\varepsilon < \langle x_i - x_{n+2}, f_j\rangle < 2\varepsilon$ whenever $i \neq j$, and $1 \leq i, j \leq n+1$.

From Lemma 2.10 (since ε can be arbitrarily small, if necessary we can normalize $x_i - x_{n+2}$ to use Lemma 2.10), for $0 < \varepsilon < 1$, there are $\{y_i\} \subseteq S_X$ and $\{g_i\} \subseteq S_{X^*}, i = 1, 2, \ldots, n+1$ such that $g_n \in \nabla_{y_n}$, $\|(x_i - x_{n+2}) - y_i\| < 2\sqrt{\varepsilon}$ and $\|f_i - g_i\| < 2\sqrt{\varepsilon}$ for $i = 1, 2, \ldots, n+1$.

Since $\langle y_i, g_j\rangle = \langle x_i - x_{n+2}, f_j\rangle - \langle (x_i - x_{n+2}) - y_i, f_j\rangle + \langle y_i, g_j - f_j\rangle$ for $1 \leq i, j \leq n+1$, we have $-2\varepsilon - 4\sqrt{\varepsilon} \leq \langle y_i, g_j\rangle \leq 2\varepsilon + 4\sqrt{\varepsilon}$ whenever $i \neq j$.

Similar to the proof of Theorem 2.13, we have

$$\det\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ \langle y_1, g_2\rangle & \langle y_2, g_2\rangle & \cdots & \langle y_n, g_2\rangle & \langle y_{n+1}, g_2\rangle \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \langle y_1, g_{n+1}\rangle & \langle y_2, g_{n+1}\rangle & \cdots & \langle y_n, g_{n+1}\rangle & \langle y_{n+1}, g_{n+1}\rangle \end{bmatrix} \geq 1 - d\varepsilon,$$

where d is a constant.

So,

$$v(y_1, y_2, \ldots, y_{n+1}) := \sup\{\det m(y_1, y_2, \ldots, y_{n+1}; g_2, g_3, \ldots, g_{n+1}) :$$

$$g_2 \in \nabla_{y_2}, g_3 \in \nabla_{y_3}, \dots, g_{n+1} \in \nabla_{y_{n+1}}\} \geq 1 - d\varepsilon.$$

On the other hand, since

$$\frac{\|y_1 + y_2 + \cdots + y_{n+1}\|}{n+1} \geq \|\frac{x_1 + x_2 + \cdots + x_{n+1}}{n+1} - x_{n+2}\| - \varepsilon > 1 - 2\varepsilon,$$

we have

$$1 - \frac{\|y_1 + y_2 + \cdots + y_{n+1}\|}{n+1} < 2\varepsilon.$$

From the definition of $U_X^n(\varepsilon)$, we have $U_X^n(1 - d\varepsilon) < 2\varepsilon$.

Since ε can be arbitrarily small, we have $U_X^n(1) = 0$. □

We consider the uniform normal structure.

Let $\mathcal{F}$ be a filter of an index set I, and let $\{x_i\}_{i\in I}$ be a subset in a Hausdorff topological space X, $\{x_i\}_{i\in I}$ is said to *converge to x with respect to $\mathcal{F}$*, denoted by $\lim_{\mathcal{F}} x_i = x$, if for each neighborhood U of x, $\{i \in I : x_i \in U\} \in \mathcal{F}$. A filter $\mathcal{U}$ on I is called an *ultrafilter* if it is maximal with respect to the ordering of the set inclusion. An ultrafilter is called *trivial* if it is of the form $\{A : A \subseteq I, i_0 \in A\}$ for some $i_0 \in I$. We will use the fact that if $\mathcal{U}$ is an ultrafilter, then

(i) for any $A \subseteq I$, either $A \subseteq U$ or $I - A \subseteq U$;

(ii) if $\{x_i\}_{i\in I}$ has a cluster point x, then $\lim_{\mathcal{U}} x_i$ exists and equals to x.

Let $\{X_i\}_{i\in I}$ be a family of Banach spaces and let $l_\infty(I, X_i)$ denote the subspace of the product space equipped with the norm $\|(x_i)\| = \sup_{i\in I} \|x_i\| < \infty$.

Definition 2.19. [5, 21]. Let $\mathcal{U}$ be an ultrafilter on I and let $N_U = \{(x_i) \in l_\infty(I, X_i) : \lim_{\mathcal{U}} \|x_i\| = 0\}$. The *ultraproduct* of $\{X_i\}_{i\in I}$ is the quotient space $l_\infty(I, X_i)/N_{\mathcal{U}}$ equipped with the quotient norm.

We will use $(x_i)_{\mathcal{U}}$ to denote the element of the ultraproduct. It follows from remark (ii) above, and the definition of quotient norm that

$$\|(x_i)_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|x_i\| \tag{2.1}$$

In the following we will restrict our index set I to be $\mathbb{N}$, the set of natural numbers, and let $X_i = X, i \in \mathbb{N}$ for some Banach space X. For an ultrafilter $\mathcal{U}$ on $\mathbb{N}$, we use $X_{\mathcal{U}}$ to denote the ultraproduct. Note that if $\mathcal{U}$ is nontrivial, then X can be embedded into $X_{\mathcal{U}}$ isometrically.

Lemma 2.20. [21]. *Suppose that $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$ and X is a Banach space. Then $(X^*)_{\mathcal{U}} \cong (X_{\mathcal{U}})^*$ if and only if X is superreflexive; and in this case, the mapping J defined by*

$$\langle (x_i)_{\mathcal{U}}, J((f_i)_{\mathcal{U}})\rangle = \lim_{\mathcal{U}} \langle x_i, f_i\rangle, \quad \text{for all } (x_i)_{\mathcal{U}} \in X_{\mathcal{U}}$$

is the canonical isometric isomorphism from $(X^*)_{\mathcal{U}}$ onto $(X_{\mathcal{U}})^*$.

Theorem 2.21. *Let X be a superreflexive Banach space. Then for any nontrivial ultrafilter $\mathcal{U}$ on $\mathbb{N}$, and for all $n \in \mathbb{N}$ and $\varepsilon > 0$, we have $U^n_{X_{\mathcal{U}}}(\varepsilon) = U^n_X(\varepsilon)$.*

Proof. Since X can be embedded into $X_{\mathcal{U}}$ isometrically, we may consider X as a subspace of $X_{\mathcal{U}}$. From the definition of $U_X^n(\varepsilon)$, we have $U_{X_{\mathcal{U}}}^n(\varepsilon) \le U_X^n(\varepsilon)$.

We prove the reverse inequality. For any very small $\eta > 0$, from the definition of $U_{X_{\mathcal{U}}}^n(\varepsilon)$, let $(x_i^1)_{\mathcal{U}}$, $(x_i^2)_{\mathcal{U}}$, $\ldots$, $(x_i^n)_{\mathcal{U}}$, $(x_i^{n+1})_{\mathcal{U}}$ belong to $S_{X_{\mathcal{U}}}$, and let $(f_i^2)_{\mathcal{U}} \in \nabla_{(x_i^2)_{\mathcal{U}}}$, $(f_i^3)_{\mathcal{U}} \in \nabla_{(x_i^3)_{\mathcal{U}}}, \ldots$, $(f_i^n)_{\mathcal{U}} \in \nabla_{(x_i^n)_{\mathcal{U}}}$, $(f_i^{n+1})_{\mathcal{U}} \in \nabla_{(x_i^{n+1})_{\mathcal{U}}}$ be such that

$$m((x_i^1)_{\mathcal{U}}, (x_i^2)_{\mathcal{U}}, \ldots, (x_i^n)_{\mathcal{U}}, (x_i^{n+1})_{\mathcal{U}}; (f_i^2)_{\mathcal{U}}, (f_i^3)_{\mathcal{U}}, \ldots, (f_i^n)_{\mathcal{U}}, (f_i^{n+1})_{\mathcal{U}}) \ge \varepsilon,$$

but

$$1 - \frac{\|(x_i^1)_{\mathcal{U}} + (x_i^2)_{\mathcal{U}} + \cdots + (x_i^n)_{\mathcal{U}} + (x_i^{n+1})_{\mathcal{U}}\|}{n+1} < U_{X_{\mathcal{U}}}^n(\varepsilon) + \eta.$$

Without loss of generality, from (2.1) we may assume

$$1 - \eta < \|(x_i^j)_{\mathcal{U}}\| < 1 + \eta,$$

for $1 \le j \le n+1$,

$$1 - \eta < \|(f_i^j)_{\mathcal{U}}\| < 1 + \eta$$

and

$$1 - \eta < \langle (x_i^{j+1})_{\mathcal{U}}, (f_i^{j+1})_{\mathcal{U}} \rangle < 1 + \eta,$$

for all $1 \le j \le n$.

From the property of ultraproduct, we know the subsets

$$P = \{i : m((x_i^1)_{\mathcal{U}}, (x_i^2)_{\mathcal{U}}, \ldots, (x_i^n)_{\mathcal{U}}, (x_i^{n+1})_{\mathcal{U}}; (f_i^2)_{\mathcal{U}}, (f_i^3)_{\mathcal{U}}, \ldots, (f_i^n)_{\mathcal{U}}, (f_i^{n+1})_{\mathcal{U}}) \ge \varepsilon\}$$

and

$$Q = \{i : 1 - \frac{\|(x_i^1)_{\mathcal{U}} + (x_i^2)_{\mathcal{U}} + \cdots + (x_i^n)_{\mathcal{U}} + (x_i^{n+1})_{\mathcal{U}}\|}{n+1} < U_{X_{\mathcal{U}}}^n(\varepsilon) + \eta\}$$

are all in $\mathcal{U}$. So the intersection $P \cap Q$ is in $\mathcal{U}$ too, and is hence not empty.

Let $i \in P \cap Q$. For this fixed i, we have

$$1 - \eta < \|x_i^j\| < 1 + \eta,$$

for $1 \le j \le n+1$;

$$1 - \eta < \|f_i^j\| < 1 + \eta,$$
$$1 - \eta < \langle x_i^{j+1}, f_i^{j+1} \rangle < 1 + \eta,$$

for $1 \le j \le n$;

$$m(x_i^1, x_i^2, \ldots, x_i^n, x_i^{n+1}; f_i^2, f_i^3, \ldots, f_i^n, f_i^{n+1}) \ge \varepsilon,$$

and

$$1 - \frac{\|x_i^1 + x_i^2 + \cdots + x_i^n + x_i^{n+1}\|}{n+1} < U_{X_U}^n(\varepsilon) + \eta.$$

From Lemma 2.10, for $0 < \eta < 1$ (since η can be arbitrarily small, if necessary we can normalize x_i^j and f_i^j to use Lemma 2.10) there are $\{y_j\} \subseteq S_X$, for $1 \le j \le n+1$, and $\{g_j\} \subseteq S_{X^*}$, for $2 \le j \le n+1$, such that $g_j \in \nabla_{y_j}$, for $2 \le j \le n+1$, $\|x_i^j - y_j\| < \eta$, for $1 \le j \le n+1$, and $\|f_i^j - g_j\| < \eta$ for $j = 2, \ldots, n+1$.

Similar to the proof of Theorem 2.13, we have

$$m(y_1, y_2, \ldots, y_n, y_{n+1}; g_2, g_3, \ldots, g_n, g_{n+1}) \ge \varepsilon - c\eta,$$

and $1 - \frac{\|y_1+y_2+\cdots+y_n+y_{n+1}\|}{n+1} < U^n_{X_{\mathcal{U}}}(\varepsilon) + d\eta$, where c and d are constants.

Since $\eta > 0$ can be arbitrarily small, we have $U^n_X(\varepsilon) \leq U^n_{X_{\mathcal{U}}}(\varepsilon)$. □

Lemma 2.22. [13] *If X is a superreflexive Banach space, then X has uniform normal structure if and only if $X_{\mathcal{U}}$ has normal structure.*

Theorem 2.23. *If X is a Banach space with $U^n_X(1) > 0$, then X has uniform normal structure.*

Proof. It follows directly from Theorems 2.15, 2.18, 2.21 and Lemma 2.22. □

Theorem 1.2, Theorem 1.4, and Theorem 1.6 in Section 1 are all corollaries of Theorem 2.15 and Theorem 2.23. The existing results are improved.

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