Fixed Point Theory, 16(2015), No. 2, 371-380
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

A FAST CONVERGING ITERATIVE METHOD FOR VOLTERRA INTEGRAL EQUATIONS OF THE SECOND KIND WITH DELAYED ARGUMENTS

SANDA MICULA

Department of Mathematics and Computer Science
Babeş-Bolyai University
Cluj-Napoca, Romania
E-mail: smicula@math.ubbcluj.ro

Abstract. In this paper we apply Mann's iterative algorithm to nonlinear Volterra integral equations of the second kind with delayed argument. This proves the existence and the uniqueness of the solution and gives a better error estimate than the classical Banach Fixed Point Theorem. The paper concludes with a numerical example.
Key Words and Phrases: Volterra nonlinear integral equations, delayed argument, fixed point theorems, Altman's algorithm, Mann's iterative algorithm.
2010 Mathematics Subject Classification: 45G10, 47H10, 47N20, 65R20.

1. Introduction

We recall the main results for fixed point theory on a Banach space.

Definition 1.1 Let $(X, ||\cdot||)$ be a Banach space. A mapping $T : X \to X$ is called a $q-$*contraction* if $0 \le q < 1$ and

$$||Tx - Ty|| \le q||x - y||, \tag{1.1}$$

for all $x, y \in X$.

We have the classical result, the contraction principle on a Banach space.

Theorem 1.2 *Let $(X, ||\cdot||)$ be a Banach space and $T : X \to X$ be a $q-$contraction. Then*

(a) *equation $x = Tx$ has exactly one solution $x^* \in X$;*
(b) *the sequence of successive approximations $x_{n+1} = Tx_n, n \in \mathbb{N}$, converges to the solution x^*, for any arbitrary choice of initial point $x_0 \in X$;*
(c) *the error estimate*

$$||x_n - x^*|| \quad \le \quad \frac{q^n}{1-q}\, ||x_0 - Tx_0|| \tag{1.2}$$

holds for every $n \in \mathbb{N}$.

Altman (see [1]) gave a stronger fixed point result:

Theorem 1.3 *Let $(X, ||\cdot||)$ be a Banach space and $T : X \to X$ be a $q-$contraction. Let $0 < \varepsilon_n \leq 1$ be a sequence of numbers satisfying*

$$\sum_{n=0}^{\infty} \varepsilon_n = \infty. \tag{1.3}$$

Then

(a) *equation $x = Tx$ has exactly one solution $x^* \in X$;*
(b) *the sequence of successive approximations*

$$x_{n+1} = (1 - \varepsilon_n)x_n + \varepsilon_n T x_n, \quad n = 0, 1, \dots \tag{1.4}$$

converges to the solution x^*, for any arbitrary choice of initial point $x_0 \in X$;
(c) *for every $n \in \mathbb{N}$, there holds the error estimate*

$$||x_n - x^*|| \quad \leq \quad \frac{e^{1-q}}{1-q}\ ||x_0 - Tx_0||\ e^{-(1-q)y_n}, \tag{1.5}$$

where $y_0 = 0$, $y_n = \displaystyle\sum_{i=0}^{n-1} \varepsilon_i$, for $n \geq 1$.

Remark 1.4 *Theorem 1.3 still holds true if instead of X, we consider any closed convex subset $Y \subset X$.*

Obviously, the error estimate in (1.5) is better than the one in (1.2) and the iterative method (1.4) converges faster than the classical one.

For more considerations on iterative algorithms, see [4]. The aim of this paper is to apply Altman's Theorem 1.3 to Volterra integral equations of the second kind with delayed arguments.

2. An iterative method for Volterra integral equations

Integral equations arise in many applications in the fields of mathematics, engineering, physics, mechanics, electrochemistry. They provide an important tool for modeling various phenomena and processes occurring in actuarial sciences, statistical study of dynamic living population, elasticity theory, diffraction problems, quantum mechanics, etc. Also, a large class of initial and boundary value problems can be converted to Volterra integral equations. Finding efficient and rapidly convergent algorithms for solving Volterra integral equations has been a long time goal for scientists in many areas of research (see e.g. [2], [3], [8]).

For more details on functional integral equations, we refer the reader to [6], [5] and [7].

We consider Volterra integral equations of the form

$$x(t) = \begin{cases} \varphi(0) + h(t) + \displaystyle\int_0^t K\big(t, s, x(s), x(s-\delta)\big)\ ds, & t \in [0, b], \\ \varphi(t), \quad t \in [-\delta, 0], \end{cases} \tag{2.1}$$

where $\delta > 0$, $K \in C\big([0,b]\times[0,b]\times\mathbb{R}^2\big)$, $\varphi \in C[-\delta,0]$, $h \in C[0,b]$ and $h(0) = 0$. Other assumptions will be made on K, h and φ later on.
As is well known, the solvability of (2.1) is based on fixed point theory. We define the operator $F : C[-\delta,b] \to C[-\delta,b]$ by

$$Fx(t) = \begin{cases} \varphi(0) + h(t) + \displaystyle\int_0^t K\big(t,s,x(s),x(s-\delta)\big)\, ds, & t \in [0,b], \\ \varphi(t), \quad t \in [-\delta,0]. \end{cases} \tag{2.2}$$

Then finding a solution of the integral equation (2.1) is equivalent to finding a fixed point for the operator F:

$$x = Fx. \tag{2.3}$$

We want to apply Altman's iterative algorithm to the operator equation (2.3). To this end, we consider the space $X = C[-\delta,b]$ equipped with the Bielecki norm

$$||x||_\tau := \max_{t\in[-\delta,b]} |x(t)|\, e^{-\tau t}, \quad x \in X, \tag{2.4}$$

for some suitable $\tau > 0$. Then $(X, ||\cdot||_\tau)$ is a Banach space and we have the following:

Theorem 2.1 *Assume that there exist constants $l_1, l_2 > 0$ such that*

$$|K(t,s,u_1,v_1) - K(t,s,u_2,v_2)| \le l_1|u_1-u_2| + l_2|v_1-v_2|, \tag{2.5}$$

for all $t,s \in [0,b]$ and all $u_1,u_2,v_1,v_2 \in \mathbb{R}$. Then

(a) *equation (2.3) has exactly one solution $x^* \in X$;*
(b) *the sequence of successive approximations*

$$x_{n+1} = \left(1 - \frac{1}{n+1}\right) x_n + \frac{1}{n+1} F x_n, \quad n = 0,1,\ldots \tag{2.6}$$

converges to the solution x^*, for any arbitrary initial point $x_0 \in X$;
(c) *for every $n \in \mathbb{N}$, there holds the error estimate*

$$||x_n - x^*||_\tau \quad \le \quad \frac{e^{1-q}}{1-q}\, ||x_0 - Fx_0||_\tau\; e^{-(1-q)y_n}, \tag{2.7}$$

where $y_0 = 0$, $y_n = \displaystyle\sum_{i=0}^{n-1} \frac{1}{i+1}$, for $n \ge 1$ and $q = \dfrac{l_1+l_2}{\tau} < 1$.

Proof. We want to use Theorem 1.3 for $T = F$ and $\varepsilon_n = \dfrac{1}{n+1}$, which obviously satisfies condition (1.3). Since $(X, ||\cdot||_\tau)$ is a Banach space, all that is left to show is that F is a $q-$contraction.

For $t \in [-\delta, 0]$, $|(Fx - Fy)(t)| = 0$. Let $t \in [0, b]$ be fixed. By (2.2) and (2.5), we have

$$\begin{aligned}
|(Fx-Fy)(t)| &\leq \int_0^t \left|K\big(t,s,x(s),x(s-\delta)\big) - K\big(t,s,y(s),y(s-\delta)\big)\right| \, ds \\
&\leq l_1 \int_0^t |x(s)-y(s)| \, ds + l_2 \int_0^t |x(s-\delta)-y(s-\delta)| \, ds \\
&= l_1 \int_0^t |x(s)-y(s)| e^{-\tau s} e^{\tau s} \, ds \\
&+ l_2 \int_0^t |x(s-\delta)-y(s-\delta)| \, e^{-\tau(s-\delta)} e^{\tau(s-\delta)} \, ds \\
&\leq l_1 ||x-y||_\tau \int_0^t e^{\tau s} \, ds + l_2 ||x-y||_\tau \int_0^t e^{\tau(s-\delta)} \, ds \\
&\leq \frac{l_1+l_2}{\tau} ||x-y||_\tau e^{\tau t}.
\end{aligned}$$

Hence,

$$||Fx-Fy||_\tau \; = \; \max_{t \in [a,b]} \left(|(Fx-Fy)(t)| \; e^{-\tau t} \right) \; \leq \; q||x-y||_\tau$$

and since τ can always be chosen so that $q < 1$, it follows that F is a $q-$contraction. Now, our result follows from Altman's Theorem. $\square$

Remark 2.2 *We denote by $|| \cdot ||$ the max norm on $C[-\delta, b]$ and for some $R > 0$, consider the set $Y_R \subset C[-\delta, b]$ defined by $Y_R := \{x \in C[-\delta, b] \; \big| \; ||x - \tilde{\varphi}|| \leq R\}$, where $\tilde{\varphi}(t) = \varphi(t)$, for $t \in [-\delta, 0]$ and $\tilde{\varphi}(t) = \varphi(0) + h(t)$, for $t \in [0, b]$. If $F(Y_R) \subseteq Y_R$, then it follows that $x^* \in Y_R$.* Thus, we have the following result:

Theorem 2.3 *Assume that there exist constants $l_1, l_2 > 0$ such that*

$$|K(t,s,u_1,v_1) - K(t,s,u_2,v_2)| \leq l_1|u_1-u_2| + l_2|v_1-v_2|, \tag{2.8}$$

for all $t, s \in [0, b]$ and all $u_1, u_2, v_1, v_2 \in [R_1 - R, R_2 + R]$, where

$$R_1 := \min_{t \in [-\delta, b]} \tilde{\varphi}(t), \quad R_2 := \max_{t \in [-\delta, b]} \tilde{\varphi}(t).$$

Further assume that

$$bM \leq R, \tag{2.9}$$

(the condition for the invariance of the ball Y_R), where $M := \max |K(t,s,u,v)|$ over all $t, s \in [0, b]$ and all $u, v \in [R_1 - R, R_2 + R]$. Then the conclusions of Theorem 2.1 still hold on Y_R.

3. Computation of the iterates in (2.6)

The successive iterates in (2.6) contain integrals that need to be approximated numerically.

Recall the composite trapezoidal rule for approximating integrals:

$$\int_c^d f(s)\,ds = \frac{d-c}{2m}\left[f(c)+\sum_{j=1}^{m-1} f(s_j)+f(d)\right]+Rf, \tag{3.1}$$

where the $m+1$ nodes are $s_j = c+\frac{d-c}{m}j,\ j=\overline{0,m}$ and the remainder is given by

$$Rf = -\frac{(d-c)^3}{12m^2}f''(\eta),\quad \eta\in(c,d). \tag{3.2}$$

Now, we consider $m+1$ nodes over the interval $[-\delta,b]$, $t_k=-\delta+\frac{b+\delta}{m}k,\ k=\overline{0,m}$, where m is taken so that one of the nodes is 0, say $t_{k_0}=0$ for some $k_0\in\{0,\ldots,m\}$, i.e. $m=\frac{b+\delta}{\delta}k_0$. Then for $k=\overline{0,k_0}$,

$$x_{n+1}(t_k)=\varphi(t_k),\ n=0,1,\ldots$$

and for $l=\overline{1,m-k_0}$ (i.e. $k=k_0+l\in\{k_0+1,\ldots,m\}$), we approximate

$$\begin{aligned} x_{n+1}(t_{k_0+l}) &= \left(1-\frac{1}{n+1}\right)x_n(t_{k_0+l}) \\ &+ \frac{1}{n+1}\left(\varphi(0)+h(t_{k_0+l})+\int_0^{t_{k_0+l}} K\left(t_{k_0+l},s,x_n(s),x_n(s-\delta)\right)ds\right), \end{aligned} \tag{3.3}$$

using the quadrature formula (3.1) with the initial approximation $x_0(t)=\varphi(0)+h(t)$, for $t\in[0,b]$ and $x_0(t)=\varphi(t)$, for $t\in[-\delta,0]$. Since for each $l=\overline{1,m-k_0}$ and each $j=\overline{0,l}$, $t_{k_0+j}-\delta=t_{k_0+j}-t_0=t_j$, we have

$$\begin{aligned} x_{n+1}(t_{k_0+l}) &= \left(1-\frac{1}{n+1}\right)x_n(t_{k_0+l}) \\ &+ \frac{1}{n+1}\Big[\varphi(0)+h(t_{k_0+l})+\frac{b+\delta}{2m}\Big(K\left(t_{k_0+l},0,x_n(0),x_n(-\delta)\right) \\ &+ 2\sum_{j=1}^{l-1}K\left(t_{k_0+l},t_{k_0+j},x_n(t_{k_0+j}),x_n(t_j)\right) \\ &+ K\left(t_{k_0+l},t_{k_0+l},x_n(t_{k_0+l}),x_n(t_l)\right)\Big)+R_{n+1,k_0+l}\Big],\ n=0,1,\ldots \end{aligned} \tag{3.4}$$

For the error, we need the second derivative $[K(t_k, s, x_n(s), x_n(s-\delta))]_s^{''}$

$$\begin{aligned}
[K(t_k, s, u, v)]_s^{'} &= \frac{\partial K}{\partial s} + \frac{\partial K}{\partial u}u' + \frac{\partial K}{\partial v}v' \\
[K(t_k, s, u, v)]_s^{''} &= \frac{\partial^2 K}{\partial s^2} + 2\frac{\partial^2 K}{\partial s \partial u}u' + 2\frac{\partial^2 K}{\partial s \partial v}v' + 2\frac{\partial^2 K}{\partial u \partial v}u'v' \\
&+ \frac{\partial^2 K}{\partial u^2}(u')^2 + \frac{\partial^2 K}{\partial v^2}(v')^2 + \frac{\partial K}{\partial u}u'' + \frac{\partial K}{\partial v}v''
\end{aligned}$$

So

$$\begin{aligned}
[K(t_k, s, x_n(s), x_n(s-\delta))]_s^{''} &= \frac{\partial^2 K}{\partial s^2} + 2\frac{\partial^2 K}{\partial s \partial u}x_n'(s) + 2\frac{\partial^2 K}{\partial s \partial v}x_n'(s-\delta) \\
&+ 2\frac{\partial^2 K}{\partial u \partial v}x_n'(s)x_n'(s-\delta) + \frac{\partial^2 K}{\partial u^2}(x_n'(s))^2 \qquad (3.5) \\
&+ \frac{\partial^2 K}{\partial v^2}(x_n'(s-\delta))^2 + \frac{\partial K}{\partial u}x_n''(s) + \frac{\partial K}{\partial v}x_n''(s-\delta)
\end{aligned}$$

Now, for $t \in [0, b]$,

$$x_n(t) = \left(1 - \frac{1}{n}\right)x_{n-1}(t) + \frac{1}{n}\left(\varphi(0) + h(t) + \int_0^t K\left(t, s, x_{n-1}(s), x_{n-1}(s-\delta)\right)\right),$$

$$\begin{aligned}
x_n'(t) &= \left(1 - \frac{1}{n}\right)x_{n-1}'(t) + \frac{1}{n}\Big(h'(t) + K(t, t, x_{n-1}(t), x_{n-1}(t-\delta)) \\
&+ \int_0^t \frac{\partial K}{\partial t}\left(t, s, x_{n-1}(s), x_{n-1}(s-\delta)\right)\ ds\Big),
\end{aligned}$$

$$\begin{aligned}
x_n''(t) &= \left(1 - \frac{1}{n}\right)x_{n-1}''(t) + \frac{1}{n}\Big(h''(t) + 2\frac{\partial K}{\partial t}\left(t, t, x_{n-1}(t), x_{n-1}(t-\delta)\right) \\
&+ \frac{\partial K}{\partial s}\left(t, t, x_{n-1}(t), x_{n-1}(t-\delta)\right) + \frac{\partial K}{\partial u}\left(t, t, x_{n-1}(t), x_{n-1}(t-\delta)\right)) x_{n-1}'(t) \\
&+ \frac{\partial K}{\partial v}\left(t, t, x_{n-1}(t), x_{n-1}(t-\delta)\right) x_{n-1}'(t-\delta) \\
&+ \int_0^t \frac{\partial^2 K}{\partial t^2}\left(t, s, x_{n-1}(s), x_{n-1}(s-\delta)\right)\ ds\Big)
\end{aligned}$$

Let

$$
\begin{aligned}
M_1 &= \max_{\substack{|\alpha|\le 2\\ t,s\in[0,b]\\ u,v\in\mathbb{R}}} \Big|\frac{\partial^\alpha K(t,s,u,v)}{\partial t^{\alpha_1}\partial s^{\alpha_2}\partial u^{\alpha_3}\partial v^{\alpha_4}}\Big|,\\
M_2 &= \max_{\substack{|\alpha|\le 2\\ t\in[-\delta,0]}} \big|\varphi^{(\alpha)}\big|, \qquad (3.6)\\
M_3 &= \max_{\substack{|\alpha|\le 2\\ t\in[0,b]}} \big|h^{(\alpha)}\big|.
\end{aligned}
$$

Then, by induction, we have the following bounds:

$$
|x_n(t)| \;=\; |\varphi(t)| \;\le\; M_2 \;\le\; bM_1+M_2+M_3, \quad t\in[-\delta,0],
$$

$$
\begin{aligned}
|x_n(t)| &= \Big|\left(1-\frac{1}{n}\right)x_{n-1}(t)\\
&+ \frac{1}{n}\left(\varphi(0)+h(t)+\int_0^t K\left(t,s,x_{n-1}(s),x_{n-1}(s-\delta)\right)\,ds\right)\Big|\\
&\le \left(1-\frac{1}{n}\right)(bM_1+M_2+M_3)+\frac{1}{n}\left(M_2+M_3+bM_1\right)\\
&= bM_1+M_2+M_3,\ t\in[0,b]. \qquad (3.7)
\end{aligned}
$$

$$
\begin{aligned}
|x_n'(t)| &= |\varphi'(t)| \;\le\; M_2 \;\le\; (b+1)M_1+M_2+M_3, \quad t\in[-\delta,0]\\
|x_n'(t)| &= \Big|\left(1-\frac{1}{n}\right)x_{n-1}'(t)+\frac{1}{n}\Big(h'(t)+K(t,t,x_{n-1}(t),x_{n-1}(t-\delta))\\
&+ \int_0^t \frac{\partial K}{\partial t}\left(t,s,x_{n-1}(s),x_{n-1}(s-\delta)\right)\,ds\Big)\Big|,\\
&\le \left(1-\frac{1}{n}\right)((b+1)M_1+M_2+M_3)+\frac{1}{n}\left(M_3+M_1+bM_1+M_2\right)\\
&= (b+1)M_1+M_2+M_3,\ t\in[0,b]. \qquad (3.8)
\end{aligned}
$$

Let $M'' := 3M_1+2M_1(M_1+M_2)+bM_1(2M_1+1)+M_2+M_3$. We have:

$$
\begin{aligned}
|x_n''(t)| &= |\varphi''(t)| \;\le\; M_2 \;\le\; M'', \quad t\in[-\delta,0]\\
|x_n''(t)| &\le \left(1-\frac{1}{n}\right)M''\\
&+ \frac{1}{n}\left(M_3+3M_1+2M_1((b+1)M_1+M_2)+bM_1+M_2\right)\\
&= M'',\ t\in[0,b]. \qquad (3.9)
\end{aligned}
$$

Let $M_4=\max\{bM_1+M_2+M_3,(b+1)M_1+M_2+M_3,M''\}$. Then

$$
|x_n(t)|,|x_n'(t)|,|x_n''(t)|\le M_4, \quad \text{for all } \ t\in[-\delta,b].
$$

So by (3.5), for all $k = \overline{k_0, m}$ and all $s \in [0, b]$,

$$\begin{aligned} \Big| [K(t_k, s, x_n(s), x_n(s-\delta))]''_s \Big| &\leq M_1 + 2M_1M_4 + 2M_1M_4 \\ &+ 2M_1M_4^2 + 2M_1M_4^2 + 2M_1M_4 \\ &=: M_0. \end{aligned} \tag{3.10}$$

Thus, for the remainders in (3.4), we have the bound

$$|R_{n,k}| \leq \frac{b^3}{12m^2} M_0, \tag{3.11}$$

for all $k = \overline{k_0, m},\ n = 0, 1, \dots$ Note that the bound above does not depend on n or k.

Remark 3.1 *It is clear now that in order to use the quadrature formula* (3.1) *and to approximate the remainder in* (3.2)*, we need to make some considerations on the smoothness of the iterations* x_n*. To this end, let* $X_0 = \{x \in C[-\delta, b] \ \big|\ x\big|_{[-\delta,0]} = 0\}$*. Then:*

(1) $F(X_0) \subseteq X_0$;
(2) *if* $K, h \in C^1$ *and* $K(0,0,0,0) = h'(0) = 0$*, then* $F\left(X_0 \cap C^1[-\delta, b]\right) \subseteq X_0 \cap C^1[-\delta, b]$;
(3) *if* $K, h \in C^2$ *and* $K(0,0,0,0) = \frac{\partial K}{\partial t}(0,0,0,0) = \frac{\partial K}{\partial s}(0,0,0,0) = h'(0) = h''(0) = 0$*, then* $F\left(X_0 \cap C^2[-\delta, b]\right) \subseteq X_0 \cap C^2[-\delta, b]$.

Then the values at the nodes $x_n(t_k)$ are approximated by $\tilde{x}_n(t_k)$, by using the quadrature formula (3.4) and, by the work above, the errors satisfy

$$\big|x_n(t_k) - \tilde{x}_n(t_k)\big| \leq \frac{b^3}{12m^2} M_0. \tag{3.12}$$

Now we can give error bounds for our approximations:

Theorem 3.2 *Assume the conditions in Theorem 2.3 and Remark 3.1 are satisfied. Then we can choose* $x_0 \in X_0 \cap C^2[-\delta, b] \cap Y_R$*, such that the sequence defined in* (2.6) *has the following properties:*

(a) $x_n \in X_0 \cap C^2[-\delta, b] \cap Y_R$, $||x_n - h|| \leq R$;
(b) $\{x'_n\}$ *and* $\{x''_n\}$ *are bounded sequences;*
(c) *for all* $k = \overline{0, m}$ *and all* $n = 0, 1, \dots,$ *the following error bound holds:*

$$\big|x^*(t_k) - \tilde{x}_n(t_k)\big| \leq \frac{e^{\tau b + 1 - q}}{1 - q}\ ||x_0 - Fx_0||_\tau\ e^{-(1-q)y_n} + \frac{b^3}{12m^2} M_0, \tag{3.13}$$

where $y_0 = 0$, $y_n = \sum_{i=0}^{n-1} \frac{1}{i+1}$*, for* $n \geq 1$, $q = \frac{l_1 + l_2}{\tau} < 1$ *and* M_0 *defined in* (3.10) *is independent of* n.

Proof. Conclusions (a) and (b) follow from our considerations in (3.6)-(3.9) and Remark 3.1. For (c), by (2.7) and (3.12), we have

$$\begin{aligned}
\left|x^*(t_k)-\tilde{x}_n(t_k)\right| &\leq \left|x^*(t_k)-x_n(t_k)\right|+\left|x_n(t_k)-\tilde{x}_n(t_k)\right| \\
&= \left|x^*(t_k)-x_n(t_k)\right|e^{-\tau t}e^{\tau t}+\left|x_n(t_k)-\tilde{x}_n(t_k)\right| \\
&\leq ||x_n-x^*||_\tau e^{\tau b}+\left|x_n(t_k)-\tilde{x}_n(t_k)\right| \\
&\leq \frac{e^{\tau b+1-q}}{1-q}\,||x_0-Fx_0||_\tau\; e^{-(1-q)y_n}+\frac{b^3}{12m^2}M_0.
\end{aligned}$$

□

4. Example

Consider the integral equation (see [9])

$$x(t)=\begin{cases} h(t)+\dfrac{35}{34}\displaystyle\int_0^t t^2\Big(x(s)-1\Big)\Big(x(s-1)+1\Big)\,ds, & t\in[0,2], \\ 0, \quad t\in[-1,0], \end{cases} \tag{4.1}$$

where $h(t)=-\dfrac{35^2}{34}\left(35\left(\dfrac{1}{7}t^9-\dfrac{1}{2}t^8-\dfrac{3}{5}t^7-\dfrac{1}{4}t^6\right)+t^5-\dfrac{3}{2}t^4\right)$. The exact solution of (4.1) is $x^*(t)=35t^3$, for $t\in[0,2]$ and $x^*(t)=0$, for $t\in[-1,0]$. We take $m=12$ and the nodes $t_k=-1+\dfrac{1}{4}k,\ k=\overline{0,12}$. Notice that $t_4=0$. Table 1 contains the errors $||\tilde{x}_n-x^*||_\infty=\max\limits_{k=\overline{0,m}}|\tilde{x}_n(t_k)-x^*(t_k)|$, with initial approximations $x_0(t_k)=0$, for $k=\overline{0,4}$ and $x_0(t_k)=h(t_k)$, for $k=\overline{5,12}$.

Table 1. Error estimates for Example 4.1

n	$\|\|\tilde{x}_n-x^*\|\|_\infty$
1	$2.792740e-00$
2	$3.235162e-01$
5	$8.539002e-02$
10	$2.576896e-02$
20	$4.482729e-03$

References

[1] M. Altman, *A Stronger Fixed Point Theorem for Contraction Mappings*, Preprint, 1981.

[2] K. Balachandran, S. Ilamaran, *An Existence Theorem for a Volterra Integral Equation with Deviating Argument*, J. Appl. Math. Stochastic Anal., **3**(1990), no. 3, 155-162.

[3] A. Bellen, N. Guglielmi, *Solving neutral delay differential equations with state-dependent delays*, J. Comp. Appl. Math., **229**(2009), 350-362.

[4] V. Berinde, *Iterative Approximation of Fixed Points*, Lecture Notes in Mathematics, Springer Berlin/Heidelberg/New York, 2007.

[5] G. Coman, G. Pavel, I. Rus, I. A. Rus, *Introducere în teoria ecuaţiilor operatoriale*, Dacia, Cluj-Napoca, 1976.

[6] D. Guo, V.L. Lakshimikanthan, X. Lin, *Nonlinear Integral Equations in Abstract Spaces*, Kluwer Academic, Dordrecht, 1996.

[7] V. Mureşan, *Functional-Integral Equations*, Mediamira, Cluj-Napoca, 2003.

[8] A.V. Plotnikov, N.V. Skripnik, *Existence and Uniqueness Theorem for Set-Valued Volterra Integral Equations*, American J. Appl. Math. Stat., **1**(2013), no. 3, 41-45.
[9] A.D. Polyanin, A.V. Manzhirov, *Handbook of Integral Equations*, CRC Press, Boca Raton, 1998.

Received: October 13, 2013; Accepted: March 15, 2014.