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NOTE ON MULTIFUNCTIONS CONDENSING IN THE HYPERSPACE

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Abstract. We fill the gap in understanding the relationship between mappings which are condensing w.r.t. the measure of noncompactness defined on the hyperspace and multifunctions condensing in the ordinary sense.

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1. Introduction

Mappings which are condensing w.r.t. the m.n.c. in a hyperspace, called hyper-condensing further, were proposed by L. Górniewicz around 2000 as a possible way to unify the topological degree theory for condensing maps with a version of the degree theory for multifunctions which are compact in the hyperspace though may have noncompact values (cf. [16, 15]). The problem of comparing an m.n.c. defined for a suitable hyperspace of sets with an m.n.c. defined on a larger hyperspace was and continues to be one of the main obstacles in achieving the full synthesis of the subject.

It turned out that at least the theory of invariant sets and maximal attractors of hyper-condensing maps can be build (cf. [23]). This raised hope for the full generalization of the results from [2, 5] along the lines of [3], so that both classes of maps: condensing and Nadler contractions with noncompact values, could be dealt jointly. Condensing maps include only Nadler contractions with compact values (cf. [5]). Therefore Nadler contractions with noncompact closed bounded values naturally invite the notion of hyper-condensing multifunction. Our goal here is to establish the precise relation between hyper-condensing and condensing multifunctions.

2. Measures of noncompactness in hyperspaces

Let (X, d) be a complete metric space with distance d. The closure of $A \subset X$ is denoted by $\overline{A}$. The open *r-neighbourhood* of $A \subset X$ is

$$N_r A = \{x \in X : \exists_{a \in A} \ d(x,a) < r\};$$

being empty for $A = \emptyset$. The *Hausdorff distance* between two sets $A, B \subset X$ is

$$h(A,B) = \inf\{r > 0 : B \subset N_r A, A \subset N_r B\},$$

with the convention that $\inf \emptyset = \infty$. Obviously $h(A, \overline{B}) = h(A,B)$, and $h(\emptyset, A) = \infty$ for $A \neq \emptyset$.

We designate the following families: $\mathcal{P}(X)$ – nonempty sets, $\mathcal{F}(X)$ – closed nonempty sets, $\mathcal{K}(X)$ – compacta, and equip them with the Hausdorff distance. For more informations about topologized collections of sets, called hyperspaces, the reader can consult [11, 17].

A functional $\mu : \mathcal{P}(X) \cup \{\emptyset\} \to [0, \infty]$ will be called the *measure of noncompactness* (m.n.c. for short) in the space X provided the following axioms hold:

(μ-0) $\mu(\emptyset) = 0$;
(μ-1) $\mu(\overline{A}) = \mu(A)$;
(μ-2) (*limit-regularity*) $\mu(\{x_m : m \geq n\}) \underset{n\to\infty}{\longrightarrow} 0 \Rightarrow \{x_m : m \geq 1\}$ — precompact;
(μ-3) (*quasimonotonicity*) $\exists_{k>0} \ A \subset B \Rightarrow \mu(A) \leq k\,\mu(B)$;

where $A, B \subset X$, $x_m \in X$. One can replace limit-regularity by the standard, seemingly stronger yet equivalent property:

(μ-2') (*regularity*) $\mu(A) = 0 \Rightarrow A$ — precompact.

Moreover, from the above set of conditions one can deduce the Kuratowski intersection theorem (e.g., [24], cf. [20, 13]).

Theorem 2.1. *If $A_n \supset A_{n+1}$, $n = 1, 2, \ldots$, is a descending sequence of closed sets such that $\mu(A_n) \to 0$ when $n \to \infty$, then the intersection $A = \bigcap_{n=1}^{\infty} A_n$ is a compact set and $A_n \underset{n\to\infty}{\longrightarrow} A$ w.r.t. the Hausdorff distance.*

Unlike in this work, in many applications (e.g., [8, 4]) one requires

(μ-4) (*ultra-additivity*) $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$.

(Note that an ultra-additive m.n.c. is monotone). This explains the use of the term 'measure' for a functional μ in agreement with the Choquet tradition of non-additive monotone measures (see [25] and the references therein).

The minimalistic system of axioms (μ-0)–(μ-3) is enough for studying invariant sets in dynamical systems (e.g., [24, 22, 3, 27]) though it is not appropriate for studying fixed points (because it lacks the Darbo formula, cf. [1, Definition 1.2.1,Theorem 1.1.5], [29, Definition 3.60, Theorem 3.61], [8, Theorems 2.4, 3.6]). The examples of m.n.c.'s, usually excluded from considerations and allowed here, provide the relative and inner Hausdorff m.n.c. and the diameter of a set. More on m.n.c.'s can be found in standard references [1, 8, 9].

Accidentally we shall make use of the following compactness criterion in the hyperspace $(\mathcal{F}(X), h)$.

Theorem 2.2 (Blaschke-Zarankiewicz, [17] Chapter 1 Theorem 1.34, [11] Theorem 3.2.4)**.** *Let $C \subset X$ be a nonempty compact and C_n, $n \geq 1$, a sequence of closed nonempty sets such that*

$$\forall_{\varepsilon>0}\ \exists_{n_0}\ \forall_{n\geq n_0}\ C_n \subset N_\varepsilon C.$$

Then C_n has a subsequence convergent in the sense of the Hausdorff distance.

Proof. Standard formulation of the theorem says that any sequence of sets which lies in compact C admits a convergent subsequence.

Given any $D \subset N_\varepsilon C$ we can find nearby its closed copy $\widetilde{D} \subset C$, $h(\widetilde{D}, D) \leq \varepsilon$. By assumption there exists indexing k_n such that $C_{k_n} \subset N_{\frac{1}{n}} C$. Thus copy sequence $\widetilde{C_{k_n}} \subset C$, $\varepsilon = \frac{1}{k_n}$, admits a convergent subsequence $\widetilde{C_{k_{l_n}}}$ due to the standard formulation of our theorem. Hence $C_{k_{l_n}}$ forms a convergent subsequence of C_n, as $h(C_{k_{l_n}}, \widetilde{C_{k_{l_n}}}) \leq \frac{1}{k_{l_n}}$. □

Remark 2.3. Associating Zarankiewicz's name with the above theorem might be justified by combination of Theorems 5.2.12, 5.2.6 and Corollary 5.1.11 in [11].

Remark 2.4. One can view also the above theorem as the generalization of a known fact that a sequence of points attracted (in the sense of distance) to a compact set admits a convergent subsequence.

Similarly as the family of nonempty closed sets $(\mathcal{F}(X), h)$ constitutes the hyperspace of (X, d), we can form a collection of closed (w.r.t. h) families $(\mathcal{F}(\mathcal{F}(X)), H)$ endowed with its respective Hausdorff distance H based on h.

Note the following convenient

Lemma 2.5 (on closing unions, [23] Lemmas 2, 3)**.** *Let $\mathcal{A} \subset \mathcal{P}(X)$, $\mathcal{B} \subset \mathcal{F}(X)$. Then*

(a) $\overline{\bigcup \mathcal{A}} = \overline{\bigcup \mathcal{A}^\sharp}$,

(b) $\overline{\bigcup \mathcal{B}} = \overline{\bigcup \overline{\mathcal{B}}^h}$,

where $\mathcal{A}^\sharp = \{\overline{A} : A \in \mathcal{A}\}$, and $\overline{\mathcal{B}}^h$ stands for the closure of $\mathcal{B}$ w.r.t. h.

In the same fashion we also consider m.n.c.'s in $\mathcal{F}(X)$, $\nu : \mathcal{P}(\mathcal{F}(X)) \cup \{\emptyset\} \to [0, \infty]$. To every m.n.c. μ in X there corresponds in a canonical way an m.n.c. $\mu^\sharp$ in $\mathcal{F}(X)$.

Theorem 2.6. *If $\mu : \mathcal{P}(X) \cup \{\emptyset\} \to [0, \infty]$ is an m.n.c. in X, then $\mu^\sharp : \mathcal{P}(\mathcal{F}(X)) \cup \{\emptyset\} \to [0, \infty]$ defined via*

$$\forall_{\mathcal{A}\subset\mathcal{F}(X)}\ \mu^\sharp(\mathcal{A}) = \mu(\bigcup \mathcal{A})$$

constitutes an m.n.c. in $\mathcal{F}(X)$. Moreover, if μ is ultra-additive, then so is $\mu^\sharp$.

Proof. Obviously $\mu^\sharp$ satisfies (μ-0) because $\bigcup \emptyset = \emptyset$ and (μ-1) because of Lemma 2.5.

If $\mathcal{A} \subset \mathcal{B} \subset \mathcal{F}(X)$, then

$$\mu^\sharp(\mathcal{A}) = \mu(\bigcup \mathcal{A}) \leq k \cdot \mu(\bigcup \mathcal{B}) = k \cdot \mu^\sharp(\mathcal{B}),$$

since $\bigcup \mathcal{A} \subset \bigcup \mathcal{B}$, which establishes ($\mu$-3) for $\mu^\sharp$.

Ultra-additivity condition (μ-4) is preserved thanks to the evident set-algebraic identity $\bigcup(\mathcal{A} \cup \mathcal{B}) = (\bigcup \mathcal{A}) \cup (\bigcup \mathcal{B})$.

The verification of (μ-2) goes as follows. Fix a sequence of sets $A_n \in \mathcal{F}(X)$ with

$$\mu^\sharp(\{A_m : m \geq n\}) \underset{n\to\infty}{\longrightarrow} 0.$$

Hence from the definition of $\mu^\sharp$ and (μ-1) for μ

$$\mu(\overline{\bigcup_{m\geq n} A_m}) \underset{n\to\infty}{\longrightarrow} 0.$$

Now we are in position to apply the Kuratowski intersection theorem (which is valid for μ by assumptions) and infer that

$$\overline{\bigcup_{m\geq n} A_m} \to A = \bigcap_{n\geq 1} \overline{\bigcup_{m\geq n} A_m}$$

with respect to h and A is nonempty compact. In consequence A attracts A_n:

$$\forall_{\varepsilon>0}\ \exists_{n_0}\ \forall_{n\geq n_0}\ A_n \subset \overline{\bigcup_{m\geq n} A_m} \subset N_\varepsilon A.$$

In view of the Blaschke-Zarankiewicz criterion we obtain that $\{A_n : n \geq 1\}$ is precompact in $(\mathcal{F}(X), h)$. □

Remark 2.7. If μ is the Hausdorff m.n.c. in X, then the corresponding $\mu^\sharp$ is the relative Hausdorff m.n.c. in $(\mathcal{F}(X), h)$ with centers in $\mathcal{K}(X) \subset \mathcal{F}(X)$ ([23] Lemma 4).

Remark 2.8. Any $\mu^\sharp$ in $\mathcal{F}(X)$ induced by m.n.c. μ in X has kernel ([9]) consisting of precompact sets. Indeed for $\mathcal{A} \subset \mathcal{F}(X)$, if $\mu(\bigcup \mathcal{A}) = \mu^\sharp(\mathcal{A}) = 0$, then $A \subset \bigcup \mathcal{A}$ is precompact for every $A \in \mathcal{A}$, because μ is quasi-monotone and regular. This among others means that not every m.n.c. in $\mathcal{F}(X)$ can be canonically generated from some m.n.c. in X; for instance the Hausdorff m.n.c. in $(\mathcal{F}(X), h)$ cannot be obtained from any m.n.c. in X. Also, excluding trivial $\mu \equiv 0$, there always exists $A \subset X$ such that $\mu^\sharp(\{A\}) = \mu(A) > 0$, which means that the canonically induced m.n.c. $\mu^\sharp$ is singular.

In view of the definition of $\mu^\sharp$ we comment upon the case when m.n.c. assumes the value of infinity. If $\mu^\sharp(\mathcal{A}) < \infty$, then $\mu(A) < \infty$ for all $A \in \mathcal{A}$. If $\mu(A) < \infty$, then $\mu^\sharp(\{A\}) < \infty$. These implications readily follow from the definition of $\mu^\sharp$ due to the quasi-monotonicity condition. The consistency of the proof of Theorem 3.4 relies on the careful consideration of infinities.

3. Relation between condensing and hyper-condensing multifunctions

A correspondence $\varphi : X \to \mathcal{P}(X)$ shall be called the *multifunction* (with nonempty values), e.g., [17, 11, 4]. Single-valued map $f : X \to X$ is simply regarded to be a multifunction $\{f\} : X \to \mathcal{P}(X)$, $\{f\}(x) = \{f(x)\} \subset X$, $x \in X$.

Following [16, 15] we define two kinds of the image:

- $\varphi(A) = \bigcup_{a\in A} \varphi(a)$ for $A \subset X$,
- $\varphi^\sharp(\mathcal{A}) = \{\overline{\varphi(A)} : A \in \mathcal{A}\}$ for $\mathcal{A} \subset \mathcal{F}(X)$.

This shall allow us to deal with multifunctions possessing noncompact values.

The *Barnsley-Hutchinson* operator $F : \mathcal{F}(X) \to \mathcal{F}(X)$ associated with φ is given by $F(A) = \overline{\varphi(A)}$ for $A \in \mathcal{F}(X)$, e.g., [21, 2, 24].

Now we introduce the key notions for this work (cf. [3, 23, 24]). A multifunction $\varphi : X \to \mathcal{P}(X)$ is

- *condensing* w.r.t. the m.n.c. μ in X (shortly μ-condensing), if for all $A \subset X$ with $\mu(A) < \infty$ holds

$$\mu(\,\varphi(A)\,) \begin{cases} < \mu(A), & \mu(A) > 0, \\ = 0, & \mu(A) = 0, \end{cases}$$

- *strongly condensing* w.r.t. the m.n.c. μ in X (shortly strongly μ-condensing), if it is μ-condensing and additionally $\lim_{n\to\infty} \mu(F^n(A)) = 0$ for all $A \subset X$ with $\mu(A) < \infty$, where F^n stands for the n-fold composition of F,
- *hyper-condensing* w.r.t. the m.n.c. ν in $\mathcal{F}(X)$ (shortly ν-hyper-condensing), if for all $\mathcal{A} \subset \mathcal{F}(X)$ with $\nu(\mathcal{A}) < \infty$ holds

$$\nu(\,\varphi^\sharp(\mathcal{A})\,) \begin{cases} < \nu(\mathcal{A}), & \nu(\mathcal{A}) > 0, \\ = 0, & \nu(\mathcal{A}) = 0, \end{cases}$$

- *strongly hyper-condensing* w.r.t. the m.n.c. ν in $\mathcal{F}(X)$ (shortly strongly ν-hyper-condensing), if it is ν-hyper-condensing and $\lim_{n\to\infty} \nu((\varphi^\sharp)^n(\mathcal{A})) = 0$ for all $\mathcal{A} \subset \mathcal{F}(X)$ with $\nu(\mathcal{A}) < \infty$, where $(\varphi^\sharp)^n$ stands for the n-fold composition of $\varphi^\sharp$.

Remark 3.1. In the case μ is nonsingular, i.e., $\mu(A \cup \{x\}) = \mu(A)$ for all $A \subset X$ and $x \in X$, a μ-condensing multifunction necessarily has precompact values.

Remark 3.2. Any map condensing w.r.t. either the Hausdorff or Kuratowski m.n.c. is automatically strongly condensing. This is a delicate result established in [1] (Lemma 1.6.11) and rewritten for multifunctions in [3] (Lemma 5). Such a property was found to be useful in dynamics in [27, 3, 23, 19].

Remark 3.3. Remark 2.8 ensures that the class of hyper-condensing multifunctions is essentially larger than the class of condensing maps (see also the family of examples below).

The class of μ-condensing correspondences, where μ is the Hausdorff m.n.c in X, contains all compact multifunctions and Nadler multivalued weak contractions with compact values (e.g., [3]). The class of ν-hyper-condensing correspondences, where ν is the relative Hausdorff m.n.c. in $\mathcal{F}(X)$ with centers in $\mathcal{K}(X)$, contains two classes of maps with closed bounded *noncompact* values: Nadler multivalued weak contractions (cf. [2]) and DeBlasi-Myjak h-compact multifunctions (cf. [16, 15]), see [23].

We can prove now the main result of this article.

Theorem 3.4. *A multifunction $\varphi : X \to \mathcal{P}(X)$ is*

(a) *μ-condensing if and only if it is $\mu^\sharp$-hyper-condensing,*
(b) *strongly μ-condensing if and only if it is strongly $\mu^\sharp$-hyper-condensing.*

Proof. ad (a). Fix $\mathcal{A} \subset \mathcal{F}(X)$. Using the definitions of $\mu^\sharp$ and $\varphi^\sharp$ together with Lemma 2.5 we have that

$$\mu^\sharp(\varphi^\sharp(\mathcal{A})) = \mu(\bigcup \varphi^\sharp(\mathcal{A})) = \mu(\overline{\bigcup_{A\in\mathcal{A}} \overline{\varphi(A)}}) = \mu(\overline{\varphi(\bigcup \mathcal{A})}) = \mu(\varphi(\bigcup \mathcal{A})).$$

Thus $\mu^\sharp(\varphi^\sharp(\mathcal{A})) < \mu^\sharp(\mathcal{A})$ iff $\mu(\varphi(\bigcup\mathcal{A})) < \mu(\bigcup\mathcal{A})$. In particular, for every $A \in \mathcal{F}(X)$ we can put $\mathcal{A} = \{A\}$ which obeys $\bigcup\mathcal{A} = A$.

ad (b). Let φ be strongly $\mu^\sharp$-hyper-condensing and $A \in \mathcal{F}(X)$. Then

$$\mu(\bigcup\{F^n(A)\}) = \mu^\sharp((\varphi^\sharp)^n(\{A\})) \to 0,$$

because $\bigcup\{F^n(A)\} = (\varphi^\sharp)^n(\{A\})$. This shows that φ is μ-condensing.

Reversely, let φ be μ-condensing and $\mathcal{A} \subset \mathcal{F}(X)$. Observation

$$(\varphi^\sharp)^n(\mathcal{A}) = \{F^n(A) : A \in \mathcal{A}\}$$

yields that $\bigcup(\varphi^\sharp)^n(\mathcal{A}) \subset F^n(\bigcup\mathcal{A})$. Therefore

$$\mu^\sharp((\varphi^\sharp)^n(\mathcal{A})) \le k \cdot F^n(\bigcup\mathcal{A}) \to 0$$

by the definition of $\mu^\sharp$ and quasi-monotonicity of μ. □

4. A class of examples of hyper-condensing multifunctions

As a complement to the previous section we provide here a general construction of hyper-condensing maps in the case of a (infinite dimensional) Banach space X. This can be viewed as a far analog of the Krasnoselskiĭ "compact + contraction" map (see also [6]).

We recall that

$$\forall_{A\subset X} \;\; \beta(A) = \inf\{r > 0 : A \subset N_r S \text{ for some finite } S \subset X\}$$

is the *Hausdorff m.n.c.* in X and denote the respective Hausdorff m.n.c. in $(\mathcal{F}(X), h)$ by β^h. (Note that $\beta^h \ne \beta^\sharp$ are different m.n.c.'s according to Remark 2.7).

A multifunction $\varphi : X \to \mathcal{P}(X)$ with closed bounded values is said to be

- a *Nadler contraction* with constant $0 < L < 1$, if

$$\forall_{x_1,x_2\in X} \;\; h(\varphi(x_1), \varphi(x_2)) \le L \cdot d(x_1, x_2),$$

- an *h-compact* map, if

$$\{\overline{\varphi(x)} : x \in X\} \subset \mathcal{F}(X)$$

  is precompact in the hyperspace $(\mathcal{F}(X), h)$ (i.e., w.r.t. the Hausdorff metric).

An h-compact multifunction with compact values is necessarily compact ([16]) so only h-compact multifunctions with noncompact values are of interest here. For Nadler contractions one can browse, e.g., [2]. Both kinds of multifunctions, h-compact and Nadler contractions, are β^h-hyper-condensing in the same manner as (weak) Nadler contractions with compact values and compact maps are β-condensing ([3]). Moreover, β^h-hyper-condensing multifunctions are strongly β^h-hyper-condensing.

The closed ball at x_0 in X is

$$D(x_0, r) = \{x \in X : d(x, x_0) \le r\}.$$

In infinite dimensional spaces closed balls are noncompact ("not nice" [11]) yet still have reasonable properties which we employ below.

Theorem 4.1. *Let $f : X \to X$ be a compact map (i.e., $\overline{f(X)} \subset X$ is compact), and $r > 0$. Then the multifunction $\psi : X \to \mathcal{F}(X) \subset \mathcal{P}(X)$ given for $x \in X$ via $\psi(x) = D(f(x), r)$ is an h-compact multifunction with noncompact values.*

Proof. It is enough to perform a simple check that for $x_i \in X$, $y_i = f(x_i)$, $i = 1, 2$, the following inequality holds:

$$h(D(y_1, r), D(y_2, r)) \leq d(y_1, y_2).$$

(This is the place which involves linear structure of X). □

Taking unions of multifunctions we can build new hyper-condensing maps from the old ones.

Theorem 4.2. *If $\varphi, \psi : X \to \mathcal{F}(X)$ are β^h-hyper-condensing, then their union $\varphi \cup \psi : X \to \mathcal{F}(X)$, $(\varphi \cup \psi)(x) = \varphi(x) \cup \psi(x)$, $x \in X$, is such.*

Proof. Fix $\mathcal{B}, \mathcal{C} \subset \mathcal{F}(X)$ and denote

$$\mathcal{B} \vee \mathcal{C} = \{B \cup C : B \in \mathcal{B}, C \in \mathcal{C}\}.$$

It is not hard to see that

$$\beta^h(\mathcal{B} \vee \mathcal{C}) = \max\{\beta^h(\mathcal{B}), \beta^h(\mathcal{C})\}.$$

Indeed one simply exploits the standard inequality

$$h(B \cup C, S_1 \cup S_2) \leq \max\{h(B, S_1), h(C, S_2)\},$$

$B, C, S_1, S_2 \subset X$. Noting that

$$(\varphi \cup \psi)^\sharp(\mathcal{A}) \subset \varphi^\sharp(\mathcal{A}) \vee \psi^\sharp(\mathcal{A})$$

for $\mathcal{A} \subset \mathcal{F}(X)$, completes the verification of the statement. □

5. Possible applications

One could repeat here the results of [23] under suitable assumptions using general m.n.c.'s defined axiomatically as in this paper. Therefore we can cover in a unified way the existence of attractors for all classes of multifunctions considered in [2, 3, 23, 12, 5]. The price we pay is that we loose uniqueness of the invariant set/attractor entirely connected to metric contractions. Recent research on projective iterated function systems ([10, 30]) suggests that the uniqueness criterion should not be considered the part of a good definition of the notion of attractor.

The original motivation of L. Górniewicz for introducing hyper-condensing multifunctions was to build in an elementary way a topological degree for maps with convex noncompact values along the lines of [16, 15]. However the reasonable relations between a concrete (say Hausdorff or Kuratowski) m.n.c. and their extensions on the appropriate hyperspaces: closed bounded sets, convex closed bounded sets, absolute retracts, turned out to be complicated (e.g., [23] Lemma 5). Homological approach to the definition of the topological degree for maps with noncompact nonconvex values was subject of the research conducted by A. Dawidowicz [14]. However, it seems that this theory is not widely recognized and still awaits a more thorough investigations with applications in mind, for instance in the theory of dissipative dynamical systems on Banach spaces with noncompact attractors.

Question: Denote by $\mathcal{F}_c(X)$ the family of convex closed bounded subsets of the Banach space X endowed with the Hausdorff distance h. Let $J : \mathcal{F}_c(X) \to \mathcal{F}_c(X)$, $I(C) = C$ for $C \in \mathcal{F}_c(X)$, stand for the identity operator. Does J admit a selection $j : \mathcal{F}_c(X) \to X$ fulfilling the following condensity-like condition: $\beta(j(\mathcal{C})) < \beta^h(\mathcal{C})$ for $\mathcal{C} \subset \mathcal{F}_c(X)$ with $0 < \beta^h(\mathcal{C}) < \infty$, and $\beta(j(\mathcal{C})) = 0$ when $\beta^h(\mathcal{C}) = 0$? (Symbols β, β^h are defined in the previous section). Observe that due to the known obstacles (e.g., [7, chapter 1 sections 7,8,9], cf. also [26, 31, 18, 28] for a panorama of recent results) one cannot simply find a nonexpansive selection.

The existence of such a selection $j \in J$ would allow us to repeat the elementary construction of the topological degree given in [15], to cover a wide class of multifunctions including fields of h-compact and condensing maps with convex values.

References

[1] R.R. Akhmerov, M.I. Kamenskiĭ, A.S. Potapov, A.E. Rodkina, B.N. Sadovskiĭ, *Measures of Noncompactness and Condensing Operators*, Birkhäuser, Basel, 1992.

[2] J. Andres, J. Fišer, *Metric and topological multivalued fractals*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., **14**(2004), 1277–1289.

[3] J. Andres, J. Fišer, G. Gabor, K. Leśniak, *Multivalued fractals*, Chaos Solitons Fractals, **24**(2005), no. 3, 665–700.

[4] J. Andres, L. Górniewicz, *Topological Fixed Point Principles for Boundary Value Problems*, Kluwer, Dordrecht, 2003.

[5] J. Andres, M. Rypka, *Multivalued fractals and hyperfractals*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., **22**(2012), no. 1, 27 pp.

[6] J. Andres, M. Väth, *Lefschetz and Nielsen number in hyperspaces for fractals and dynamical systems*, Proc. Amer. Math. Soc., **135**(2007), 479–487.

[7] J.-P. Aubin, A. Cellina, *Differential Inclusions. Set-Valued Maps and Viability Theory*, Springer, Berlin, 1984.

[8] J.M. Ayerbe Toledano, T. Domínguez Benavides, G. López Acedo, *Measures of Noncompactness in Metric Fixed Point Theory*, Birkhäuser, Basel, 1997.

[9] J. Banaś, K. Goebel, *Measures of Noncompactness in Banach Spaces*, M. Dekker, New York-Basel, 1980.

[10] M.F. Barnsley, A. Vince, *Real projective iterated function systems*, J. Geom. Anal., **22**(2012), no. 4, 1137–1172.

[11] G. Beer, *Topologies on Closed and Closed Convex Sets*, Kluwer, 1993.

[12] C. Chifu, A. Petruşel, *Multivalued fractals and generalized multivalued contractions*, Chaos Solitons Fractals, **36**(2008), no. 2, 203–210.

[13] G. Conti, V. Obukhovskii, P. Zecca, *On the topological structure of the solution set for a semilinear functional-differential inclusion in a Banach space*, Topology in Nonlinear Analysis, (Eds. K. Gęba, L. Górniewicz), Vol. 35, Banach Center Publications, Polish Acad. Sci. Inst. Math., Warszawa, 1996, 159–169.

[14] A. Dawidowicz, *Méthodes homologiques dans la théorie des applications et des champs de vecteurs sphériques dans les espaces de Banach* (French), Dissertationes Math. (Rozprawy Mat.), **326**(1993), 50 pp.

[15] F.S. De Blasi, P.Gr. Georgiev, *Hukuhara's topological degree for non compact valued multifunctions*, Publ. Res. Inst. Math. Sci., **39**(2003), no. 1, 183–203.

[16] F.S. De Blasi, J. Myjak, *A remark on the definition of topological degree for set-valued mappings*, J. Math. Anal. Appl., **92**(1983), no. 2, 445–451.

[17] S. Hu, N.S. Papageorgiou, *Handbook of Multivalued Analysis. Vol. I: Theory*, Kluwer, Dordrecht, 1997.

[18] P.N. Ivanshin, *Properties of two selections in metric spaces of nonpositive curvature*, Asian-Eur. J. Math., **1**(2008), no. 3, 383–395.

[19] L. Kapitanski, S. Živanović Gonzalez, *Attractors in hyperspace*, report no. 8, Preprint Series - Hamiltonians in Magnetic Fields, Institut Mittag-Leffler, Djursholm, 2012.

[20] C. Kuratowski, *Topologie vol.I* (French), Polish Scientific Publishers, Warszawa,1958.

[21] A. Lasota, J. Myjak, *Attractors of multifunctions*, Bull. Pol. Ac. Sci. Math., **48**(2000), 319–334.

[22] K. Leśniak, *Stability and invariance of multivalued iterated function systems*, Math. Slovaca, **53**(2003), no. 4, 393–405.

[23] K. Leśniak, *Fixed points of the Barnsley-Hutchinson operators induced by hyper-condensing maps*, Matematiche (Catania), **60**(2005), no. 1, 67–80.

[24] K. Leśniak, *Invariant sets and Knaster-Tarski principle*, Cent. Eur. J. Math., **10**(2012), no. 6, 2077–2087.

[25] E.P. Klement, R. Mesiar, E. Pap, *A nonlinear integral which generalizes both the Choquet and the Sugeno integral*, Quantitative Logic and Soft Computing 2010. Vol. 2, (Eds. B.-Y. Cao, G.-J. Wang, S.-L. Chen, S.-Z. Guo), Springer, 39–52.

[26] J.P. Moreno, R. Schneider, *Lipschitz selections of the diametric completion mapping in Minkowski spaces*, Adv. Math., **233**(2013), 248–267.

[27] V. Šeda, *On condensing discrete dynamical systems*, Math. Bohem., **125**(2000), no. 3, 275–306.

[28] P. Shvartsman, *Barycentric selectors and a Steiner-type point of a convex body in a Banach space*, J. Funct. Anal., **210**(2004), no. 1, 1–42.

[29] M. Väth, *Topological analysis. From the basics to the triple degree for nonlinear Fredholm inclusions*, De Gruyter, Berlin, 2012.

[30] A. Vince, *Möbius iterated function systems*, Trans. Amer. Math. Soc., **365**(2013), no. 1, 491–509.

[31] J.-Z. Xiao, X.-H. Zhu, *The Chebyshev selections and fixed points of set-valued mappings in Banach spaces with some uniform convexity*, Math. Comput. Modelling, **54**(2011), no. 5–6, 1576–1583.

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