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FIXED POINT THEOREMS FOR MULTIVALUED CONTRACTIVE OPERATORS ON GENERALIZED METRIC SPACES

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Abstract. Using the concept of generalized w-distance we prove some interesting results on the existence of fixed points for multivalued contractive type operators in the setting of generalizes metric spaces. A data dependence result and Ulam-Hyers stability of fixed point inclusions are also presented.

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1. Introduction

The well-known Banach contraction principle has been extended in many different directions. Investigations on the existence of fixed points of multi-valued contraction maps in the setting of metric spaces were initiated by Nadler [14]. Using the concept of Hausdorff metric, he proved a multivalued version of the well known Banach contraction principle. Since then, many authors have been using the Hausdorff metric to obtain fixed point results for multivalued maps. In fact, without using the Hausdorff metric, the existence part can be proved under much less stringent conditions. Most recently, Feng and Liu [3] proved a fixed point result for multivalued contractive maps without using the concept of the Hausdorff metric and extended the Nadler's fixed point result.

On the other hand, in [10], Kada et al. have introduced a notion of w-distance on a metric space and improved several results replacing the involved metric by a generalized distance. Recently, Suzuki and Takahashi [25] introduced notions of single-valued and multivalued weakly contractive maps with respect to w-distance and improved

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Nadler's fixed point result for such maps. Recent fixed point results concerning w-distance can be found in [5, 11, 12, 13, 15, 25].

The concept of multivalued weakly Picard operator (briefly MWP operator) was introduced in close connection with the successive approximation method and the data dependence phenomenon for the fixed point set of multivalued operators on complete metric space, by Rus et al. [21]. The theory of multivalued weakly Picard operators in L-spaces appeared in [19].

In [16], Perov introduced the concept of generalized metric space and obtained a generalization of the Banach principle for contractive operators on spaces endowed with such vector-valued metrics. Using the concept of the generalizes metric spaces, Filip and Petruşel [4] extended some fixed point results of [17]. Most recently, Guran [6] introduced a concept of generalized w-distance on generalized metric spaces and proved some fixed point results for multivalued operators.

The Ulam stability of various functional equations have been investigated by many authors (see [1], [2], [7], [8], [9], [18], [20], [23], [24]).

In this paper, using the concept of generalized w-distance we prove some interesting results on the existence of fixed points for multivalued contractive type operators in the setting of generalizes metric spaces. We also present a data dependence result. Then we define the notions of Ulam-Hyers stability with respect to a w-distance, multivalued c_w-weakly Picard operator and we establish a connection between these notions.

2. Preliminaries

Let (X, d) be a metric space. Let $P(X)$ be denote a collection of nonempty subsets of X, $P_{cl}(X)$ a collection of nonempty closed subsets of X, $P_b(X)$ a collection of nonempty bounded subsets of X and $P_{b,cl}(X$ a collection of nonempty closed bounded subsets of X. We also consider the following functionals:

(a) the diameter functional $\delta : P(X) \times P(X) \to \mathbb{R}_+$,

$$\delta(A, B) := \sup\{d(a, b) |\ x \in A, b \in B\}$$

(b) the generalized Hausdorff functional $H : P_{cl}(X) \times P_{cl}(X) \to \mathbb{R}_+ \cup \{+\infty\}$,

$$H(A, B) = \max\{\sup_{x\in A} D(x, B), \sup_{y\in B} D(y, A)\},$$

where $D(x, B) = \inf_{y\in B} d(x, y)$.

It is well known that if (X, d) is a complete metric space, then the pair $(P_{cl}(X), H)$ is a complete generalized metric space.

Let $T : X \to P(X)$ be a multivalued operator and $Y \in X$. Then:

(a) $f : X \to Y$ *is a selection for* $T : X \to P(Y)$ if $f(x) \in T(x)$, for each $x \in X$;

(b) $Graph(T) := \{(x, y) \in X \times Y \mid x \in T(x)\}$- *the graphic of* T;

(c) An element $x \in X$ is called a *fixed point* of the multivalued map T if $x \in T(x)$. We denote $Fix(T) = \{x \in X : x \in T(x)\}$.

(d) A map $f : X \to \mathbb{R}$ is called *lower semi-continuous* if for any sequence $\{x_n\} \subset X$ with $x_n \to x \in X$ imply that $f(x) \leq \lim_{n\to\infty} inf f(x_n)$.

We also denote by $\mathbb{N}$ the set of all natural numbers and by $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$. Now, we recall the concept of L-space, (see [19]).

Definition 2.1. Let X be a nonempty set and $s(X) := \{(x_n)_{n\in\mathbb{N}} | \; x_n \in X, n \in \mathbb{N}\}$. Let $c(X) \subset s(X)$ a subset of $s(X)$ and $Lim : c(X) \to X$ an operator. By definition the triple $(X, c(X), Lim)$ is called an **L-space** if the following conditions are satisfied:

(i) If $x_n = x$, for all $n \in \mathbb{N}$, then $(x_n)_{n\in\mathbb{N}} \in c(X)$ and $Lim(x_n)_{n\in\mathbb{N}} = x$.

(ii) If $(x_n)_{n\in\mathbb{N}} \in c(X)$ and $Lim(x_n)_{n\in\mathbb{N}} = x$, then for all subsequences, $(x_{n_i})_{i\in\mathbb{N}}$, of $(x_n)_{n\in\mathbb{N}}$ we have that $(x_{n_i})_{i\in\mathbb{N}} \in c(X)$ and $Lim(x_{n_i})_{i\in\mathbb{N}} = x$.

Note that an element of $c(X)$ is convergent and $x := Lim(x_n)_{n\in\mathbb{N}}$ is the limit of this sequence and we can write $x_n \to x$ as $n \to \infty$. We will denote an L-space by $(X, \to)$.

For the following notations see I.A. Rus [22] and [23], I.A. Rus, A. Petruşel, A. Sîntămărian [21] and A. Petruşel [19].

Definition 2.2. Let $(X, \to)$ be an L-space. Then $T : X \to P(X)$ is called *multivalued weakly Picard operator*(briefly MWP operator)if for each $x \in X$ and each $y \in T(x)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that:

(i) $x_0 = x$, $x_1 = y$;

(ii) $x_{n+1} \in T(x_n)$, for all $n \in \mathbb{N}$;

(iii) the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent and its limit is a fixed point of T.

Remark 2.1. A sequence $(x_n)_{n\in\mathbb{N}}$ satisfying the condition (i) and (ii) in the Definition 2 is called a sequence of successive approximations of F starting from $(x, y) \in Graph(F)$.

For examples of MWP operators, see [19, 21].

In [10], Kada et al. introduced a concept of w-distance on a metric space (X, d) as follows:

A function $w : X \times X \to [0, \infty)$ is a *w-distance* on X if it satisfies the following conditions for any $x, y, z \in X$:

(1) $w(x, z) \le w(x, y) + w(y, z)$;

(2) the map $w(x, .) : X \to [0, \infty)$ is lower semicontinuous;

(3) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $w(z, x) \le \delta$ and $w(z, y) \le \delta$ imply $d(x, y) \le \varepsilon$.

In the sequel we consider the natural order relation in $\mathbb{R}^m$; that is, if $v, r \in \mathbb{R}^m$, $v := (v_1, v_2, \cdots, v_m)$ and $r := (r_1, r_2, \cdots, r_m)$, then $v \le r$ (resp., $v < r$) means $v_i \le r_i$ (resp.,$v_i < r_i$) , for each $i \in \{1, 2, \cdots, m\}$. Also, $|v| := (|v_1|, |v_2|, \cdots, |v_m|)$ and, if $c \in \mathbb{R}$ then $v \le c$ means $v_i \le c$, for each $i \in \{1, 2, \cdots, m\}$.

The concept of vector-valued metric was introduced by Perov [16] as follows:

Let X be a nonempty set. A mapping $\widetilde{d} : X \times X \to \mathbb{R}^m$ is called vector-valued metric on X if the following properties are satisfied:

(1) $\widetilde{d}(x, y) \ge 0$ for all $x, y \in X$; and $\widetilde{d}(x, y) = 0$ implies $x = y$;

(2) $\widetilde{d}(x, y) = \widetilde{d}(y, x)$;

(3) $\widetilde{d}(x, y) \le \widetilde{d}(x, z) + \widetilde{d}(z, y)$ for all $x, y, z \in X$.

A set X equipped with a vector-valued metric $\widetilde{d}$ is called a generalized metric space and we will denote it by $(X, \widetilde{d})$.

Throughout this paper we will denote by $M_{m,m}(\mathbb{R}_+)$ the set of all $m \times m$ matrices with positive elements, by Θ the zero $m \times m$ matrix, by I the identity $m \times m$ matrix and by U the unity $m \times m$ matrix. If $A \in M_{m,m}(\mathbb{R}_+)$, then the symbol A^τ stands for the transpose matrix of A.

Recall that a matrix A is said to be convergent to zero if and only if $A^n \to 0$ as $n \to \infty$.

3. Main results

First of all, let us recall the notion of generalized w-distance in the setting of generalized metric spaces defined in [6] by L. Guran.

Definition 3.1. Let $(X, \widetilde{d})$ be a generalized metric space. The mapping $\widetilde{w} : X \times X \to \mathbb{R}^m_+$ is called generalized w-distance on X if it satisfies the following conditions:

(1) $\widetilde{w}(x,y) \le \widetilde{w}(x,z) + \widetilde{w}(z,y)$, for every $x,y,z \in X$;

(2) $\widetilde{w}$ is lower semicontinuous in its second variable.;

(3) For any $\varepsilon := (\varepsilon_1, \varepsilon_2, ..., \varepsilon_m) > 0$, there exists $\delta := (\delta_1, \delta_2, ..., \delta_m) > 0$ such that $\widetilde{w}(z,x) \le \delta$ and $\widetilde{w}(z,y) \le \delta$ implies $\widetilde{d}(x,y) \le \varepsilon$.

Examples of generalized w-distance and some of its useful properties are also given [6]. Let us recall the following useful result.

Lemma 3.1. *Let $(X, \widetilde{d})$ be a generalized metric space, and let $\widetilde{w} : X \times X \to \mathbb{R}^m_+$ be a generalized w-distance on X. Let (x_n) and (y_n) be two sequences in X, let $\alpha_n = (\alpha_n^{(1)}, \alpha_n^{(2)}, ..., \alpha_n^{(m)}) \in \mathbb{R}^m_+$ and $\beta_n = (\beta_n^{(1)}, \beta_n^{(2)}, ..., \beta_n^{(m)}) \in \mathbb{R}^m_+$ be two sequences such that $\alpha_n^{(i)}$ and $\beta_n^{(i)}$ converge to zero for each $i \in \{1, 2, \ldots, m\}$. Let $x, y, z \in X$. Then the following hold for every $x, y, z \in X$:*

(a) If $\widetilde{w}(x_n, y) \le \alpha_n$ and $\widetilde{w}(x_n, z) \le \beta_n$ for any $n \in \mathbb{N}$, then $y = z$.

(b) If $\widetilde{w}(x_n, y_n) \le \alpha_n$ and $\widetilde{w}(x_n, z) \le \beta_n$ for any $n \in \mathbb{N}$, then (y_n) converges to z.

(c) If $\widetilde{w}(x_n, x_m) \le \alpha_n$ for any $n, m \in \mathbb{N}$ with $m > n$, then (x_n) is a Cauchy sequence.

(d) If $\widetilde{w}(y, x_n) \le \alpha_n$ for any $n \in \mathbb{N}$, then (x_n) is a Cauchy sequence.

If $(X, \widetilde{d})$ is a generalized metric space, $\widetilde{w}$ is a generalized w-distance on X, $x_0 \in X$ and $r := (r_i)_{i=1}^m$ with $r_i > 0$ for each $i = \{1, 2, ..., m\}$, then we denote:

(1) $B^r_{\widetilde{w}}(x_0; r) := \{x \in X | \widetilde{w}(x_0, x) < r\}$-the right open ball centered at x_0 with radius r with respect to $\widetilde{w}$.

(2) $B^l_{\widetilde{w}}(x_0; r) := \{x \in X | \widetilde{w}(x_0, x) < r\}$-the left open ball centered at x_0 with radius r with respect to $\widetilde{w}$.

(3) $\widetilde{B}^r_{\widetilde{w}}(x_0; r) := \{x \in X | \widetilde{w}(x_0, x) \le r\}$-the right closed ball centered at x_0 with radius r with respect to $\widetilde{w}$.

(4) $\widetilde{B}^l_{\widetilde{w}}(x_0; r) := \{x \in X | \widetilde{w}(x_0, x) \le r\}$-the left closed ball centered at x_0 with radius r with respect to $\widetilde{w}$.

(5) $\overline{B^r_{\widetilde{w}}(x_0; r)} := \widetilde{B}^r_{\widetilde{w}}(x_0; r) \cup$ {*the set of limit points with respect to the convergence induced by metric d of all sequences from $\widetilde{B}^r_{\widetilde{w}}(x_0; r)$*}

Then, we prove a local fixed point result for multivalued contractive maps in the setting of generalized metric spaces.

Theorem 3.1. *Let $(X,\widetilde{d})$ be a complete generalized metric space and let $\widetilde{w}$ be a generalized w-distance on X. Let $x_0 \in X$, $r := (r_i)_{i=1}^m \in \mathbb{R}_+^m$ and let $T : \overline{B_{\widetilde{w}}^r(x_0;r)} \to P(X)$ be a multivalued operator such that for every $x, y \in X$ and $u \in T(x)$, there exists $v \in T(y)$ such that*

$$\widetilde{w}(u,v) \leq A\widetilde{w}(x,y) + BD_{\widetilde{w}}(y,T(y)),$$

where A, B, $C \in M_{m,m}(\mathbb{R}_+), B \neq I$, $C := (I-B)^{-1}A$ and $D_{\widetilde{w}}(x,T(x)) := \inf\{\widetilde{w}(x,y) : y \in T(x)\}$. Further, suppose that

(a) $C := (I-B)^{-1}A$ is a metric that converges to zero;

(b) $\inf\{\widetilde{w}(x,y) + D_{\widetilde{w}}(x,T(x))\} > 0$, for every $x, y \in X$ and $y \notin T(y)$;

(c) There exists $x_1 \in T(x_0)$ such that $\widetilde{w}(x_0,x_1)(I-C)^{-1} \leq r$;

(d) If $u \in \mathbb{R}_+^m$ is such that $u(I-C)^{-1} \leq (I-C)^{-1}r$, then $u \leq r$.

Then $FixT \neq \emptyset$.

Proof. By (c) there exists $x_1 \in T(x_0)$ such that

$$\widetilde{w}(x_0,x_1)(I-C)^{-1} \leq r \leq (I-C)^{-1} \cdot r$$

Then, by (d), $\widetilde{w}(x_0,x_1) \leq r$ which implies $x_1 \in \overline{B_{\widetilde{w}}^r(x_0;r)}$. Since $x_1 \in T(x_0)$, there exists $x_2 \in T(x_1)$ such that

$$\widetilde{w}(x_1,x_2) \leq A\widetilde{w}(x_0,x_1) + BD_{\widetilde{w}}(x_1,T(x_1))$$
$$\leq A\widetilde{w}(x_0,x_1) + B\widetilde{w}(x_1,x_2)$$

Thus $\widetilde{w}(x_1,x_2) \leq (I-B)^{-1}A\widetilde{w}(x_0,x_1)$. Then $\widetilde{w}(x_1,x_2) \leq C\widetilde{w}(x_0,x_1)$.
Thus $\widetilde{w}(x_1,x_2)(I-C)^{-1} \leq Cw(x_0,x_1)(I-C)^{-1} \leq Cr$.

Using the triangle inequality $\widetilde{w}(x_0,x_2) \leq \widetilde{w}(x_0,x_1) + \widetilde{w}(x_1,x_2)$ we have

$$w(x_0,x_2)(I-C)^{-1} \leq \widetilde{w}(x_0,x_1)(I-C)^{-1} + \widetilde{w}(x_1,x_2)(I-C)^{-1}$$
$$\leq Ir + Cr \leq (I-C)^{-1}r,$$

now by using (c) we get $\widetilde{w}(x_0,x_2) \leq r$, and thus $x_2 \in \overline{B_{\widetilde{\omega}}^r(x_0;r)}$.

By induction, we obtain the sequence $(x_n)_{n\in\mathbb{N}}$ in $\overline{B_{\widetilde{w}}^r(x_0;r)}$ having the properties:
(i) $x_{n+1} \in T(x_n), n \in \mathbb{N}$;
(ii) $\widetilde{w}(x_0,x_n)(I-C)^{-1} \leq (I-C)^{-1}r$, for each $n \in \mathbb{N}^*$;
(iii) $\widetilde{w}(x_n,x_{n+1})(I-C)^{-1} \leq C^n r$, for each $n \in \mathbb{N}$.

By (iii), for every $m, n \in \mathbb{N}$, with $m > n$, we get that

$$\widetilde{w}(x_n,x_m) \leq \widetilde{w}(x_n,x_{n+1}) + \widetilde{w}(x_{n+1},x_{n+2}) + ... + \widetilde{w}(x_{m-1},x_m)$$

Then $\widetilde{w}(x_n,x_m)(I-A)^{-1} \leq C^n r + C^{n+1}r + ... + C^{m-1}r \leq$

$$\leq C^n(I + C + C^2 + ... + C^{m-1-n}) \leq C^n(I-C)^{-1}r.$$

By Lemma 3(3) the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy sequence with respect to the generalized metric $\widetilde{d}$ in the closed set $\widetilde{B}_{\widetilde{w}}(x_0;r) \subset \overline{B_{\widetilde{w}}^r(x_0;r)} \subset X$. Since $\overline{B_{\widetilde{w}}^r(x_0;r)}$ is closed with respect to the metric d, there exists $x^* \in \overline{B_{\widetilde{w}}^r(x_0;r)}$ such that $(x_n) \xrightarrow{d} x^*$.

Assume that $x^* \notin T(x^*)$. Fix $n \in \mathbb{N}$. Since $(x_m)_{m\in\mathbb{N}}$ is a sequence in $\overline{B_{\widetilde{w}}^r(x_0;r)}$ which converge to x^* and $\widetilde{w}(x_n,\cdot)$ is lower semicontinuous we have

$$\widetilde{w}(x_n,x^*) \leq \lim_{m\to\infty}\inf \widetilde{w}(x_n,x_m) \leq C^n r, \text{ for every } n \in \mathbb{N}.$$

Therefore, by hypothesis (b) and using above inequality we have

$$\begin{aligned} 0 &< \inf\{\widetilde{w}(x,x^*) + D_{\widetilde{w}}(x,T(x)) : x \in X\} \\ &\le \inf\{\widetilde{w}(x_n,x^*) + \widetilde{w}(x_n,x_{n+1}) : n \in \mathbb{N}\} \\ &\le \inf\{2C^n r\} = 0, \end{aligned}$$

which is impossible and hence we conclude that $x^* \in T(x^*)$.

Remark 3.1. The same result can be obtained and for the case of the left ball $B^l_{\widetilde{w}}(x_0;r)$.

A global version of the Theorem 3.1 is the following result.

Theorem 3.2. *Let $(X,\widetilde{d})$ be a complete generalized metric space and let $\widetilde{w}$ be a generalized w-distance on X. Let $r := (r_i)_{i=1}^m \in \mathbb{R}^m_+$ and let $T : X \to P(X)$ be a multivalued operator such that for every $x, y \in X$ and $u \in T(x)$, there exists $v \in T(y)$ such that*

$$\widetilde{w}(u,v) \le A\widetilde{w}(x,y) + BD_{\widetilde{w}}(y,T(y)),$$

where $A, B, C \in M_{m,m}(\mathbb{R}_+), B \neq I$, where $C := A(I-B)^{-1}$ and $D_{\widetilde{w}}(x,T(x)) :=$ $\inf\{\widetilde{w}(x,y) : y \in T(x)\}$. Further, suppose that

(a) $C := A(I-B)^{-1}$ is a metric that converges to zero;

(b) $\inf\{\widetilde{w}(x,y) + D_{\widetilde{w}}(x,T(x))\} > 0$, for every $x, y \in X$ and $y \notin T(y)$.

Then

(1) $FixT \neq \emptyset$.

(2) There exists a sequence $(x_n)_{n\in\mathbb{N}} \in X$ such that $x_{n+1} \in T(x_n)$, for all $n \in \mathbb{N}$ and converges to a fixed point of T.

(3) One has the estimation $\widetilde{w}(x_n,x^*) \le C^n\widetilde{w}(x_0,x_1)$ where $C \in \mathcal{M}_{m,m}(\mathbb{R})$ and $x^* \in FixT$.

Remark 3.1. In Theorem 3.2, we observe that T is a MWP operator.

4. Data dependence theorem for weakly contractive type operators in generalized metric spaces

The main result of this section is the following data dependence theorem with respect to the Theorem 3.

Theorem 4.1. *Let $(X,\widetilde{d})$ be a complete generalized metric space and let $\widetilde{w}$ be a generalized w-distance on X. Let $r := (r_i)_{i=1}^m \in \mathbb{R}^m_+$ and let $T_1, T_2 : X \to P(X)$ be a multivalued operator with the property that there exists $A,\ B\ C \in \mathcal{M}_{m,m}(\mathbb{R}_+)$, where $C := A(I-B)^{-1}$ is a matrix convergent to zero such that, for every $x, y \in X$ and $u \in T_j(x)$, for every $j \in \{1,2\}$, there exists $v \in T_j(y)$ such that*

$$\widetilde{w}(u,v) \le A\widetilde{w}(x,y) + BD_{\widetilde{w}}(y,T_j(y)),$$

where $D_{\widetilde{w}}(x,T_j(x)) := inf\{\widetilde{w}(x,y) : y \in T_j(x)\}$. Further suppose that the following are true:

(a) $FixT_1 \neq \emptyset \neq FixT_2$;

(b) there exists $\eta := (\eta_i)_{i=1}^n$, for each $i = \{1,2,...,m\}$, with $\eta > 0$, such that for every $u \in T_1(x)$ there exists $v \in T_2(x)$ such that $\widetilde{w}(u,v) \le \eta$, (respectively for every $v \in T_2(x)$ there exists $u \in T_1(x)$ such that $\widetilde{w}(v,u) \le \eta$);

(c) $\inf\{\widetilde{w}(x,y)+D_{\widetilde{w}}(x,T_j(x))\}>0$ *for each* $j\in\{1,2\}$*, for every* $x,y\in X$ *and* $y\notin T_j(y)$.
Then for every $u^*\in FixT_1$ *there exists* $v^*\in FixT_2$ *such that*

$$\widetilde{w}(u^*,v^*)\leq I(I-C)^{-1}\eta,\ \textit{where}\ C\in\mathcal{M}_{m,m}(\mathbb{R}),\ C:=A(I-B)^{-1};$$

(respectively for every $v^*\in FixT_2$ *there exists* $u^*\in FixT_1$ *such that*

$$\widetilde{w}(v^*,u^*)\leq I(I-C)^{-1}\eta,\ \textit{where}\ C\in\mathcal{M}_{m,m}(\mathbb{R}),\ C:=A(I-B)^{-1}).$$

Proof. Let $u_0\in FixT_1$, then $u_0\in T_1(u_0)$. By the hypothesis (b) there exists $u_1\in T_2(u_0)$ such that $\widetilde{w}(u_0,u_1)\leq\eta$. Now, for every $u_0,u_1\in X$ with $u_1\in T_2(u_0)$ there exists $u_2\in T_2(u_1)$ such that

$$\widetilde{w}(u_1,u_2)\leq A\widetilde{w}(u_0,u_1)+BD_{\widetilde{w}}(u_1,T_2(u_1))\leq A\widetilde{w}(u_0,u_1)+B\widetilde{w}(u_1,u_2).$$

Then $\widetilde{w}(u_1,u_2)\leq A(I-B)^{-1}\widetilde{w}(u_0,u_1)$. Thus $\widetilde{w}(u_1,u_2)\leq C\widetilde{w}(u_0,u_1)$.
By induction we obtain a sequence $(u_n)_{n\in\mathbb{N}}\in X$ such that
(i) $u_{n+1}\in T_2(u_n)$, for every $n\in\mathbb{N}$;
(ii) $\widetilde{w}(u_n,u_{n+1})\leq C^n\widetilde{w}(u_0,u_1)$
For $n,p\in\mathbb{N}$ we have the inequality

$$\widetilde{w}(u_n,u_{n+p})\leq C^n(I-C)^{-1}\widetilde{w}(u_0,u_1).$$

By the Lemma 3(3) we have that the sequence $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Since (X,d) is a complete metric space we have that there exists $v^*\in X$ such that $u_n\xrightarrow{d}v^*$.

Assume that $v^*\notin T_2(v^*)$. Fix $n\in\mathbb{N}$. By the lower semicontinuity of $\widetilde{w}(x,\cdot):X\to[0,\infty)$ we have

$$\widetilde{w}(u_n,v^*)\leq\lim_{p\to\infty}\inf\widetilde{w}(u_n,u_{n+p})\leq C^n(I-C)^{-1}\widetilde{w}(u_0,u_1)\tag{4.1}$$

Therefore, by hypothesis (c) and using the relation 4.1 we have the inequality:

$$\begin{aligned}0&<\inf\{\widetilde{w}(u,v^*)+D_{\widetilde{w}}(u,T_2(u)):x\in X\}\\&\leq\inf\{\widetilde{w}(u_n,v^*)+\widetilde{w}(u_n,u_{n+1}):n\in\mathbb{N}\}\\&\leq\inf\{C^n(I-C)^{-1}\widetilde{w}(u_0,u_1)+C^n\widetilde{w}(u_0,u_1):n\in\mathbb{N}\}=0,\end{aligned}$$

which is a contradiction. Thus we conclude that $v^*\in T_2(v^*)$. Note that

$$\widetilde{w}(u_0,v^*)\leq I(I-C)^{-1}\widetilde{w}(u_0,u_1)\leq I(I-C)^{-1}\eta,$$

which complete the proof.

5. Ulam-Hyers stability of fixed point inclusions for weakly contractive type operators in generalized metric spaces

First, let us recall some useful notions of theory of weakly Picard operators for the multivalued case (see [19] and [21]).

If $T:X\to P(X)$ is a MWP operator we define

$$T^\infty:Graph(T)\to P(Fix(T))$$

by the formula $T^\infty(x,y):=\{z\in Fix(T)\mid$ *there exists a sequence of successive approximations of* T *starting from* (x,y) *that converges to* $z\}$

Definition 5.1. Let (X,d) be a metric space and $T:X\to P(X)$ be a MWP operator. Then T is called c-multivalued weakly Picard operator (briefly c-MWP operator) if and only if there exists a selection f^∞ of T^∞ such that $d(x, f^\infty(x,y)) \leq cd(x,y),$ for all $(x,y) \in Graph(T)$.

In [23] are given the definition of Ulam-Hyers stability as follows.

Definition 5.2. Let (X,d) be a metric space and $T : X \to P(X)$ be a multivalued operator. By definition, the fixed point equation

$$x \in T(x) \tag{5.1}$$

is Ulam-Hyers stable if there exists a real number $c > 0$ such that: for each $\varepsilon > 0$ and each solution y^* of the inequation

$$D_d(y, T(y)) \leq \varepsilon \tag{5.2}$$

there exists a solution x^* of the equation 5.1 such that

$$d(y^*, x^*) \leq c\varepsilon.$$

Remark 5.1. If T is a multivalued c-weakly Picard operator, then the fixed point equation 5.1 is Ulam-Hyers stable.

Let us denote a multivalued c-weakly Picard operator with respect to a generalized w-distance by multivalued $C_{\widetilde{w}}$-weakly Picard operator (briefly $C_{\widetilde{w}}$-MWP operator). Next we define this notion.

Definition 5.3. Let (X, $\widetilde{d}$) be a generalized metric space, let $\widetilde{w}$ be a generalized w-distance on X and $C \in M_{m,m}(\mathbb{R}_+)$ be a real positive matrix. $T : X \to P(X)$ is a multivalued $C_{\widetilde{w}}$-weakly Picard operator if there exists a selection f^∞ for T^∞ such that

$$\widetilde{w}(x, f^\infty(x,y)) \leq C\widetilde{w}(x,y), \text{ for all } (x,y) \in Graph(T).$$

Theorem 5.1. *Let $(X, \widetilde{d})$ be a complete generalized metric space and let $\widetilde{w}$ be a generalized w-distance on X. Let $T : X \to P(X)$ be a multivalued operator such that for every $x, y \in X$ and $u \in T(x)$, there exists $v \in T(y)$ such that*

$$\widetilde{w}(u,v) \leq A\widetilde{w}(x,y) + BD_{\widetilde{w}}(y, T(y)),$$

where $A,\ B,\ M \in M_{m,m}(\mathbb{R}_+), B \neq I,\ M := (I-B)^{-1}A$ is a matrix convergent to zero and $D_{\widetilde{w}}(x, T(x)) := \inf\{\widetilde{w}(x,y) : y \in T(x)\}$. Then T is a multivalued $C_{\widetilde{w}}$-weakly Picard operator with $C_{\widetilde{w}} = I + M(I-M)^{-1}$.

Proof. For $x_0 \in X$ fixed and $x_1 \in T(x_0)$, there exists $x_2 \in T(x_1)$ such that

$$\begin{aligned}\widetilde{w}(x_1, x_2) &\leq A\widetilde{w}(x_0, x_1) + BD_{\widetilde{w}}(x_1, T(x_1)) \\ &\leq A\widetilde{w}(x_0, x_1) + B\widetilde{w}(x_1, x_2)\end{aligned}$$

Thus $\widetilde{w}(x_1, x_2) \leq (I-B)^{-1}A\widetilde{w}(x_0, x_1)$. Then $\widetilde{w}(x_1, x_2) \leq M\widetilde{w}(x_0, x_1)$.

By induction, we obtain the sequence $(x_n)_{n\in\mathbb{N}} \in X$ having the properties, for every $n \in \mathbb{N}$:

(i) $x_{n+1} \in T(x_n), n \in \mathbb{N}$;

(ii) $\widetilde{w}(x_n, x_{n+1}) \leq M^n\widetilde{w}(x_0, x_1),$ for each $n \in \mathbb{N}$.

By (ii), for every $m, n \in \mathbb{N}$, with $m > n$, we get that

$$\begin{aligned}\widetilde{w}(x_n, x_m) &\leq \widetilde{w}(x_n, x_{n+1}) + \widetilde{w}(x_{n+1}, x_{n+2}) + ... + \widetilde{w}(x_{m-1}, x_m) \\ &\leq M^n\widetilde{w}(x_0, x_1) + M^{n+1}\widetilde{w}(x_0, x_1) + ... + M^{m-1}\widetilde{w}(x_0, x_1) \\ &\leq M^n(I + M + M^2 + ... + M^{m-1-n})\widetilde{w}(x_0, x_1)\end{aligned}$$

$$\leq M^n(I-M)^{-1}\widetilde{w}(x_0,x_1).$$

Since $(X,\widetilde{d})$ is a generalized complete metric space the sequence $(x_n)_{n\in\mathbb{N}}$ has a limit with respect to the generalized metric $\widetilde{d}$. Let $f^\infty(x_0,x_1)=\lim\limits_{x\to\infty}x_n$ be the limit of the sequence x_n, where f^∞ is a selection of the operator T^∞.

Fix $n\in\mathbb{N}$. Since $(x_m)_{m\in\mathbb{N}}$ is a sequence in X which converge to the limit $f^\infty(x_0,x_1)$ and $\widetilde{w}(x_n,\cdot)$ is lower semicontinuous we have $\widetilde{w}(x_n,f^\infty(x_0,x_1))\leq\lim\limits_{m\to\infty}\inf\widetilde{w}(x_n,x_m)\leq M^n(I-M)^{-1}\widetilde{w}(x_0,x_1)$, for every $n\in\mathbb{N}$.

Then, by triangle inequality we obtain

$$\widetilde{w}(x_0,f^\infty(x_0,x_1))\leq\widetilde{w}(x_0,x_n)+\widetilde{w}(x_n,f^\infty(x_0,x_1)).$$

Then $\widetilde{w}(x_0,f^\infty(x_0,x_1))\leq\widetilde{w}(x_0,x_n)+M^n(I-M)^{-1}w(x_0,x_1)$.

If we make $n\to 1$ we have

$$\widetilde{w}(x_0,f^\infty(x_0,x_1))\leq\widetilde{w}(x_0,x_1)+M(I-M)^{-1}w(x_0,x_1).$$

Then $\widetilde{w}(x_0,f^\infty(x_0,x_1))\leq(I+M(I-M)^{-1})w(x_0,x_1)$.

Then T is a $C_{\widetilde{w}}$-MWP operator with $C_{\widetilde{w}}:=I+M(I-M)^{-1}$.

Let us extend the Ulam-Hyers stability of fixed point inclusions for the case of multivalued operators on generalized metric spaces.

Definition 5.4. Let $(X,\widetilde{d})$ be a generalized metric space and $T:X\to P(X)$ be a multivalued operator. By definition, the fixed point equation

$$x\in T(x) \tag{5.3}$$

is Ulam-Hyers stable if there exists a real positive matrix $C\in M_{m,m}(\mathbb{R}_+)$ such that: for each $\varepsilon>0$ and each solution y^* of the inequation

$$D_{\widetilde{d}}(y,T(y))\leq\varepsilon I \tag{5.4}$$

there exists a solution x^* of the equation 5.3 such that

$$\widetilde{d}(y^*,x^*)\leq C\varepsilon I.$$

Theorem 5.2. *Let $(X,\widetilde{d})$ be a complete generalized metric space and $\widetilde{w}$ be a generalized w-distance on X. Let $T:X\to P(X)$ be a $C_{\widetilde{w}}$-MWP operator such that, there exists a positive matrix $A\in M_{m,m}(\mathbb{R}_+)$ such that $\widetilde{d}(x,y)\leq A\widetilde{w}(x,y)$, for every $(x,y)\in GraphT$.*

Then the fixed point inequality $x\in T(x)$ is Ulam-Hyers stable.

Proof. Let $\varepsilon>0$ and let y^* be a solution of the inequation 5.4.

Let $z\in T(y^*)$ be such that $D_{\widetilde{d}}(y^*,T(y^*))=d(y^*,z)$.

We take a solution of equation 5.3 such that $x^*:=f^\infty(y^*,z)$.

Then we have

$$\widetilde{d}(y^*,x^*)\leq A\widetilde{w}(y^*,x^*)=A\widetilde{w}(y^*,f^\infty(y^*,z))$$

$$\leq AC_{\widetilde{w}}\widetilde{w}(y^*,z)\leq AC_{\widetilde{w}}A^{-1}\widetilde{d}(y^*,z)\leq C_{\widetilde{w}}\varepsilon I.$$

□

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