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ON THE APPROXIMATE CONTINUITY OF VECTOR FUNCTIONS

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Abstract. We prove an equivalent condition to the approximate continuity with respect to some differentiation bases of vector functions. This condition is an counterpart of Lipiński condition from [12] for real functions $f : \mathbb{R} \to \mathbb{R}$.
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1. Introduction

Approximate continuity and existence of fixed points. It is well known that every Baire 1 Darboux function $f : [0,1] \to [0,1]$ has a fixed point. Since the graphs of Darboux Baire 1 functions on $[0,1]$ are connected, such functions are "connectivity maps" and we observe that the answer to Nash's problem in [13] for such functions is affirmative. In [6] R. Gibson and T. Natkaniec mentioned the question of K. Ciesielski, which asks whether the composition of two derivatives $f, g : [0,1] \to [0,1]$ must always have a fixed point. Some affirmative answers to this questions are given in [3] or in [4]. It is well known that, using a result of Maximoff ([15]), Ciesielski's question is equivalent to asking the same question with derivatives by Darboux Baire 1 functions or approximately continuous functions. Earlier some partial answers to Ciesielski problem were obtained in [9].

Of the second hand there are possible some questions about the existence of fixed points for approximately continuous (with respect to different differentiation bases) functions $f : [0,1]^2 \to [0,1]^2$ or some multiple-valued transformations (F. B. Fuller, [5] or O. H. Hamilton [8]).

In this article I prove some equivalent condition to the approximate continuity with respect to some differentiation bases of vector functions.

Maybe this result will be helpfull for obtain some new theorems from the theory of fixed point in Banach spaces.

Let $\mathbb{R}$ be the set of all reals and let $(X, \mathcal{M}, \mu)$ be a measurable space with σ-finite complete measure μ defined on a σ-algebra $\mathcal{M}$ of subsets of X such that $\mu(X) > 0$. Moreover suppose that in $(X, \mathcal{M}, \mu)$ is defined a differentiation basis $(\Phi, \Longrightarrow)$ (for the definition of differentiation basis see [2], p. 30), i.e., a couple $(\Phi, \Longrightarrow)$, where Φ is a family of subsets $A \in \mathcal{M}$ with $0 < \mu(A) < \infty$ and the symbol $\Longrightarrow$ denotes a convergence of generalized sequences of Moore-Smith of subsets from Φ to points of X such that

(a) for each point $x \in X$ there is at least one generalized sequence (I_α) of subsets from Φ convergent in the sense $\Longrightarrow$ to x and
(b) every cofinal subsequence I_{α_s} of a sequence (I_α) contracting in the sense $\Longrightarrow$ to x also contracts in the sense $\Longrightarrow$ to x.

The differentiation bases are very important in the theory of differentiation of integrals in Euclidean and abstract spaces ([2]). In well know book [16] S. Saks considers in $\mathbb{R}^2$ the ordinary differentiation basis $(\Phi, \Longrightarrow)$, where Φ is the family of all discs or squares and a sequence (I_n) of subsets of Φ contracts to a point x iff $x \in I_n$ for $n \geq 1$ and the sequence $(diam(I_n))$ of the diameters of I_n converges to 0. If Φ is the family of all two dimensional intervals then he obtain the strong differentiation basis. Moreover, in [16] the author investigates also the differentiation basis $(\Phi, \Longrightarrow)$, where Φ is the family of all rectangles in $\mathbb{R}^2$. In [2] the author describes many other examples of differentiation bases in infinite dimensional spaces and in abstract measure spaces and also a special type of differentiation bases called net structures.

In the article I will used the Bochner integral constructed in [10]. With the help of it we will integrate vector functions with values in a separable Banach space Y. Then bounded μ-measurable functions will be integrable on the sets from Φ. So we assume that $(Y, ||.||)$ is a separable Banach space with the norm $||.||$. A function $f : X \to Y$ is said μ-measurable if $f^{-1}(B)$ belongs to $\mathcal{M}$ for each Borel set $B \subset Y$. A μ-measurable function $f : X \to Y$ integrable in the Bochner sense on all sets belonging to Φ is said a derivative at a point $x \in X$ with respect to $(\Phi, \Longrightarrow)$ (compare [2] for real functions) if for each generalized sequence (I_α) of subsets from Φ convergent in the sense $\Longrightarrow$ to x the corresponding generalized sequence $\left(\frac{\int_{I_\alpha} f}{\mu(I_\alpha)}\right)$ converges to $f(x)$.

In the investigation of derivatives the notions of density point and approximate continuity is very important ([1]).

A point $x \in X$ is said a density point of a set $A \in \mathcal{M}$ with respect to $(\Phi, \Longrightarrow)$ (see [2]) if for every generalized sequence (I_α) of subsets from Φ convergent in the sense $\Longrightarrow$ to x the corresponding generalized sequence $\left(\frac{\mu(A \cap I_\alpha)}{\mu(I_\alpha)}\right)$ converges to 1. Moreover we assume that if x is a density point of a set A with respect to $(\Phi, \Longrightarrow)$ then x is also a density point of every set $G \supset A$. A μ-measurable function $f : X \to Y$ is said approximately continuous at a point $x \in X$ with respect to $(\Phi, \Longrightarrow)$ if for each open set $U \subset Y$ containing $f(x)$ the point x is a density point of $f^{-1}(U)$ with respect to $(\Phi, \Longrightarrow)$.

In 1958 J. S. Lipiński proved in [12] the following theorem:

Theorem 1.1. *A Lebesgue measurable function* $f : \mathbb{R} \to [-\infty, \infty]$ *is approximately continuous at each point* $x \in \mathbb{R}$ *if and only if for all reals* r, s *with* $r < s$*, the functions* $f_r^s = \min(s, \max(r, f))$ *are derivatives.*

As a generalization of above Lipiński theorem in [7] it is proved the following.

Theorem 1.2. *A μ-measurable function $f : X \to \mathbb{R}$ is approximately continuous at each point $x \in X$ with respect to $(\Phi, \Longrightarrow)$ if and only if for all reals r, s with $r < s$ the functions $f_r^s = \min(s, \max(r, f))$ are derivatives with respect to $(\Phi, \Longrightarrow)$.*

In this article we prove an counterpart of that result for vector functions $f : X \to Y$.

2. Main results

Theorem 2.1. *A μ-measurable function $f : X \to Y$ is approximately continuous at each point $x \in X$ with respect to $(\Phi, \Longrightarrow)$ if and only if for each bounded continuous function $\phi : Y \to \mathbb{R}$ the superposition $(\phi(f))$ is a derivative with respect to $(\Phi, \Longrightarrow)$.*

Proof. Necessity. If $f : X \to Y$ is an approximately continuous function with respect to $(\Phi, \Longrightarrow)$ and $\phi : Y \to \mathbb{R}$ is a bounded continuous function then the μ-measurable superposition $\phi(f)$ is also approximately continuous with respect to $(\Phi, \Longrightarrow)$ and consequently, as a bounded approximately continuous function with respect to $(\Phi, \Longrightarrow$) it is also ([2]) a derivative with respect to $(\Phi, \Longrightarrow)$.
Sufficiency. Suppose, contrary to our claim, that the function f is not approximately continuous with respect to $(\Phi, \Longrightarrow)$ at a point $x \in X$. Without loss of generality we can assume that $f(x) = 0$. There is an open set $U \subset Y$ containing $f(x)$ such that x is not any density point of $f^{-1}(U)$ with respect to $(\Phi, \Longrightarrow)$. Consequently, there are a generalized sequence (I_α) convergent in the sense $\Longrightarrow$ to x and a positive real η such that the limit of the generalized sequence $(\frac{\mu(f^{-1}((Y \setminus U) \cap I_\alpha)}{\mu(I_\alpha)})$ is equal η. Let $K \subset U$ be a closed ball with the center 0 and the radius $s > 0$. By Tietze theorem there is a continuous function $\phi : Y \to [0, 1]$ such that $\phi(K) = \{0\}$ and $\phi(Y \setminus U) = \{1\}$. Observe that $\phi(f) \geq 0$ and $\phi(f(t)) = 1$ for $t \in f^{-1}(Y \setminus U)$. So for each index α we have
$$\frac{\int_{I_\alpha} f}{\mu(I_\alpha)} \geq \frac{\mu(I_\alpha \cap f^{-1}(Y \setminus U))}{\mu(I_\alpha)},$$
and consequently the generalized sequence $(\frac{\int_{I_\alpha} f}{\mu(I_\alpha)})$ does not converge to $f(0) = 0$. This contradicts our assumption that $\phi(f)$ is a derivative at x with respect to $(\Phi, \Longrightarrow)$. □

3. Final Observations

In the natural case, where $X = \mathbb{R}$ and μ is the measure of Lebesgue we assume that Φ is the family of all nondegenerate bounded open intervals and the convergence of sequences (I_n) to points x denotes that $x \in I_n$ for $n = 1, 2, \ldots$ and $\lim_{n\to\infty} \mu(I_n) = 0$. Then the approximate continuity of $f : \mathbb{R} \to \mathbb{R}$ denotes ([1]) the continuity of f as an application from $(\mathbb{R}, T_d)$ to $(\mathbb{R}, T_e)$, where T_e denotes the natural topology on $\mathbb{R}$ and T_d denotes the density topology in $\mathbb{R}$, i.e., a set $A \in T_d$ iff every point $x \in A$ is a density point of A ([1]).

From Theorems 1.1 and 2.1 we obtain the following theorem.

Theorem 3.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable (in the Lebesgue sense) function. The following conditions are equivalent:*

(i) *f is approximately continuous at each point;*
(ii) *for all reals $r < s$ the functions $f_r^s = \max(r, \min(s, f))$ are derivatives;*

(iii) *for each continuous bounded function* $g : \mathbb{R} \to \mathbb{R}$ *the superposition* $g(f)$ *is a derivative.*

With the density topology T_d is connected the topology T_{ae} defined by O'Malley in [14] by the following manner: $A \in T_{ae}$ iff $A \in T_d$ and $\mu(A \setminus int(A)) = 0$, where $int(A)$ denotes the natural interior of A. It is well known ([7]) that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous as an application from $(\mathbb{R}, T_{ae})$ to $(\mathbb{R}, T_e$ iff f is approximately continuous everywhere and continuous almost everywhere with respect to Lebesgue measure μ. Similarly as Theorem 3.1 we can prove the following.

Theorem 3.2. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be an almost everywhere continuous function. Then the following conditions are equivalent:*

(j) f *is continuous as an application from* $(\mathbb{R}, T_{ae})$ *to* $(\mathbb{R}, T_e)$*;*

(jj) *for all reals* $r < s$ *the functions* $F_r^s = \max(r, \min(s, f))$ *are derivatives;*

(jjj) *for each continuous bounded function* $g : \mathbb{R} \to \mathbb{R}$ *the superposition* $g(f)$ *is a derivative.*

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