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MINIMAL SETS OF NONCYCLIC RELATIVELY NONEXPANSIVE MAPPINGS IN CONVEX METRIC SPACES

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Abstract. Let us consider a mapping $T : A \cup B \to A \cup B$ such that $T(A) \subseteq A$ and $T(B) \subseteq B$, where A and B are two nonempty subsets of a metric space (X, d). We provide sufficient conditions for the existence of a point $(p, q) \in A \times B$, called best proximity pair, which satisfies $p = Tp, q = Tq$ and $d(p, q) = dist(A, B) := \inf\{d(x, y) : (x, y) \in A \times B\}$, in the setting of convex metric spaces for noncyclic contractions. Then, we present a similar result of Goebel-Karlovitz lemma for noncyclic relatively nonexpansive mappings in convex metric spaces.
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1. Introduction and Preliminaries

Let (X, d) be a metric space and let A and B be two nonempty subsets of X. A mapping $T : A \cup B \to A \cup B$ is said to be a *noncyclic mapping* provided that $T(A) \subseteq A$ and $T(B) \subseteq B$. A point $(p, q) \in A \times B$ is said to be a *best proximity pair* for noncyclic mapping T, provided that

$$Tp = p, \quad Tq = q \quad \text{and} \quad d(p, q) = dist(A, B).$$

Let (A, B) be a nonempty pair of subsets of a metric space (X, d). A mapping $T : A \cup B \to A \cup B$ is said to be a *noncyclic relatively nonexpansive* if T is noncyclic and $d(Tx, Ty) \leq d(x, y)$ for all $(x, y) \in A \times B$.

In [3], Eldred et al. studied the existence of best proximity pairs for noncyclic relatively non-expansive mappings in Banach spaces with a geometric property, called *proximal normal structure*. They proved the following theorem.

Theorem 1.1. (Theorem 2.2 of [3]) *Let (A, B) be a nonempty, weakly compact and convex pair in a strictly convex Banach space X. Let $T : A \cup B \to A \cup B$ be a noncyclic relatively nonexpansive mapping. If the pair (A, B) has proximal normal structure. The T has a best proximity pair.*

Also, in [2], the authors investigated sufficient conditions which ensure the existence of a best proximity pair for noncyclic mappings.

The notion of convexity in metric spaces was introduced by Takahashi as follows.

Definition 1.2. ([7]) Let (X, d) be a metric space and $I := [0, 1]$. A mapping $\mathcal{W} : X \times X \times I \to X$ is said to be a convex structure on X provided that for each $(x, y; \lambda) \in X \times X \times I$ and $u \in X$,

$$d(u, \mathcal{W}(x, y; \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y).$$

A metric space (X, d) together with a convex structure $\mathcal{W}$ is called a convex metric space, which is denoted by $(X, d, \mathcal{W})$. A Banach space and each of its convex subsets are convex metric spaces. But a Frechet space is not necessary a convex metric space. The other examples of convex metric spaces which are not imbedded in any Banach space can be founded in [7].

Here, we recall some notations and definitions of [1, 7].

Definition 1.3. ([7]) A subset K of a convex metric space $(X, d, \mathcal{W})$ is said to be a convex set provided that $\mathcal{W}(x, y; \lambda) \in K$ for all $x, y \in K$ and $\lambda \in I$.

Proposition 1.4. ([7]) *Let $(X, d, \mathcal{W})$ be a convex metric space and let $B(x; r)$ denote the closed ball centered at $x \in X$ with radius $r \geq 0$. Then $B(x; r)$ is a convex subset of X.*

Proposition 1.5. ([7]) *Let $\{K_\alpha\}_{\alpha \in A}$ be a family of convex subsets of X, then $\bigcap_{\alpha \in A} K_\alpha$ is also a convex subset of X.*

Definition 1.6. ([7]) A convex metric space $(X, d, \mathcal{W})$ is said to have property (C) if every bounded decreasing net of nonempty closed convex subsets of X has a nonempty intersection.

For example every bounded, closed and convex subset of a reflexive Banach space X has property (C).

Let A and B be two nonempty subsets of a convex metric space $(X, d, \mathcal{W})$. We shall say that a pair (A, B) in a convex metric space $(X, d, \mathcal{W})$ satisfies a property if both A and B satisfy that property. For instance, (A, B) is convex if and only if both A and B are convex; $(A, B) \subseteq (C, D) \Leftrightarrow A \subseteq C$, and $B \subseteq D$. We shall also adopt the following notations.

$$\begin{aligned} \delta_x(A) &:= \sup\{d(x, y) \colon y \in A\} \text{ for all } x \in X, \\ \delta(A, B) &:= \sup\{d(x, y) \colon x \in A,\ y \in B\}, \\ \operatorname{diam}(A) &:= \delta(A, A). \end{aligned}$$

The *closed and convex hull* of a set A will be denoted by $\overline{\text{con}}(A)$ and defined as below.

$$\overline{con}(A) := \bigcap\{C : C \text{ is a closed and convex subset of } X \text{ such that } C \supseteq A\}.$$

The pair $(x, y) \in A \times B$ is said to be *proximal* in (A, B) if $d(x, y) = dist(A, B)$. Moreover, we set

$$\begin{aligned} A_0 &:= \{x \in A : d(x, y') = dist(A, B), \text{ for some } y' \in B\}, \\ B_0 &:= \{y \in B : d(x', y) = dist(A, B), \text{ for some } x' \in A\}. \end{aligned}$$

Note that if (A, B) is a nonempty weakly compact and convex pair of subsets of a Banach space X, then also is the pair (A_0, B_0) and it is easy to see that $dist(A, B) = dist(A_0, B_0)$.

Definition 1.7. A pair of sets (A, B) is said to be proximal if $A = A_0$ and $B = B_0$.

The following result follows from the proof of Theorems 2.2 in [3].

Lemma 1.8. *Let (A, B) be a nonempty weakly compact convex pair of a Banach space X and $T\colon A \cup B \to A \cup B$ a noncyclic relatively nonexpansive mapping. Then there exists $(K_1, K_2) \subseteq (A_0, B_0) \subseteq (A, B)$ which is minimal with respect to being nonempty closed convex and T-invariant pair of subsets of (A, B) such that*

$$\operatorname{dist}(K_1, K_2) = \operatorname{dist}(A, B).$$

Moreover, the pair (K_1, K_2) is proximal.

In this article, we study sufficient conditions which ensure the existence of best proximity pairs for noncyclic contractions in convex metric spaces. In this way, we obtain a similar result of Goebel-Karlovitz lemma [4, 5] which is a key lemma in fixed point theory.

Lemma 1.9. (Goebel-Karlovitz lemma [4, 5]) *Let A be a nonempty, weakly compact, convex subset of a Banach space X and let $T : A \to A$ be a nonexpansive mapping. Assume that K is a subset of A which is minimal with respect to being nonempty, weakly compact, convex and T-invariant, and suppose $\{x_n\}$ is a sequence in K such that*

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Then, for each $x \in K$, $\lim_{n\to\infty} \|x - x_n\| = \operatorname{diam}(K)$.

2. Existence of best proximity pairs for noncyclic contractions

In this section, we prove a best proximity pair theorem for *noncyclic contractions* in convex metric spaces. We begin our main result with the following notion.

Definition 2.1. A convex metric space $(X, d, \mathcal{W})$ is said to have property (D) provided that for each x_1, x_2, y in X with $x_1 \neq x_2$ we have

$$d(\mathcal{W}(x_1, x_2, \frac{1}{2}), y) < \frac{1}{2}[d(x_1, y) + d(x_2, y)].$$

It is clear that every strictly convex Banach space is a convex metric space which satisfies the property (D).

We now state the main result of this section.

Theorem 2.2. *Let (A, B) be a nonempty, bounded, closed and convex pair in a convex metric space $(X, d, \mathcal{W})$. Suppose that $T : A \cup B \to A \cup B$ is a noncyclic contraction, that is, there exists $\alpha \in (0, 1)$ such that*

$$d(Tx, Ty) \leq \alpha d(x, y) + (1 - \alpha) dist(A, B), \tag{2.1}$$

for each $(x, y) \in A \times B$. If X satisfies the properties (C) and (D) then T has a best proximity pair.

Proof. Let Σ denote the set of all nonempty, bounded, closed and convex pairs (E, F) which are subsets of (A, B) and such that T is noncyclic on $E \cup F$. Note that $(A, B) \in \Sigma$. Also, Σ is partially ordered by revers inclusion, that is $(E_1, E_2) \leq (F_1, F_2) \Leftrightarrow (F_1, F_2) \subseteq (E_1, E_2))$. Since X has the property (C), every increasing chain in Σ is bounded above. By using Zorn's lemma, we obtain a minimal element say $(E, F) \in \Sigma$. Note that $(\overline{con}(T(E)), \overline{con}(T(F)))$ is a nonempty, bounded, closed and convex subset of (E, F). By the fact that T is noncyclic,

$$T(\overline{con}(T(E))) \subseteq T(E) \subseteq \overline{con}(T(E)),$$

and also,

$$T(\overline{con}(T(F))) \subseteq \overline{con}(T(F)).$$

So, T is noncyclic on $\overline{con}(T(E)) \cup \overline{con}(T(F))$. The minimality of (E, F) implies that

$$\overline{con}(T(E)) = E\,,\ \overline{con}(T(F)) = F.$$

Let $x \in E$, then $F \subseteq B(x; \delta_x(F))$. Now, if $y \in F$ we have

$$\begin{aligned} d(Tx, Ty) &\leq \alpha d(x, y) + (1-\alpha)dist(A, B) \\ &\leq \alpha\delta(E, F) + (1-\alpha)dist(A, B). \end{aligned}$$

Thus, for all $y \in F$ we have

$$Ty \in B(Tx; \alpha\delta(E, F) + (1-\alpha)dist(A, B)),$$

and so,

$$T(F) \subseteq B(Tx; \alpha\delta(E, F) + (1-\alpha)dist(A, B)).$$

Then,

$$F = \overline{con}(T(F)) \subseteq B(Tx; \alpha\delta(E, F) + (1-\alpha)dist(A, B)),$$

which implies that

$$d(z, Tx) \leq \alpha\delta(E, F) + (1-\alpha)dist(A, B), \quad \forall z \in F.$$

Hence,

$$\delta_{Tx}(F) \leq \alpha\delta(E, F) + (1-\alpha)dist(A, B). \tag{2.2}$$

Similarly, if $y \in F$ we obtain

$$\delta_{Ty}(E) \leq \alpha\delta(E, F) + (1-\alpha)dist(A, B). \tag{2.3}$$

Put,

$$\begin{aligned} E' &:= \{x \in E : \delta_x(F) \leq \alpha\delta(F, F) + (1-\alpha)dist(A, B)\}, \\ F' &:= \{y \in F : \delta_y(E) \leq \alpha\delta(E, F) + (1-\alpha)dist(A, B)\}. \end{aligned}$$

We now have $T(E) \subseteq E'$ and $T(F) \subseteq F'$ and it is easy to see that

$$E' = \bigcap_{y \in F} B(y; \alpha\delta(E, F) + (1-\alpha)dist(A, B)) \cap E,$$

$$F' = \bigcap_{x \in E} B(x; \alpha\delta(E, F) + (1-\alpha)dist(A, B)) \cap F.$$

We note that by Propositions 1.4 and 1.5 the pair (E', F') is convex. Moreover, by relations (2.2) and (2.3) we conclude that T is noncyclic on $E' \cup F'$. Minimality of (E, F) guarantees that $E' = E$ and $F' = F$. Therefore,

$$\delta_x(F) \leq \alpha\delta(E, F) + (1-\alpha)dist(A, B), \quad \forall x \in E.$$

Thus,

$$\delta(E, F) = dist(A, B). \tag{2.4}$$

Let $(p, q) \in E \times F$. It now follows from (2.4) that $d(p, q) = dist(A, B)$. We claim that E and F are singleton. Assume that $p \neq p' \in E$ and $q \in F$. Since E is a convex set, $(\mathcal{W}(p,p',\frac{1}{2}),\mathcal{W}(q,q',\frac{1}{2}))\in E\times F$. Now, by the fact that X satisfies the property (D), we deduce that

$$\begin{aligned} dist(A,B) &\leq d(\mathcal{W}(p,p',\frac{1}{2}),q) \\ &< \frac{1}{2}[d(p,q)+d(p',q)] \leq \delta(E,F) = dist(A,B), \end{aligned}$$

which is a contradiction. Hence, E and F are singleton. This completes the proof of theorem.

Remark 2.3. Theorem 2.2 holds once the minimal sets E and F have been fixed and the noncyclic mapping $T : A\cup B \to A\cup B$, satisfies the condition that there exists $\alpha\in[0,1)$ such that

$$d(Tx,Ty) \leq \alpha\delta(E,F) + (1-\alpha)dist(A,B), \tag{2.5}$$

for all $(x,y)\in A\times B$.

The next result obtains from Theorem 2.2, directly.

Corollary 2.4. *Let (A,B) be a nonempty, bounded, closed and convex pair in a reflexive and strictly convex Banach space X. Suppose that $T : A\cup B \to A\cup B$ is a noncyclic contraction. Then T has a best proximity pair.*

3. Goebel-Karlovitz lemma for noncyclic relatively nonexpansive mappings

The purpose of this section is to give a similar result of Goebel-Karlovitz lemma for noncyclic relatively nonexpansive mappings in convex metric spaces. We start our results of this section by the next definitions.

Definition 3.1. Let (A,B) be a nonempty pair of subsets of a metric space (X,d). We say that the pair (A,B) is a proximal compactness pair provided that every net $\{(x_\alpha,y_\alpha)\}$ of $A\times B$ satisfying the condition that $d(x_\alpha,y_\alpha)\to dist(A,B)$, has a convergent subnet in $A\times B$. Also, we say that A is semi-compactness if (A,A) is proximal compactness.

It is clear that if (A,B) is a compact pair in a metric space (X,d) then (A,B) is proximal compactness.

Definition 3.2. Let (A,B) be a nonempty pair of sets in a Banach space X. A point p in A (q in B) is said to be a diametral point with respect to B (w.r.t. A) if $\delta_p(B)=\delta(A,B)$ ($\delta_q(A)=\delta(A,B)$). A pair (p,q) in $A\times B$ is diametral if both points p and q are diametral.

The following result is another version of Lemma 1.8 for noncyclic mappings in convex metric spaces.

Lemma 3.3. *Let (A,B) be a nonempty, bounded, closed and convex pair of a convex metric space $(X,d,\mathcal{W})$ such that A_0 is nonempty and (A,B) is a proximal compactness pair. Assume that $T\colon A\cup B\to A\cup B$ is a noncyclic relatively nonexpansive mapping. If X has the property (C) then there exists a pair $(K_1,K_2)\subseteq(A,B)$ which is minimal with respect to being nonempty, closed, convex and T-invariant pair of subsets of (A,B) such that*

$$\mathrm{dist}(K_1,K_2)=\mathrm{dist}(A,B).$$

Proof. Let Σ denote the set of all nonempty, closed and convex pairs (E, F) which are subsets of (A, B) and such that T is noncyclic on $E \cup F$ and $d(x, y) = dist(A, B)$ for some $(x, y) \in E \times F$. Since A_0 is nonempty set, $(A, B) \in \Sigma$. Moreover, Σ is partially ordered by revers inclusion. Suppose $\{(E_\alpha, F_\alpha)\}_\alpha$ is a descending chain in Σ. Put $E := \bigcap E_\alpha$ and $F := \bigcap F_\alpha$. By the fact that X has the property (C), we conclude that (E, F) is a nonempty pair. By Proposition 1.5, (E, F) is a convex pair. Also,

$$T(E) = T(\bigcap E_\alpha) \subseteq \bigcap T(E_\alpha) \subseteq \bigcap E_\alpha = E.$$

Similarly, $T(F) \subseteq F$, that is, T is noncyclic on $E \cup F$. Now, let $(x_\alpha, y_\alpha) \in E_\alpha \times F_\alpha$ be such that $d(x_\alpha, y_\alpha) = dist(A, B)$. Since (A, B) is proximal compactness, (x_α, y_α) has a convergent subsequence say $(x_{\alpha_i}, y_{\alpha_i})$ such that $x_{\alpha_i} \to x \in A$ and $y_{\alpha_i} \to y \in B$. Thus,

$$d(x, y) = \lim_i d(x_{\alpha_i}, y_{\alpha_i}) = dist(A, B).$$

Therefore, there exists an element $(x, y) \in E \times F$ such that $d(x, y) = dist(A, B)$. So, every increasing chain in Σ is bounded above with respect to revers inclusion relation. Thus, by using Zorn's Lemma we can get an element say (K_1, K_2) which is minimal with respect to being nonempty, closed, convex and T-invariant pair of subsets of (A, B) such that

$$\mathrm{dist}(K_1, K_2) = \mathrm{dist}(A, B).$$

Lemma 3.4. *Let (A, B) be a nonempty, bounded, closed and convex pair of a convex metric space $(X, d, \mathcal{W})$ such that A_0 is nonempty. Suppose that X has the property (C) and (A, B) is a proximal compactness pair. Let $T\colon A \cup B \to A \cup B$ be a noncyclic relatively nonexpansive mapping. Suppose that $(K_1, K_2) \subseteq (A, B)$ is a minimal closed convex pair which is T-invariant and such that $dist(K_1, K_2) = dist(A, B)$. Then each pair $(p, q) \in K_1 \times K_2$ with $d(p, q) = dist(A, B)$ is a diametral pair (with respect to (K_1, K_2)), that is,*

$$\delta_p(K_2) = \delta_q(K_1) = \delta(K_1, K_2).$$

Proof. By the similar argument of Theorem 2.2 we conclude that T is noncyclic on $\overline{\mathrm{con}}(T(K_1)) \cup \overline{\mathrm{con}}(T(K_2))$. Let $(p, q) \in K_1 \times K_2$ be such that $d(p, q) = \mathrm{dist}(A, B)$ and suppose

$$\min\{\delta_p(K_2), \delta_q(K_1)\} < \delta(K_1, K_2). \tag{3.1}$$

Since T is noncyclic relatively nonexpansive,

$$\mathrm{dist}(\overline{\mathrm{con}}(T(K_1)), \overline{\mathrm{con}}(T(K_2)) = \mathrm{dist}(A, B).$$

Minimality of (K_1, K_2) concludes that

$$\overline{\mathrm{con}}(T(K_1)) = K_1, \quad \overline{\mathrm{con}}(T(K_2)) = K_2.$$

Put, $r_1 := \delta_p(K_2)$ and $r_2 := \delta_q(K_1)$. So, $\min\{r_1, r_2\} < \delta(K_1, K_2)$. Let

$$K_1^* := K_1 \bigcap (\cap_{x \in K_2} B(x; r_1)), \quad K_2^* := K_2 \bigcap (\cap_{x \in K_1} B(x; r_2)).$$

Then (K_1^*, K_2^*) is a nonempty, closed and convex pair in X by Propositions 1.4 and 1.5. Also, since $(p, q) \in (K_1^*, K_2^*)$,

$$\mathrm{dist}(K_1^*, K_2^*) = \mathrm{dist}(A, B).$$

It is easy to see that for $(x, y) \in K_1 \times K_2$,

$$(x, y) \in (K_1^*, K_2^*) \Leftrightarrow K_2 \subseteq B(x; r_1), \quad K_1 \subseteq B(y; r_2).$$

We now claim that T is noncyclic on $K_1^* \cup K_2^*$. Let $x \in K_1^*$. We must show that $Tx \in K_1^*$, or equivalently, $K_2 \subseteq B(Tx; r_1)$. For $y \in K_2$, the relatively nonexpansiveness of T deduces that

$$d(Tx, Ty) \leq d(x, y) \leq r_1,$$

then $Ty \in B(Tx; r_1)$ which implies that $T(K_2) \subseteq B(Tx; r_1)$. Therefore, $K_2 = \overline{\text{con}}(T(K_2)) \subseteq B(Tx; r_1)$ and hence, $Tx \in K_1^*$. Thus, $T(K_1^*) \subseteq K_1^*$. Similarly, we can see that $T(K_2^*) \subseteq K_2^*$. Therefore, T is noncyclic on $K_1^* \cup K_2^*$. It now follows from the minimality of (K_1, K_2) that $K_1^* = K_1$ and $K_2^* = K_2$. Then, $K_1 \subseteq \bigcap_{x \in K_2} B(x; r_1)$ and so, for each $x \in K_1$, $\delta_x(K_2) \leq r_1$. We now conclude that

$$\delta(K_1, K_2) = \sup_{x \in K_1} \delta_x(K_2) \leq r_1.$$

Similarly, we can see that $\delta(K_1, K_2) \leq r_2$, which is a contradiction with the relation (3.1) and this completes the proof.

Here, we introduce the notion of *proximal approximate fixed point sequence* for noncyclic mappings as follows.

Definition 3.5. Let (A, B) be a nonempty pair of subsets of a metric space (X, d). Suppose that $T : A \cup B \to A \cup B$ is a noncyclic mapping. Then a sequence $\{(x_n, y_n)\}$ in $A \times B$ is said to be a proximal approximate fixed point sequence for T if

$$d(x_n, Tx_n) \to 0, \quad d(y_n, Ty_n) \to 0 \;\text{ and }\; d(x_n, y_n) \to dist(A, B).$$

The following lemma guarantees the existence of a proximal approximate fixed point sequence for noncyclic relatively nonexpansive mappings.

Lemma 3.6. *Let (A, B) be a nonempty, bounded, closed and convex pair of a convex metric space $(X, d, \mathcal{W})$ such that A_0 is nonempty, X has the properties (C) and (D) and (A, B) is a proximal compactness pair. Let $T\colon A \cup B \to A \cup B$ be a noncyclic relatively nonexpansive mapping. Then there exists a proximal approximate fixed point sequence for T.*

Proof. By using Lemma 3.3, there exists a pair $(K_1, K_2) \subseteq (A, B)$ which is minimal with respect to being nonempty, closed, convex and T-invariant pair of subsets of (A, B) and there exists $(p, q) \in K_1 \times K_2$ such that

$$dist(K_1, K_2) = d(p, q) = \text{dist}(A, B).$$

For any $\alpha \in (0, 1)$ let $r := 2\alpha - \alpha^2$. Then $r < 1$. Define the mapping $T_\alpha : A \cup B \to A \cup B$ with

$$T_\alpha(x) = \begin{cases} \mathcal{W}(Tx, p, \alpha); & x \in A, \\ \mathcal{W}(Tx, q, \alpha); & x \in B. \end{cases}$$

Since T is noncyclic and (A, B) is a convex pair in convex metric space $(X, d, \mathcal{W})$, we conclude that T_α is noncyclic on $A \cup B$. Now, for each $(x, y) \in A \times B$ we have

$$\begin{aligned} d(T_\alpha x, T_\alpha y) &= d(\mathcal{W}(Tx, p, \alpha), \mathcal{W}(Ty, q, \alpha)) \\ &\leq \alpha d(\mathcal{W}(Tx, p, \alpha), Ty) + (1 - \alpha) d(\mathcal{W}(Tx, p, \alpha), q) \end{aligned}$$

$$\le \alpha[\alpha d(Ty,Tx) + (1-\alpha)d(Ty,p)] + (1-\alpha)[\alpha d(Tx,q) + (1-\alpha)d(p,q)]$$
$$\le \alpha^2 d(x,y) + \alpha(1-\alpha)d(Ty,p) + \alpha(1-\alpha)d(q,Tx) + (1-\alpha)^2 d(p,q)$$
$$\le \alpha^2\delta(K_1,K_2) + 2\alpha\delta(K_1,K_2) - 2\alpha^2\delta(K_1,K_2) + (1-\alpha)^2 dist(A,B)$$
$$= (2\alpha-\alpha^2)\delta(K_1,K_2) + [1-(2\alpha-\alpha^2)]dist(A,B)$$
$$= r\delta(K_1,K_2) + (1-r)dist(A,B).$$

Hence, for each $\alpha \in (0,1)$ we have

$$d(T_\alpha x, T_\alpha y) \le r\delta(K_1,K_2) + (1-r)dist(A,B).$$

It now follows from Remark 2.3 that for each $\alpha \in (0,1)$ the noncyclic mapping T_α has a best proximity pair say $(p_\alpha, q_\alpha) \in A \times B$. That is, for each $\alpha \in (0,1)$ there exists $(p_\alpha, q_\alpha) \in A \times B$ such that

$$p_\alpha = T_\alpha(p_\alpha), \quad q_\alpha = T_\alpha(q_\alpha) \;\text{ and }\; d(p_\alpha, q_\alpha) = dist(A,B).$$

We now have

$$d(p_\alpha, T(p_\alpha)) = d(T_\alpha(p_\alpha), T(p_\alpha)) = d(\mathcal{W}(Tp_\alpha, p, \alpha), Tp_\alpha)$$
$$\le (1-\alpha)d(p, Tp_\alpha) \le (1-\alpha)diam(A).$$

Now, if $\alpha \to 1^-$ in above relation, we conclude that

$$d(p_\alpha, Tp_\alpha) \to 0.$$

Similarly, we can see that $d(q_\alpha, Tq_\alpha) \to 0$. Therefore, there exists a sequence $(\{x_n\},\{y_n\}) \in A \times B$ such that

$$d(x_n, Tx_n) \to 0, \quad d(y_n, Ty_n) \to 0 \;\text{ and }\; d(x_n, y_n) \to dist(A,B).$$

The next result is a new version of Goebel-Karlovitz lemma for noncyclic mappings in convex metric spaces.

Theorem 3.7. *Let (A,B) be a nonempty, bounded, closed and convex pair of a convex metric space $(X,d,\mathcal{W})$ such that X has the properties (C) and (D). Assume that A_0 is nonempty and (A,B) is a proximal compactness pair. Let $T\colon A\cup B \to A\cup B$ be a noncyclic relatively nonexpansive mapping. Suppose $(K_1,K_2) \subseteq (A,B)$ is a minimal closed and convex pair which is T-invariant and such that* $\mathrm{dist}(K_1,K_2) = \mathrm{dist}(A,B)$ *and let $(\{x_n\},\{y_n\}) \in A\times B$ be a proximal approximate sequence in $A\times B$. Then for each $(p,q) \in K_1 \times K_2$ with $d(p,q) = dist(A,B)$ we have*

$$\limsup_{n\to\infty} d(x_n, q) = \limsup_{n\to\infty} d(p, y_n) = \delta(K_1,K_2).$$

Proof. The existence of the proximal approximate fixed point sequence for T obtains from Lemma 3.6. By this reality that (A,B) is proximal compactness, there exists a subsequence $(\{x_{n_k}\},\{y_{n_k}\})$ of the sequence $(\{x_n\},\{y_n\})$ such that $x_{n_k} \to p^*$ and $y_{n_k} \to q^*$ for some $(p^*,q^*) \in K_1 \times K_2$. Hence,

$$d(p^*,q^*) = \lim_{k\to\infty} d(x_{n_k}, y_{n_k}) = dist(A,B).$$

It follows from Lemma 3.4 that (p^*,q^*) is a diametral pair. Let

$$r_1 := \limsup_{n\to\infty} d(x_n, q), \quad r_2 := \limsup_{n\to\infty} d(p, y_n).$$

We claim that

$$r_1 = r_2 = \delta(K_1, K_2). \tag{3.2}$$

Suppose that $r_1 < \delta(K_1, K_2)$. Set,

$$K_1^* := \{x \in K_1 : \limsup_{n\to\infty} d(x, y_n) \leq r_1\}, \quad K_2^* := \{y \in K_2 : \limsup_{n\to\infty} d(x_n, y) \leq r_2\}.$$

Note that $(p, q) \in K_1^* \times K_2^*$ and (K_1^*, K_2^*) is a closed pair in X. Moreover, (K_1^*, K_2^*) is a convex pair in X. In fact, if $x_1, x_2 \in K_1^*$, then

$$\limsup_{n\to\infty} d(\mathcal{W}(x_1, x_2, \alpha), y_n) \leq \limsup_{n\to\infty}[\alpha d(x_1, y_n) + (1-\alpha)d(x_2, y_n)] \leq r_1.$$

Thus, $\mathcal{W}(x_1, x_2, \alpha) \in K_1^*$, that is, K_1^* is convex. Similarly, we can see that K_2^* is convex. Further, $T(K_1^*) \subseteq K_1^*$. Indeed, if $x \in K_1^*$, then

$$\begin{aligned}\limsup_{n\to\infty} d(Tx, y_n) &\leq \limsup_{n\to\infty}[d(Tx, Ty_n) + d(Ty_n, y_n)] \\ &\leq \limsup_{n\to\infty} d(x, y_n) \leq r_1,\end{aligned}$$

which concludes that $Tx \in K_1^*$. Similarly, we can see that $T(K_2^*) \subseteq K_2^*$. Therefore, T is noncyclic on $K_1^* \cup K_2^*$. It now follows from the minimality of (K_1, K_2) that $(K_1, K_2) = (K_1^*, K_2^*)$. Then for each $y \in K_2$ we have

$$d(p^*, y) = \lim_{k\to\infty} d(x_{n_k}, y) \leq \limsup_{n\to\infty} d(x_n, y) \leq r_1.$$

Hence, $\delta_{p^*}(K_2) \leq r_1 < \delta(K_1, K_2)$ which is a contradiction by the fact that p^* is a diametral point with respect to K_2. By the similar way, we can see that if $r_2 < \delta(K_1, K_2)$, then we get a contradiction. That is, (3.2) holds.

Corollary 3.8. *Under the conditions of Theorem 3.7 if, in addition, the sequence* $\{x_n\}$ *is converges to* $p^* \in A$ *then* T *has a best proximity pair.*

Proof. By Theorem 3.7, if $d(p^*, q^*) = d(A, B)$ for some $q^* \in K_2$, we have

$$dist(K_1, K_2) = dist(A, B) = d(p^*, q^*) = \limsup_{n\to\infty} d(x_n, q^*) = \delta(K_1, K_2).$$

Now, by the fact that the convex metric space X has the property (D) we conclude that K_1 and K_2 are singleton and the result follows.

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