MINIMAL SETS OF NONCYCLIC RELATIVELY
NONEXPANSIVE MAPPINGS IN CONVEX METRIC SPACES

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Abstract. Let us consider a mapping \( T : A \cup B \to A \cup B \) such that \( T(A) \subseteq A \) and \( T(B) \subseteq B \), where \( A \) and \( B \) are two nonempty subsets of a metric space \((X, d)\). We provide sufficient conditions for the existence of a point \((p, q) \in A \times B\), called best proximity pair, which satisfies \( Tp = p, Tq = q \) and \( d(p, q) = \text{dist}(A, B) := \inf \{d(x, y) : (x, y) \in A \times B\} \), in the setting of convex metric spaces for noncyclic contractions. Then, we present a similar result of Goebel-Karlovitz lemma for noncyclic relatively nonexpansive mappings in convex metric spaces.

Key Words and Phrases: Best proximity pair, noncyclic contraction, relatively nonexpansive mapping, convex metric space, Goebel-Karlovitz lemma.

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1. Introduction and Preliminaries

Let \((X, d)\) be a metric space and let \( A \) and \( B \) be two nonempty subsets of \( X \). A mapping \( T : A \cup B \to A \cup B \) is said to be a noncyclic mapping provided that \( T(A) \subseteq A \) and \( T(B) \subseteq B \). A point \((p, q) \in A \times B\) is said to be a best proximity pair for noncyclic mapping \( T \), provided that

\[ Tp = p, \quad Tq = q \quad \text{and} \quad d(p, q) = \text{dist}(A, B). \]

Let \((A, B)\) be a nonempty pair of subsets of a metric space \((X, d)\). A mapping \( T : A \cup B \to A \cup B \) is said to be a noncyclic relatively nonexpansive if \( T \) is noncyclic and \( d(Tx, Ty) \leq d(x, y) \) for all \((x, y) \in A \times B\).

In [3], Eldred et al. studied the existence of best proximity pairs for noncyclic relatively non-expansive mappings in Banach spaces with a geometric property, called proximal normal structure. They proved the following theorem.

**Theorem 1.1.** (Theorem 2.2 of [3]) Let \((A, B)\) be a nonempty, weakly compact and convex pair in a strictly convex Banach space \(X\). Let \( T : A \cup B \to A \cup B \) be a noncyclic relatively nonexpansive mapping. If the pair \((A, B)\) has proximal normal structure. The \( T \) has a best proximity pair.

Also, in [2], the authors investigated sufficient conditions which ensure the existence of a best proximity pair for noncyclic mappings.

The notion of convexity in metric spaces was introduced by Takahashi as follows.
Definition 1.2. ([7]) Let \((X, d)\) be a metric space and \(I := [0, 1]\). A mapping \(W : X \times X \times I \to X\) is said to be a convex structure on \(X\) provided that for each \((x, y; \lambda) \in X \times X \times I\) and \(u \in X\),
\[
d(u, W(x, y; \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y).
\]

A metric space \((X, d)\) together with a convex structure \(W\) is called a convex metric space, which is denoted by \((X, d, W)\). A Banach space and each of its convex subsets are convex metric spaces. But a Frechet space is not necessary a convex metric space. The other examples of convex metric spaces which are not imbedded in any Banach space can be founded in [7].

Here, we recall some notations and definitions of [1, 7].

Definition 1.3. ([7]) A subset \(K\) of a convex metric space \((X, d, W)\) is said to be a convex set provided that \(W(x, y; \lambda) \in K\) for all \(x, y \in K\) and \(\lambda \in I\).

Proposition 1.4. ([7]) Let \((X, d, W)\) be a convex metric space and let \(B(x; r)\) denote the closed ball centered at \(x \in X\) with radius \(r \geq 0\). Then \(B(x; r)\) is a convex subset of \(X\).

Proposition 1.5. ([7]) Let \(\{K_\alpha\}_{\alpha \in A}\) be a family of convex subsets of \(X\), then \(\bigcap_{\alpha \in A} K_\alpha\) is also a convex subset of \(X\).

Definition 1.6. ([7]) A convex metric space \((X, d, W)\) is said to have property (C) if every bounded decreasing net of nonempty closed convex subsets of \(X\) has a nonempty intersection.

For example, every bounded, closed and convex subset of a reflexive Banach space \(X\) has property (C).

Let \(A\) and \(B\) be two nonempty subsets of a convex metric space \((X, d, W)\). We shall say that a pair \((A, B)\) in a convex metric space \((X, d, W)\) satisfies a property if both \(A\) and \(B\) satisfy that property. For instance, \((A, B)\) is convex if and only if both \(A\) and \(B\) are convex; \((A, B) \subseteq (C, D) \Leftrightarrow A \subseteq C, \text{ and } B \subseteq D\). We shall also adopt the following notations.

\[
\delta_x(A) := \sup\{d(x, y) : y \in A\} \text{ for all } x \in X,
\]
\[
\delta(A, B) := \sup\{d(x, y) : x \in A, y \in B\},
\]
\[
\text{diam}(A) := \delta(A, A).
\]

The closed and convex hull of a set \(A\) will be denoted by \(\overline{\text{conv}}(A)\) and defined as below.

\[
\overline{\text{conv}}(A) := \bigcap\{C : C \text{ is a closed and convex subset of } X \text{ such that } C \supseteq A\}.
\]

The pair \((x, y) \in A \times B\) is said to be proximal in \((A, B)\) if \(d(x, y) = \text{dist}(A, B)\). Moreover, we set
\[
A_0 := \{x \in A : d(x, y') = \text{dist}(A, B), \text{ for some } y' \in B\},
\]
\[
B_0 := \{y \in B : d(x', y) = \text{dist}(A, B), \text{ for some } x' \in A\}.
\]

Note that if \((A, B)\) is a nonempty weakly compact and convex pair of subsets of a Banach space \(X\), then also is the pair \((A_0, B_0)\) and it is easy to see that \(\text{dist}(A, B) = \text{dist}(A_0, B_0)\).

Definition 1.7. A pair of sets \((A, B)\) is said to be proximal if \(A = A_0\) and \(B = B_0\).

The following result follows from the proof of Theorems 2.2 in [3].
Lemma 1.8. Let \((A, B)\) be a nonempty weakly compact convex pair of a Banach space \(X\) and \(T : A \cup B \to A \cup B\) a noncyclic relatively nonexpansive mapping. Then there exists \((K_1, K_2) \subseteq (A_0, B_0) \subseteq (A, B)\) which is minimal with respect to being nonempty closed convex and \(T\)-invariant pair of subsets of \((A, B)\) such that
\[
\text{dist}(K_1, K_2) = \text{dist}(A, B).
\]
Moreover, the pair \((K_1, K_2)\) is proximal.

In this article, we study sufficient conditions which ensure the existence of best proximity pairs for noncyclic contractions in convex metric spaces. In this way, we obtain a similar result of Goebel-Karlovitz lemma [4, 5] which is a key lemma in fixed point theory.

Lemma 1.9. (Goebel-Karlovitz lemma [4, 5]) Let \(A\) be a nonempty, weakly compact, convex subset of a Banach space \(X\) and let \(T : A \to A\) be a nonexpansive mapping. Assume that \(K\) is a subset of \(A\) which is minimal with respect to being nonempty, weakly compact, convex and \(T\)-invariant, and suppose \(\{x_n\}\) is a sequence in \(K\) such that
\[
\lim_{n \to \infty} \|x_n - Tx_n\| = 0.
\]
Then, for each \(x \in K\), \(\lim_{n \to \infty} \|x - x_n\| = \text{diam}(K)\).

2. Existence of best proximity pairs for noncyclic contractions

In this section, we prove a best proximity pair theorem for noncyclic contractions in convex metric spaces. We begin our main result with the following notion.

Definition 2.1. A convex metric space \((X, d, W)\) is said to have property \((D)\) provided that for each \(x_1, x_2, y\) in \(X\) with \(x_1 \neq x_2\) we have
\[
d(W(x_1, x_2, \frac{1}{2}), y) < \frac{1}{2}[d(x_1, y) + d(x_2, y)].
\]

It is clear that every strictly convex Banach space is a convex metric space which satisfies the property \((D)\).

We now state the main result of this section.

Theorem 2.2. Let \((A, B)\) be a nonempty, bounded, closed and convex pair in a convex metric space \((X, d, W)\). Suppose that \(T : A \cup B \to A \cup B\) is a noncyclic contraction, that is, there exists \(\alpha \in (0, 1)\) such that
\[
d(Tx, Ty) \leq \alpha d(x, y) + (1 - \alpha) \text{dist}(A, B),
\]
for each \((x, y)\) in \(A \times B\). If \(X\) satisfies the properties \((C)\) and \((D)\) then \(T\) has a best proximity pair.

Proof. Let \(\Sigma\) denote the set of all nonempty, bounded, closed and convex pairs \((E, F)\) which are subsets of \((A, B)\) and such that \(T\) is noncyclic on \(E \cup F\). Note that \((A, B) \in \Sigma\). Also, \(\Sigma\) is partially ordered by reverse inclusion, that is \((E_1, E_2) \subseteq (F_1, F_2) \Rightarrow (F_1, F_2) \subseteq (E_1, E_2))\). Since \(X\) has the property \((C)\), every increasing chain in \(\Sigma\) is bounded above. By using Zorn’s lemma, we obtain a minimal element say \((E, F)\) in \(\Sigma\). Note that \((\overline{\text{conv}}(T(E)), \overline{\text{conv}}(T(F)))\) is a nonempty, bounded, closed and convex subset of \((E, F)\). By the fact that \(T\) is noncyclic,
\[
T(\overline{\text{conv}}(T(E))) \subseteq T(E) \subseteq \overline{\text{conv}}(T(E)),
\]
and also,
\[ T(\overline{\text{con}}(T(F))) \subseteq \overline{\text{con}}(T(F)). \]

So, \( T \) is noncyclic on \( \overline{\text{con}}(T(E)) \cup \overline{\text{con}}(T(F)) \). The minimality of \( (E, F) \) implies that \( \overline{\text{con}}(T(E)) = E \), \( \overline{\text{con}}(T(F)) = F \).

Let \( x \in E \), then \( F \subseteq B(x; \delta_x(F)) \). Now, if \( y \in F \) we have
\[
d(Tx, Ty) \leq \alpha d(x, y) + (1 - \alpha) \text{dist}(A, B)
\]
\[
\leq \alpha \delta(E, F) + (1 - \alpha) \text{dist}(A, B).
\]

Thus, for all \( y \in F \) we have
\[
T(y) \in B(Tx; \alpha \delta(E, F) + (1 - \alpha) \text{dist}(A, B)),
\]
and so,
\[
T(F) \subseteq B(Tx; \alpha \delta(E, F) + (1 - \alpha) \text{dist}(A, B)).
\]

Then,
\[
F = \overline{\text{con}}(T(F)) \subseteq B(Tx; \alpha \delta(E, F) + (1 - \alpha) \text{dist}(A, B)),
\]
which implies that
\[
d(z, Tx) \leq \alpha \delta(E, F) + (1 - \alpha) \text{dist}(A, B), \quad \forall z \in F.
\]

Hence,
\[
\delta_{Tx}(F) \leq \alpha \delta(E, F) + (1 - \alpha) \text{dist}(A, B). \quad (2.2)
\]

Similarly, if \( y \in F \) we obtain
\[
\delta_{Ty}(E) \leq \alpha \delta(E, F) + (1 - \alpha) \text{dist}(A, B). \quad (2.3)
\]

Put,
\[
E' := \{ x \in E : \delta_x(F) \leq \alpha \delta(E, F) + (1 - \alpha) \text{dist}(A, B) \},
\]
\[
F' := \{ y \in F : \delta_y(E) \leq \alpha \delta(E, F) + (1 - \alpha) \text{dist}(A, B) \}.
\]

We now have \( T(E) \subseteq E' \) and \( T(F) \subseteq F' \) and it is easy to see that
\[
E' = \bigcap_{y \in F} B(y; \alpha \delta(E, F) + (1 - \alpha) \text{dist}(A, B)) \cap E,
\]
\[
F' = \bigcap_{x \in E} B(x; \alpha \delta(E, F) + (1 - \alpha) \text{dist}(A, B)) \cap F.
\]

We note that by Propositions 1.4 and 1.5 the pair \( (E', F') \) is convex. Moreover, by relations (2.2) and (2.3) we conclude that \( T \) is noncyclic on \( E' \cup F' \). Minimality of \( (E, F) \) guarantees that \( E' = E \) and \( F' = F \). Therefore,
\[
\delta_x(F) \leq \alpha \delta(E, F) + (1 - \alpha) \text{dist}(A, B), \quad \forall x \in E.
\]

Thus,
\[
\delta(E, F) = \text{dist}(A, B). \quad (2.4)
\]

Let \( (p, q) \in E \times F \). It now follows from (2.4) that \( d(p, q) = \text{dist}(A, B) \). We claim that \( E \) and \( F \) are singleton. Assume that \( p \neq p' \in E \) and \( q \in F \). Since \( E \) is a convex
set, \((W(p, p', \frac{1}{2}), W(q, q', \frac{1}{2})) \in E \times F\). Now, by the fact that \(X\) satisfies the property (D), we deduce that
\[
dist(A, B) \leq d(W(p, p', \frac{1}{2}), q) < \frac{1}{2}[d(p, q) + d(p', q)] \leq \delta(E, F) = dist(A, B),
\]
which is a contradiction. Hence, \(E\) and \(F\) are singleton. This completes the proof of theorem.

**Remark 2.3.** Theorem 2.2 holds once the minimal sets \(E\) and \(F\) have been fixed and the noncyclic mapping \(T : A \cup B \to A \cup B\), satisfies the condition that there exists \(\alpha \in [0, 1)\) such that
\[
d(Tx, Ty) \leq \alpha \delta(E, F) + (1 - \alpha)dist(A, B), \tag{2.5}
\]
for all \((x, y) \in A \times B\).

The next result obtains from Theorem 2.2, directly.

**Corollary 2.4.** Let \((A, B)\) be a nonempty, bounded, closed and convex pair in a reflexive and strictly convex Banach space \(X\). Suppose that \(T : A \cup B \to A \cup B\) is a noncyclic contraction. Then \(T\) has a best proximity pair.

3. **Goebel-Karlovitz lemma for noncyclic relatively nonexpansive mappings**

The purpose of this section is to give a similar result of Goebel-Karlovitz lemma for noncyclic relatively nonexpansive mappings in convex metric spaces. We start our results of this section by the next definitions.

**Definition 3.1.** Let \((A, B)\) be a nonempty pair of subsets of a metric space \((X, d)\). We say that the pair \((A, B)\) is a proximal compactness pair provided that every net \(\{(x_{\alpha}, y_{\alpha})\}\) of \(A \times B\) satisfying the condition that \(d(x_{\alpha}, y_{\alpha}) \to dist(A, B)\), has a convergent subnet in \(A \times B\). Also, we say that \(A\) is semi-compactness if \((A, A)\) is proximal compactness.

It is clear that if \((A, B)\) is a compact pair in a metric space \((X, d)\) then \((A, B)\) is proximal compactness.

**Definition 3.2.** Let \((A, B)\) be a nonempty pair of sets in a Banach space \(X\). A point \(p\) in \(A\) (\(q\) in \(B\)) is said to be a diametral point with respect to \(B\) (w.r.t. \(A\)) if \(\delta_p(B) = \delta(A, B)\) (\(\delta_q(A) = \delta(A, B)\)). A pair \((p, q)\) in \(A \times B\) is diametral if both points \(p\) and \(q\) are diametral.

The following result is another version of Lemma 1.8 for noncyclic mappings in convex metric spaces.

**Lemma 3.3.** Let \((A, B)\) be a nonempty, bounded, closed and convex pair of a convex metric space \((X, d, W)\) such that \(A_0\) is nonempty and \((A, B)\) is a proximal compactness pair. Assume that \(T : A \cup B \to A \cup B\) is a noncyclic relatively nonexpansive mapping. If \(X\) has the property (C) then there exists a pair \((K_1, K_2) \subseteq (A, B)\) which is minimal with respect to being nonempty, closed, convex and \(T\)-invariant pair of subsets of \((A, B)\) such that
\[
dist(K_1, K_2) = dist(A, B).
\]
Proof. Let $\Sigma$ denote the set of all nonempty, closed and convex pairs $(E, F)$ which are subsets of $(A, B)$ and such that $T$ is noncyclic on $E \cup F$ and $d(x, y) = \text{dist}(A, B)$ for some $(x, y) \in E \times F$. Since $A_0$ is nonempty set, $(A, B) \in \Sigma$. Moreover, $\Sigma$ is partially ordered by reverse inclusion. Suppose $\{(E_\alpha, F_\alpha)\}_{\alpha}$ is a descending chain in $\Sigma$. Put $E := \bigcap E_\alpha$ and $F := \bigcap F_\alpha$. By the fact that $X$ has the property (C), we conclude that $(E, F)$ is a nonempty pair. By Proposition 1.5, $(E, F)$ is a convex pair. Also,

$$T(E) = T(\bigcap E_\alpha) \subseteq \bigcap T(E_\alpha) \subseteq E = E.$$  

Similarly, $T(F) \subseteq F$, that is, $T$ is noncyclic on $E \cup F$. Now, let $(x_\alpha, y_\alpha) \in E_\alpha \times F_\alpha$ be such that $d(x_\alpha, y_\alpha) = \text{dist}(A, B)$. Since $(A, B)$ is proximal compactness, $(x_\alpha, y_\alpha)$ has a convergent subsequence say $(x_\alpha, y_\alpha)$ such that $x_\alpha \to x \in A$ and $y_\alpha \to y \in B$. Thus,

$$d(x, y) = \lim_i d(x_\alpha, y_\alpha) = \text{dist}(A, B).$$

Therefore, there exists an element $(x, y) \in E \times F$ such that $d(x, y) = \text{dist}(A, B)$. So, every increasing chain in $\Sigma$ is bounded above with respect to reverse inclusion relation. Thus, by using Zorn’s Lemma we can get an element say $(K_1, K_2)$ which is minimal with respect to being nonempty, closed, convex and $T$-invariant pair of subsets of $(A, B)$ such that

$$\text{dist}(K_1, K_2) = \text{dist}(A, B).$$

**Lemma 3.4.** Let $(A, B)$ be a nonempty, bounded, closed and convex pair of a convex metric space $(X, d, W)$ such that $A_0$ is nonempty. Suppose that $X$ has the property (C) and $(A, B)$ is a proximal compactness pair. Let $T : A \cup B \to A \cup B$ be a noncyclic relatively nonexpansive mapping. Suppose that $(K_1, K_2) \subseteq (A, B)$ is a minimal closed convex pair which is $T$-invariant and such that $\text{dist}(K_1, K_2) = \text{dist}(A, B)$. Then each pair $(p, q) \in K_1 \times K_2$ with $d(p, q) = \text{dist}(A, B)$ is a diametral pair (with respect to $(K_1, K_2)$), that is,

$$\delta_p(K_2) = \delta_q(K_1) = \delta(K_1, K_2).$$

**Proof.** By the similar argument of Theorem 2.2 we conclude that $T$ is noncyclic on $\overline{\text{conv}}(T(K_1)) \cup \overline{\text{conv}}(T(K_2))$. Let $(p, q) \in K_1 \times K_2$ be such that $d(p, q) = \text{dist}(A, B)$ and suppose

$$\min\{\delta_p(K_2), \delta_q(K_1)\} < \delta(K_1, K_2). \tag{3.1}$$

Since $T$ is noncyclic relatively nonexpansive,

$$\text{dist}(\overline{\text{conv}}(T(K_1)), \overline{\text{conv}}(T(K_2))) = \text{dist}(A, B).$$

Minimality of $(K_1, K_2)$ concludes that

$$\overline{\text{conv}}(T(K_1)) = K_1, \quad \overline{\text{conv}}(T(K_2)) = K_2.$$  

Put, $r_1 := \delta_p(K_2)$ and $r_2 := \delta_q(K_1)$. So, $\min\{r_1, r_2\} < \delta(K_1, K_2)$. Let

$$K_1^* := K_1 \bigcap (\bigcap_{x \in K_1} B(x, r_1)), \quad K_2^* := K_2 \bigcap (\bigcap_{x \in K_1} B(x, r_2)).$$

Then $(K_1^*, K_2^*)$ is a nonempty, closed and convex pair in $X$ by Propositions 1.4 and 1.5. Also, since $(p, q) \in (K_1^*, K_2^*)$,

$$\text{dist}(K_1^*, K_2^*) = \text{dist}(A, B).$$
It is easy to see that for \((x, y) \in K_1 \times K_2\),
\[(x, y) \in (K_1^+, K_2^+) \iff K_2 \subseteq B(x; r_1), \quad K_1 \subseteq B(y; r_2).\]

We now claim that \(T\) is noncyclic on \(K_1^+ \cup K_2^+\). Let \(x \in K_1^+\). We must show that \(Tx \in K_1^+\), or equivalently, \(K_2 \subseteq B(Tx; r_1)\). For \(y \in K_2\), the relatively nonexpansiveness of \(T\) deduces that
\[d(Tx, Ty) \leq d(x, y) \leq r_1,\]
then \(Ty \in B(Tx; r_1)\) which implies that \(T(K_2) \subseteq B(Tx; r_1)\). Therefore, \(K_2 = \text{comm}(T(K_2)) \subseteq B(Tx; r_1)\) and hence, \(Tx \in K_1^+\). Thus, \(T(K_1^+) \subseteq K_1^+\). Similarly, we can see that \(T(K_2^+) \subseteq K_2^+\). It now follows from the minimality of \((K_1, K_2)\) that \(K_1^+ = K_1\) and \(K_2^+ = K_2\). Then, \(K_1 \subseteq \bigcap_{x \in K_2} B(x; r_1)\) and so, for each \(x \in K_1\), \(\delta_x(K_2) \leq r_1\). We now conclude that
\[\delta(K_1, K_2) = \sup_{x \in K_1} \delta_x(K_2) \leq r_1.\]

Similarly, we can see that \(\delta(K_1, K_2) \leq r_2\), which is a contradiction with the relation (3.1) and this completes the proof.

Here, we introduce the notion of proximal approximate fixed point sequence for noncyclic mappings as follows.

**Definition 3.5.** Let \((A, B)\) be a nonempty pair of subsets of a metric space \((X, d)\). Suppose that \(T : A \cup B \to A \cup B\) is a noncyclic mapping. Then a sequence \(\{(x_n, y_n)\}\) in \(A \times B\) is said to be a proximal approximate fixed point sequence for \(T\) if
\[d(x_n, Tx_n) \to 0, \quad d(y_n, Ty_n) \to 0 \quad \text{and} \quad d(x_n, y_n) \to \text{dist}(A, B).\]

The following lemma guarantees the existence of a proximal approximate fixed point sequence for noncyclic relatively nonexpansive mappings.

**Lemma 3.6.** Let \((A, B)\) be a nonempty, bounded, closed and convex pair of a convex metric space \((X, d, W)\) such that \(A_0\) is nonempty, \(X\) has the properties (C) and (D) and \((A, B)\) is a proximal compactness pair. Let \(T : A \cup B \to A \cup B\) be a noncyclic relatively nonexpansive mapping. Then there exists a proximal approximate fixed point sequence for \(T\).

**Proof.** By using Lemma 3.3, there exists a pair \((K_1, K_2) \subseteq (A, B)\) which is minimal with respect to being nonempty, closed, convex and \(T\)-invariant pair of subsets of \((A, B)\) and there exists \((p, q) \in K_1 \times K_2\) such that
\[\text{dist}(K_1, K_2) = d(p, q) = \text{dist}(A, B).\]

For any \(\alpha \in (0, 1)\) let \(r := 2\alpha - \alpha^2\). Then \(r < 1\). Define the mapping \(T_\alpha : A \cup B \to A \cup B\) with
\[T_\alpha(x) = \begin{cases} W(Tx, p, \alpha); & x \in A, \\ W(Tx, q, \alpha); & x \in B. \end{cases}\]

Since \(T\) is noncyclic and \((A, B)\) is a convex pair in convex metric space \((X, d, W)\), we conclude that \(T_\alpha\) is noncyclic on \(A \times B\). Now, for each \((x, y) \in A \times B\) we have
\[d(T_\alpha x, T_\alpha y) = d(W(Tx, p, \alpha), W(Ty, q, \alpha)) \leq \alpha d(W(Tx, p, \alpha), Ty) + (1 - \alpha) d(W(Tx, p, \alpha), q)\]
It now follows from Remark 2.3 that for each \( q \in K \)
we now have \( p \rightarrow q^* \) for some \( (p^*, q^*) \in K \times K \).

Similarly, we can see that \( d(p, y) \rightarrow 0 \).
Therefore, there exists a sequence \( (\{x_n\}, \{y_n\}) \in A \times B \) such that
\begin{align*}
d(x_n, x_m) &\rightarrow 0, \\
d(y_n, y_m) &\rightarrow 0 \\
d(x_n, y_n) &\rightarrow \text{dist}(A, B).
\end{align*}

The next result is a new version of Goebel-Karlovitz lemma for noncyclic mappings
in convex metric spaces.

**Theorem 3.7.** Let \((A, B)\) be a nonempty, bounded, closed and convex pair of a convex metric space \((X, d, W)\) such that \( X \) has the properties (C) and (D). Assume that \( A_0 \) is nonempty and \((A, B)\) is a proximal compactness pair. Let \( T : A \cup B \rightarrow A \cup B \) be a noncyclic relatively nonexpansive mapping. Suppose \((K_1, K_2) \subseteq (A, B)\) is a minimal closed and convex pair which is \( T \)-invariant and such that \( \text{dist}(K_1, K_2) = \text{dist}(A, B) \)
and let \( \{x_n\}, \{y_n\} \in A \times B \) be a proximal approximate sequence in \( A \times B \). Then for each \((p, q) \in K_1 \times K_2 \) with \( d(p, q) = \text{dist}(A, B) \) we have
\[
\limsup_{n \to \infty} d(x_n, q) = \limsup_{n \to \infty} d(p, y_n) = \delta(K_1, K_2).
\]

**Proof.** The existence of the proximal approximate fixed point sequence for \( T \) obtains from Lemma 3.6. By this reality that \((A, B)\) is proximal compactness, there exists a subsequence \( \{x_{n_k}\}, \{y_{n_k}\} \) of the sequence \( \{x_n\}, \{y_n\} \) such that \( x_{n_k} \rightarrow p^* \) and \( y_{n_k} \rightarrow q^* \) for some \((p^*, q^*) \in K_1 \times K_2 \). Hence,
\[
d(p^*, q^*) = \lim_{k \to \infty} d(x_{n_k}, y_{n_k}) = \text{dist}(A, B).
\]
We claim that
\[ r_1 = r_2 = \delta(K_1, K_2). \] (3.2)
Suppose that \( r_1 < \delta(K_1, K_2) \). Set,
\[ K^*_1 := \{ x \in K_1 : \limsup_{n \to \infty} d(x, y_n) \leq r_1 \}, \quad K^*_2 := \{ y \in K_2 : \limsup_{n \to \infty} d(x_n, y) \leq r_2 \}. \]
Note that \((p, q) \in K^*_1 \times K^*_2\) and \((K^*_1, K^*_2)\) is a closed pair in \(X\). Moreover, \((K^*_1, K^*_2)\) is a convex pair in \(X\). In fact, if \(x_1, x_2 \in K^*_1\), then
\[ \limsup_{n \to \infty} d(W(x_1, x_2, \alpha), y_n) \leq \limsup_{n \to \infty} [\alpha d(x_1, y_n) + (1 - \alpha) d(x_2, y_n)] \leq r_1. \]
Thus, \(W(x_1, x_2, \alpha) \in K^*_1\), that is, \(K^*_1\) is convex. Similarly, we can see that \(K^*_2\) is convex. Further, \(T(K^*_1) \subseteq K^*_1\). Indeed, if \(x \in K^*_1\), then
\[ \limsup_{n \to \infty} d(Tx, y_n) \leq \limsup_{n \to \infty} [d(Tx, Ty_n) + d(Ty_n, y_n)] \leq \limsup_{n \to \infty} d(x, y_n) \leq r_1, \]
which concludes that \(Tx \in K^*_1\). Similarly, we can see that \(T(K^*_2) \subseteq K^*_2\). Therefore, \(T\) is noncyclic on \(K^*_1 \cup K^*_2\). It now follows from the minimality of \((K_1, K_2)\) that \((K_1, K_2) = (K^*_1, K^*_2)\). Then for each \(y \in K_2\) we have
\[ d(p^*, y) = \lim_{k \to \infty} d(x_n, y) \leq \limsup_{n \to \infty} d(x_n, y) \leq r_1. \]
Hence, \(\delta(p^*, K_2) \leq r_1 < \delta(K_1, K_2)\) which is a contradiction by the fact that \(p^*\) is a diametral point with respect to \(K_2\). By the similar way, we can see that if \(r_2 < \delta(K_1, K_2)\), then we get a contradiction. That is, (3.2) holds.

**Corollary 3.8.** Under the conditions of Theorem 3.7 if, in addition, the sequence \(\{x_n\}\) is converges to \(p^* \in A\) then \(T\) has a best proximity pair.

**Proof.** By Theorem 3.7, if \(d(p^*, q^*) = d(A, B)\) for some \(q^* \in K_2\), we have
\[ \text{dist}(K_1, K_2) = \text{dist}(A, B) = d(p^*, q^*) = \limsup_{n \to \infty} d(x_n, q^*) = \delta(K_1, K_2). \]
Now, by the fact that the convex metric space \(X\) has the property (D) we conclude that \(K_1\) and \(K_2\) are singleton and the result follows.

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**References**


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