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FIXED POINT THEORY FOR SUZUKI TYPE (θ, L)-WEAK MULTIVALUED OPERATORS

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Abstract. Existence of a fixed point of Suzuki type (θ, L)- weak multivalued operator is obtained. As an application, we obtain homotopy and data dependence results for Suzuki type contractive multivalued operator. Our results complement and extend some very recent comparable results in the existing literature.
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1. Introduction and Preliminaries

Let (X, d) be a metric space, 2^X , $P(X)$, $P_b(X)$, $P_{cl}(X)$ and $P_{b,cl}(X)$ be the collections of all subsets of X, all nonempty subsets of X, all nonempty bounded subsets of X, all nonempty closed subsets of X and all nonempty closed bounded subsets of X, respectively. Let Y be nonempty set and $T : X \to P(Y)$ be a multivalued operator. A subset $G(T) = \{(x, y) : y \in Tx\}$ of $X \times Y$ is called a graph of $T : X \to P(Y)$. A multivalued operator $T : X \to P(Y)$ is said to be closed if $G(T)$ is a closed set in $X \times Y$. A point x in X is called a fixed point of $T : X \to P(Y)$ if $x \in Tx$. A set of all fixed points of $T : X \to P(Y)$ is denoted by $Fix(T)$.

We will denote by D, ρ and H the gap, the excess and respectively the Pompeiu-Hausdorff functional induced by the metric d (see [11], [12] for details). It is known that if (X, d) is complete metric space, then $(P_{b,cl}(X), H)$ is a complete metric space. Also H is a generalized metric on $P_{cl}(X)$.

The following result is well known.

Lemma 1.1. *Let $A, B \subset X$ and $q > 1$. Then, for every $a \in A$, there exists $b \in B$ such that $d(a, b) \leq qH(A, B)$.*

Definition 1.2. A multivalued mapping $T : X \to P(X)$ is a multivalued weakly Picard (briefly MWP) operator if for each $x \in X$ and each $y \in T(x)$, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that

(i) $x_0 = x$, $x_1 = y$,

(ii) $x_{n+1} \in T(x_n)$ for all $n \in \mathbb{N}$,
(iii) the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent and its limit is a fixed point of T.

A sequence in above definition is called a sequence of successive approximations of T starting from (x, y).

Definition 1.3. Let $T : X \to P(X)$ be a MWP operator. Define $T^{\infty} : G(T) \to P(Fix(T))$ as follows: $T^{\infty}(x, y) = \{z \in Fix(T) :$ there exists a sequence of successive approximations of T starting from (x, y) that converges to $z\}$.

Definition 1.4. A multivalued operator $T : X \to P(X)$ is called a c-multivalued weakly Picard (c-MWP) operator if $c > 0$ and there exists a selection t^{∞} of T^{∞} such that $d(x, t^{\infty}(x, y)) \leq cd(x, y),$ for all $(x, y) \in G(T)$.

The study of fixed points for multivalued contractions mappings using the Hausdorff metric was initiated by Nadler [12] and Markin [9]. Later, an interesting and rich fixed point theory for such maps was developed which has found applications in control theory, convex optimization, differential inclusion and economics. Klim and Wardowski [6] obtained existence of fixed point for set-valued contractions in complete metric spaces. Fixed points of multivalued operators also play an important role in the theory of set-valued dynamic systems. Following Aubin and Siegel ([1]), a set-valued dynamic is a pair (X, T), where system X is a metric space and $T : X \to P(X)$ is a multivalued operator. Any sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_{n+1} \in Tx_n$ for each $n \in \mathbb{N}$ is called a motion of the system T at x_0 or a dynamic process of T starting from x_0 . A fixed point of a multivalued map T may be interpreted as rest-point of the dynamic system while a strict fixed point for T can be regarded as an end-point of the system. The study set-valued dynamic systems has received more attention in the last two decades, see for example, [1], [10], [14], and references mentioned therein.

On the other hand, concept of completeness of metric spaces has interesting and important applications in classical analysis (see for example, [7], [8]) . Suzuki [13] obtained a variant of Banach's fixed point theorem that characterizes metric completeness by using different types of contractions. Subsequently, many researchers in metric fixed point theory obtained different generalizations of this result ([5], [13] and references therein).

Motivated by the work in [13] and [2], we introduce Suzuki type multivalued operator and obtain some fixed point results for such mappings. Results obtained in this paper extend or generalize many comparable results in contemporary literature. Our results can also be viewed as Suzuki type extension of Kannan, Chatterjea, Zamfirescu, and Ciric type contraction for multivalued operators (see for example, [3], [11], [4] and some references therein).

2. Main Results

2.1. **Suzuki type (θ, L)- weak multivalued operator.** We introduce Suzuki type (θ, L)-weak multivalued operator.

Definition 2.2. Let $\mu : [0, 1) \to (\frac{1}{2}, 1]$ be a function defined by $\mu(a) = \dfrac{1}{1 + a}$.
A multivalued mapping $T : X \to P_{b,cl}(X)$ is called Suzuki type (θ, L)-weak multivalued operator if $a \in [0, 1)$, there exists two constants $\theta \in (0, 1)$ and $L \geq 0$ such that

for $x, y \in X$

$$\mu(a)D(x,Tx) \leq d(x,y) \Rightarrow H(Tx,Ty) \leq \theta d(x,y) + LD(y,Tx). \tag{2.1}$$

Due to symmetric of d and H, one obtains the dual of (2.1) as follows: for $x, y \in X$

$$\mu(a)D(x,Tx) \leq d(x,y) \Rightarrow H(Tx,Ty) \leq \theta d(x,y) + LD(x,Ty). \tag{2.2}$$

First we show that any Suzuki type (θ, L)-weak multivalued operator is a MWP operator.

Theorem 2.3. *Let $\mu : [0,1) \to (\frac{1}{2}, 1]$ be a function defined by $\mu(a) = \dfrac{1}{1+a}$. Let (X,d) be a complete metric space and $T : X \to P_{b,cl}(X)$ Suzuki type (θ, L)-weak multivalued operator. Then there exists $z \in X$ such that $z \in Tz$.*

Proof. Let θ_1 be a real number such that $0 \leq \theta < \theta_1 < 1$ and $u_0 \in X$. Choose $u_1 \in Tu_0$. Clearly, if $u_1 = u_0$, then proof is finished. Suppose that $u_1 \neq u_0$. If we take $h = \dfrac{1}{\sqrt{\theta_1}}$, then there exists $u_2 \in Tu_1$ such that $d(u_1,u_2) \leq \dfrac{1}{\sqrt{\theta_1}} H(Tu_0, Tu_1)$. Again if $u_2 = u_1$, the proof is finished and so we assume $u_2 \neq u_1$. Since $\mu(a) \leq 1$, we have

$$(1+a)^{-1}D(u_0,Tu_0) \leq (1+a)^{-1}d(u_0,u_1) \leq d(u_0,u_1).$$

Hence

$$d(u_1,u_2) \leq \frac{1}{\sqrt{\theta_1}}H(Tu_0,Tu_1) \leq \sqrt{\theta_1}d(u_0,u_1) + \frac{L}{\sqrt{\theta_1}}LD(u_1,Tu_0) = \sqrt{\theta_1}d(u_0,u_1).$$

That is, there exists $u_2 \in Tu_1$ such that

$$d(u_1,u_2) \leq \sqrt{\theta_1}d(u_0,u_1). \tag{2.3}$$

Thus, we have a sequence $\{u_n\}$ in X such that $u_n \in Tu_{n-1}$ and

$$d(u_n,u_{n+1}) \leq \sqrt{\theta_1}d(u_{n-1},u_n) \leq (\sqrt{\theta_1})^2 d(u_{n-2},u_{n-1}) \leq ... \leq (\sqrt{\theta_1})^n d(u_0,u_1).$$

As

$$\sum_{n=1}^{\infty} d(u_n,u_{n+1}) \leq \sum_{n=1}^{\infty} (\sqrt{\theta_1})^n d(u_0,u_1) < \infty,$$

so $\{u_n\}$ is a Cauchy sequence. Since X is complete, there exists $z \in X$ such that $u_n \to z$. Next we show that

$$D(z,Tx) \leq \theta d(z,x) \text{ for all } x \in X - \{z\}. \tag{2.4}$$

Since $u_n \to z$, there exists $n_0 \in \mathbb{N}$ such that $d(z,u_n) \leq \frac{1}{3}d(z,x)$ for all $n \in \mathbb{N}$ with $n \geq n_0$. So we have

$$\begin{aligned} \mu(a)D(u_n,Tu_n) &\leq D(u_n,Tu_n) \leq d(u_n,u_{n+1}) \leq \frac{2}{3}d(z,x) \\ &= d(z,x) - \frac{1}{3}d(z,x) \leq d(x,z) - d(u_n,z) = d(u_n,x). \end{aligned}$$

Hence $D(u_{n+1},Tx) \leq H(Tu_n,Tx) \leq \theta d(u_n,x) + LD(x,Tu_n)$. Taking limit as $n \to \infty$, we obtain $D(z,Tx) \leq \theta d(z,x)$ for all $x \in X - \{z\}$. We next show that

$$H(Tx,Tz) \leq \theta d(x,z) + LD(z,Tx) \text{ for all } x \in X. \tag{2.5}$$

If $x = z$ then it holds obviously. Assume that $x \neq z$. Then for every $n \in \mathbb{N}$, there exists $y_n \in Tx$ such that $d(z, y_n) \leq D(z, Tx) + \dfrac{1}{n} d(x, z)$. Note that, for $n \in \mathbb{N}$, we have

$$\begin{aligned} D(x, Tx) &\leq d(x, y_n) \leq d(x, z) + d(z, y_n) \leq d(x, z) + D(z, Tx) + \frac{1}{n} d(x, z) \\ &\leq d(x, z) + a d(z, x) + \frac{1}{n} d(x, z) \leq \left(1 + a + \frac{1}{n}\right) d(x, z). \end{aligned}$$

Hence we obtain $\frac{1}{1+a} D(x, Tx) \leq d(x, z)$. Therefore the claim follows. Finally, we have

$$\begin{aligned} D(z, Tz) &\leq d(z, u_{n+1}) + D(u_{n+1}, Tz) \leq d(z, u_{n+1}) + H(Tu_n, Tz) \\ &\leq d(z, u_{n+1}) + \theta d(u_n, z) + L D(z, u_{n+1}) \end{aligned} \tag{2.6}$$

Taking limit as $n \to \infty$ in (2.6) and employing the fact that $u_{n+1} \in Tu_n$, we have $D(z, u_{n+1}) \to 0$. Therefore $D(z, Tz) = 0$ implies that $z \in Tz$ as Tz is closed.

Example 2.4. Let $[0, 1]$ be the unit interval with the usual norm and let $T : [0, 1] \to CB([0, 1])$ be Suzuki type (θ, L)-weak multivalued operator given by

$$Tx = \begin{cases} \{\frac{1}{2}\}, & x \in [0, \frac{2}{3}) \\ \{1\}, & x \in [\frac{2}{3}, 1] \end{cases}$$

T does not satisfy neither Ciric's condition (see in [3]), nor Banach, Kannan, Chatterjea and Zamfirescu contractive conditions but T satisfies the contraction condition (2.1). Indeed, contractive condition (3.10), too. Indeed, for $x, y \in [0, \frac{2}{3})$ or $x, y \in [\frac{2}{3}, 1]$, (2.1) is obvious. For $x \in [0, \frac{2}{3})$, $y \in [\frac{2}{3}, 1]$ or $y \in [0, \frac{2}{3})$, $x \in [\frac{2}{3}, 1]$ we have $H(Tx, Ty) = \frac{1}{2}$ and $D(x, Ty) = \frac{1}{6}$, in the first case, and $D(x, Ty) = \frac{1}{3}$, in the second case, which shows that it suffices to take $L = 3$ in order to ensure that (2.1) holds for all $x, y \in [0, 1]$.

Definition 2.5. Let $\mu : [0, 1) \to (\frac{1}{2}, 1]$ be a function defined by $\mu(a) = \dfrac{1}{1 + a}$. Let (X, d) be a complete metric space and $T : X \to X$ Suzuki type (θ, L)-weak contraction if

$$\mu(a) d(x, Tx) \leq d(x, y) \Rightarrow d(Tx, Ty) \leq \theta d(x, y) + L d(y, Tx) \tag{2.7}$$

which is $\theta \in (0, 1)$ and $L \geq 0$. Due to symmetric of d, one obtains the dual of (2.7) as follows: for $x, y \in X$

$$\mu(a) d(x, Tx) \leq d(x, y) \Rightarrow d(Tx, Ty) \leq \theta d(x, y) + L d(x, Ty) \tag{2.8}$$

From above Theorem 2.3, we can have following conclusion.

Corollary 2.6. *Let $\mu : [0, 1) \to (\frac{1}{2}, 1]$ be a function defined by $\mu(a) = \dfrac{1}{1 + a}$. Let (X, d) be a complete metric space and $T : X \to X$ Suzuki type (θ, L)-weak contraction. Then there exists $z \in X$ such that $z \in Tz$.*

Example 2.7. Let $\mu : [0, 1) \to (\frac{1}{2}, 1]$ be a function defined by $\mu(a) = \dfrac{1}{1 + a}$, $[0, 1]$ be the unit interval with the usual norm and let $T : [0, 1] \to [0, 1]$ be the identity map, i.e., $Tx = x$, for all $x \in [0, 1]$. Then T does not satisfy the Ciric's contractive condition (see in [3]), since $M(x, y) = |x - y|$ and $|x - y| > h\,|x - y|$; for all $x \neq y$ and $0 < h < 1$, but T satisfies condition (2.7) with $\theta \in (0, 1)$ arbitrary and $L \geq 1 - \theta$.

Indeed condition (2.7) is equivalent to $|x-y| \leq \theta|x-y| + L|y-x|$, which is true for all $x, y \in [0,1]$.

Theorem 2.8. *Let (X,d) be a complete metric space and $T_i : X \to P_{cl}(X)$ Suzuki-type (θ_i, L_i)- weak multivalued operators, for $i \in \{1,2\}$. If there exists $\lambda > 0$ such that $H(T_1(x), T_2(x)) \leq \lambda$, for all $x \in X$. Then:*

(a) *$Fix(T_i) \in P_{cl}(X)$, $i \in \{1,2\}$;*

(b) *T_1 and T_2 are MWP operators and $H(Fix(T_1), Fix(T_2)) \leq \dfrac{\lambda}{1-\max\{\theta_1,\theta_2\}}$.*

Proof. (a) From Theorem 2.3, $Fix(T_i) \neq \phi$, $i \in \{1,2\}$. Let us prove that the fixed point set of Suzuki type (θ, L)-weak multivalued operator T is closed. let $\{u_n\} \in Fix(T)$, $n \in \mathbb{N}$ such that $u_n \to u^*$ as $n \to \infty$. Since $0 = \mu(a)D(u_n, Tu_n) \leq \mu(a)d(u_n, u^*) \leq d(u_n, u^*)$, we have:

$$\begin{aligned} D(u^*, Tu^*) &\leq d(u^*, u_n) + D(u_n, Tu^*) \leq d(u^*, u_n) + H(Tu_n, Tu^*) \\ &\leq d(u^*, u_n) + \theta d(u_n, u^*) + LD(u^*, Tu_n). \end{aligned}$$

Taking limit as $n \to \infty$, we have $u^* \in Tu^*$.

(b) From the proof of Theorem 2.3, it follows that Suzuki type (θ, L)-weak multivalued operator is MWP operator. Next we show that

$$H(Fix(T_1), Fix(T_2)) \leq \frac{\lambda}{1-\max\{\theta_1,\theta_2\}}.$$

First approach. If $q > 1$, and u_0 is arbitrary fixed point of T_1, then there exists $u_1 \in T_2(u_0)$ such that $d(u_0, u_1) \leq qH(T_1u_0, T_2u_1)$. Again, for $u_1 \in T_2(u_0)$, there exists $u_2 \in T_2(u_1)$ such that $d(u_1, u_2) \leq qH(T_2u_0, T_2u_1)$. As $\mu(a)D(u_0, T_2u_0) \leq \mu(a)d(u_0, u_1) \leq d(u_0, u_1)$ so

$$d(u_1, u_2) \leq qH(T_2u_0, T_2u_1) \leq q\left[\theta_2 d(u_0, u_1) + LD(u_1, T_2u_0)\right] = q\theta_2 d(u_0, u_1).$$

Continuing this process, we obtain a sequence of successive approximations for T_2 such that

$$d(u_n, u_{n+1}) \leq (q\theta_2)^n\, d(u_0, u_1), \text{ for all } n \in \mathbb{N}.$$

It is straight forward to check that

$$d(u_n, u_{n+p}) \leq \frac{(q\theta_2)^n}{1-q\theta_2} d(u_0, u_1) \text{ for each } n \in \mathbb{N} \text{ and } p \in \mathbb{N}^*. \tag{2.9}$$

Choosing $1 < q < \min\{\frac{1}{\theta_1}, \frac{1}{\theta_2}\}$ and taking limit as $n \to \infty$, it follows that $(u_n)_{n\in\mathbb{N}}$ is Cauchy sequence in (X,d). Since X is complete metric space, $u_n \to u$ as $n \to \infty$. Now, we show that $D(u, T_2x) \leq ad(u,x)$ for each $x \in X$. If $x = u$, then the claim follows. Assume that $x \neq u$. As $u_n \to u$ as $n \to \infty$, there exists $n_0 \in \mathbb{N}$ such that $d(u_n, u) \leq \frac{1}{3}d(x,u)$ for each $n \geq n_0$. Now we have that

$$\begin{aligned} \mu(a)D(u_n, T_2u_n) &\leq D(u_n, T_2u_n) \leq d(u_n, u_{n+1}) \leq d(u_n, u) + d(u_{n+1}, u) \\ &\leq \frac{2}{3}d(u,x) \leq d(u,x) - d(u_n,u) \leq d(u_n, x). \end{aligned}$$

Hence $H(T_2u_n, T_2x) \leq \theta_2 d(u_n, x) + LD(x, T_2u_n)$ for each $n \geq n_0$.

Thus, $D(u_{n+1}, T_2x) \leq H(T_2u_n, T_2x) \leq \theta_2 d(u_n, x) + LD(x, T_2u_n)$ for each $n \geq n_0$.

Taking limit as $n \to \infty$, we obtain $D(u, T_2x) \leq \theta_2 d(u,x)$ for all $x \neq u$. Next we show that

$$H(T_2x, T_2z) \leq \theta_2 d(x,z) + LD(z, T_2x). \tag{2.10}$$

If $x = z$ then it obviously holds. Assume that $x \neq z$. Then for every $n \in \mathbb{N}$, there exists $y_n \in T_2x$ such that $d(z, y_n) \leq D(z, T_2x) + \frac{1}{n}d(x,z)$. So, for $n \in \mathbb{N}$, we have

$$\begin{aligned} D(x, T_2x) &\leq d(x, y_n) \leq d(x,z) + D(z, T_2x) + \frac{1}{n}d(x,z) \\ &\leq d(x,z) + ad(z,x) + \frac{1}{n}d(x,z) \leq \left(1 + a + \frac{1}{n}\right) d(x,z). \end{aligned}$$

Hence $\frac{1}{1+a}D(x, T_2x) \leq d(x,z)$ and the result follows. Finally, we obtain

$$\begin{aligned} D(u, T_2u) &\leq d(u, u_{n+1}) + D(u_{n+1}, T_2u) \leq d(u, u_{n+1}) + H(T_2u_n, T_2u) \\ &\leq d(u, u_{n+1}) + \theta_2 d(u_n, u) + LD(u, T_2u_n) \end{aligned} \tag{2.11}$$

Taking limit as $n \to \infty$ in (2.11) and using the fact that $u_{n+1} \in T_2u_n$, we have $D(u, T_2u) = 0$ and $u \in Fix(T_2)$ as T_2u is closed. From (2.9), letting $p \to \infty$ we obtain $d(u_n, u) \leq \dfrac{(q\theta_2)^n}{1 - q\theta_2}d(u_0, u_1)$, for each $n \in \mathbb{N}$. Taking $n = 0$ we have

$$d(u_0, u) \leq \frac{1}{1 - q\theta_2}d(u_0, u_1) \leq \frac{q^\lambda}{1 - q\theta_2}.$$

Interchanging the roles of T_1 and T_2, for each $u_0 \in Fix\,(T_2)$ there exists $u \in Fix(T_1)$ such that

$$d(u_0, u) \leq \frac{1}{1 - q\theta_1}d(u_0, u) \leq \frac{q^\lambda}{1 - q\theta_1}.$$

Hence

$$H(Fix(T_1), Fix(T_2)) \leq \frac{q^\lambda}{1 - \max\{q\theta_1, q\theta_2\}}.$$

The conclusion follows when $q \to 1$.

Second approach. Suppose that $T : X \to P_{ci}(X)$ is an (θ, L)- weak multivalued operator. We will show that T is a c-MWP operator with $c := \frac{1}{1-a}$. Then, the conclusion will follow from Theorem 3 (in [18]). Let $q > 1$, $x \in X$ and $y \in T(x)$ be a arbitrary chosen. By a similar approach to (b$_1$), we obtain a sequence of successive approximations $(u_n)_{n\in\mathbb{N}}$ and $p \in \mathbb{N}^*$. Moreover, the sequence $(u_n)_{n\in\mathbb{N}}$ is Cauchy and its limit, denote by $u := u(x,y)$, is a fixed point for T. Letting $p \to \infty$ in the above estimation we get

$$d(u_n, u) \leq \frac{(qa)^n}{1 - qa}d(u_0, u_1),$$

for each $n \in \mathbb{N}$. For $n := 0$ we obtain that

$$d(x, u) \leq \frac{1}{1 - qa}d(x, y).$$

Letting $q \to 1$ we obtain $d(x,u) \leq \frac{1}{1-a}d(x,y)$. thus T is a $\frac{1}{1-a}$-MWP operator.

Example 2.9. Let $[0,1]$ be the unit interval with the usual metric and let $T_1, T_2 : [0,1] \to CL([0,1])$ be Suzuki type (θ_i, L)-weak multivalued operator for $i = 1,2$ given by $T_1(x) = \overline{B(0, \frac{1}{5}x)} = [0, \frac{1}{5}]$, $T_2(x) = \overline{B(0, \frac{2}{5}x)} = [0, \frac{2}{5}]$. In this case,

$$H(T_1(x), T_2(x)) = \max\{\sup_{y_1 \in T_1 x}\{\inf_{y_2 \in T_2 x}|y_1 - y_2|\}, \sup_{y_2 \in T_2 x}\{\inf_{y_1 \in T_1 x}|y_2 - y_1|\}\} = \frac{2}{7} < \lambda$$

Moreover $Fix(T_1) = [0, \frac{1}{5}] \in CL([0,1])$, $Fix(T_2) = [0, \frac{2}{5}] \in CL([0,1])$. For all $x, y \in [0,1]$, $i = 1,2$, we can easily see that

$$\frac{1}{1+a}D(x, T_i x) \leq d(x,y) \Rightarrow H(T_i(x), T_i(y)) \leq \theta_i d(x,y) + LD(x, Ty),$$

that is, T_1, T_2 are MWP operator and

$$H(Fix(T_1), Fix(T_2)) \leq \frac{\lambda}{1 - \max\{\theta_1, \theta_2\}}.$$

Therefore, all the conditions of Theorem 2.8 are satisfied.

The following is a local fixed point result for Suzuki type (θ, L)- weak multivalued operator.

Theorem 2.10. *Let (X,d) be a complete metric space, $x_0 \in X$ and $r > 0$. Suppose that $T : B(x_0, r) \to P_{cl}(X)$ is Suzuki type (θ, L)- weak multivalued operator and $D(x_0, Tx_0) < (1-\theta)r$. Then $Fix(T) \neq \emptyset$.*

Proof. Let $0 < s < r$ be such that $\widetilde{B}(x_0, s) \subset B(x_0, r)$ and $D(x_0, Tx_0) < (1-\theta)s < (1-\theta)r$. If $x_1 \in Tx_0$ is such that $d(x_0, x_1) < (1-\theta)s$, then $\mu(a)D(x_0, Tx_0) \leq \mu(a)d(x_0, x_1) \leq d(x_0, x_1)$. So,

$$D(x_1, Tx_1) \leq H(Tx_0, Tx_1) \leq \theta d(x_0, x_1) + LD(x_1, Tx_0) \leq \theta d(x_0, x_1) < \theta(1-\theta)s.$$

Thus there exists $x_2 \in Tx_1$ such that $d(x_1, x_2) < \theta(1-\theta)s$. Also, we have $x_2 \in B(x_0, s)$ because $d(x_0, x_2) \leq d(x_0, x_1) + d(x_1, x_2) < (1-\theta)s + \theta(1-\theta)s = (1-\theta^2)s < s$.

In this way, we obtain inductively a sequence $(x_n)_{n\in\mathbb{N}}$ satisfying the following properties:

(i) $x_n \in B(x_0, s)$; for each $n \in \mathbb{N}$;
(ii) $x_{n+1} \in Tx_n$, for all $n \in \mathbb{N}$;
(iii) $d(x_n, x_{n+1}) \leq \theta^n(1-\theta)s$ for each $n \in \mathbb{N}$.

From (iii) the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy and hence, it converges to a certain $u \in B(x_0, r)$. Following similar arguments to those given in Theorem 2.8, we obtain $u \in Fix(T)$.

Now we present following homotopy result which extends several comparable results in the existing literature.

Theorem 2.11. *Let (X,d) be a complete metric space and U an open subset of X. Let $G : \overline{U} \times [0,1] \to P(X)$ be a multivalued operator such that the following conditions are satisfied:*

(i) *$x \notin G(x,t)$, for each $x \in \partial U$ and each $t \in [0,1]$;*
(ii) *$G(.,t) : \overline{U} \to P(X)$ is a Suzuki type (θ, L)-weak multivalued operator for each $t \in [0,1]$;*

(iii) *there exists a continuous increasing function* $\psi : [0,1] \to \mathbb{R}$ *such that*

$$H(G(x,t), G(x,s)) \leq |\psi(t) - \psi(s)| \text{ for all } t, s \in [0,1] \text{ and each } x \in \overline{U};$$

(iv) $G : \overline{U} \times [0,1] \to P(X)$ *is closed.*

Then $G(.,0)$ *has a fixed point if and only if* $G(.,1)$ *has a fixed point.*

Proof. Suppose that $G(.,0)$ has a fixed point z. From (a), $z \in U$. Define

$$\Delta = \{(t,x) \in [0,1] \times U \mid x \in G(x,t)\}.$$

Clearly $\Delta \neq \emptyset$, as $(0,z) \in \Delta$. Define a partial order on Δ as follows:

$$(t,x) \leq (s,y) \text{ if and only if } t \leq s \text{ and } d(x,y) \leq \frac{2}{1-\theta}[\psi(s) - \psi(t)].$$

Let M be a totally ordered subset of Δ and $t^* := \sup\{t \mid (t,x) \in M\}$. Consider a sequence $(t_n, x_n)_{n\in\mathbb{N}} \subset M$ such that $(t_n, x_n) \leq (t_{n+1}, x_{n+1})$ and $t_n \to t^*$ as $n \to \infty$. Then

$$d(x_m, x_n) \leq \frac{2}{1-\theta}[\psi(t_m) - \psi(t_n)], \text{ for each } m, n \in \mathbb{N},\ m > n.$$

Taking limit as $m, n \to \infty$, we obtain $d(x_m, x_n) \to 0$. Thus $(x_n)_{n\in\mathbb{N}}$ is Cauchy sequence which converges to (say) x^* in X. As $x_n \in G(x_n, t_n)$, $n \in \mathbb{N}$ and G is closed, so $x^* \in G(x^*, t^*)$. Also, from (a) we have $x^* \in U$. Hence $(t^*, x^*) \in \Delta$. Since M is totally ordered, therefore $(t,x) \leq (t^*, x^*)$, for each $(t,x) \in M$. That is, (t^*, x^*) is an upper bound of M. By Zorn's Lemma Δ admits a maximal element $(t_0, x_0) \in \Delta$. We claim that $t_0 = 1$. Suppose that $t_0 < 1$. Choose $r > 0$ and $t \in (t_0, 1]$ such that $B(x_0, r) \subset U$ and $r = \frac{2}{1-\theta}[\psi(t) - \psi(t_0)]$. Note that

$$\begin{aligned} D(x_0, G(x_0,t)) &\leq D(x_0, G(x_0,t_0)) + H(G(x_0,t_0), G(x_0,t)) \\ &\leq [\psi(t) - \psi(t_0)] = \frac{(1-\theta)r}{2} < (1-\theta)r. \end{aligned}$$

Thus $G(.,t) : B(x_0, r) \to P_{cl}(X)$ satisfies, for all $t \in [0,1]$, the assumptions of Theorem 2.1. Hence, for all $t \in [0,1]$, there exists $x \in B(x_0, r)$ such that $x \in G(x,t)$ which implies that $(t,x) \in \Delta$. Now $d(x_0, x) \leq r = \frac{2}{1-\theta}[\psi(t) - \psi(t_0)]$, gives that $(t_0, x_0) < (t, x)$, a contraction to the maximality of (t_0, x_0). Conversely if $G(.,1)$ has a fixed point, then by a similar approach we obtain that $G(.,0)$ has a fixed point.

3. An Application

The existence and uniqueness of solutions of functional equations and system of functional equations arising in dynamic programming have been studied by using various fixed point theorems. (see [15, 16, 17])The aim of this subsection is to prove the existence and the uniqueness of a solution for a class of functional equations by using Corollary 2.1.

Throughout this subsection, we assume that X and Y are Banach spaces, $A \subset X$, $B \subset Y$ and R is the field of real numbers. Let $\Delta(A)$ denote the set of all bounded real-valued functions on A . It is well-known that $\Delta(A)$ endowed with the metric

$$d_\Delta(h,k) = \sup_{x\in A} |h(x) - k(x)|, \quad h, k \in \Delta(A)$$

is complete metric space. Let A and B be the state and, respectively the decision spaces. Then the problem of dynamic programming reduces to the problem of solving the functional equation:

$$p(x) = \sup_{y\in B} H(x, y, p(\tau(x, y)))$$

where $\tau : A\times B \to A$ represents the transformation of the process and $p(x)$ represents the optimal return function with initial functional equation:

$$p(x) = \sup_{y\in B}\{g(x, y) + G(x, y, p(\tau(x, y)))\}, \quad x \in A, \tag{3.1}$$

where $g : A \times B \to R$ and $G : A \times B \times R \to R$ are bounded functions.

Let F be defined by:

$$F(h(x)) = \sup_{y\in B}\{g(x, y) + G(x, y, p(\tau(x, y)))\}, \quad h \in \Delta(A),\ x \in A.$$

Theorem 3.1. *Assume that there exist $\theta \in [0, 1)$, $r > \theta$ such that for every $(x, y) \in A \times B$, $h, k \in \Delta(A)$ and $t \in A$ the inequality*

$$|h(t) - F(h(t))| \le r\,|h(t) - k(t)| \tag{3.2}$$

implies

$$|G(x, y, h(t)) - G(x, y, k(t))| \le \theta\,|h(t) - k(t)| + L\,|h(t) - F(k(t))| \tag{3.3}$$

Then the functional equation (3.1) has a bounded solution. Moreover, if $r \ge 1$, then the solution is unique.

Proof. It is obvious that F is selfmap of $\Delta(A)$. Let γ be an arbitrary positive real number and $h_1, h_2 \in \Delta(A)$. Pick $x \in A$ and choose $y_1, y_2 \in B$ such that

$$F(h_1(x)) < g(x, y_1) + G(x, y_1, h_1(\tau(x, y_1)) + \gamma \tag{3.4}$$

$$F(h_2(x)) < g(x, y_2) + G(x, y_2, h_2(\tau(x, y_2)) + \gamma \tag{3.5}$$

From the definition of F we get

$$F(h_1(x)) \ge g(x, y_2) + G(x, y_2, h_1(\tau(x, y_2)) \tag{3.6}$$

$$F(h_2(x)) \ge g(x, y_1) + G(x, y_1, h_2(\tau(x, y_1)) \tag{3.7}$$

If the inequality (3.2) holds with $h = h_1$,$k = h_2$, then from (3.4), (3.7) and (3.3) we have

$$\begin{aligned} F(h_1(x)) - F(h_2(x)) &< G(x, y_1, h_1(\tau(x, y_1)) - G(x, y_1, h_2(\tau(x, y_2)) + \gamma \quad (3.8)\\ &\le \theta\,|h_1(t) - h_2(t)| + L\,|h_1(t) - F(h_2(t))| + \gamma. \end{aligned}$$

Similarly, from (3.5), (3.6) and (3.3) we have,

$$F(h_2(x)) - F(h_1(x)) < \theta\,|h_1(t) - h_2(t)| + L\,|h_1(t) - F(h_2(t))| + \gamma. \tag{3.9}$$

Thus, from (3.8) and (3.9) we obtain that

$$F(h_1(x)) - F(h_2(x)) \le \theta\,|h_1(t) - h_2(t)| + L\,|h_1(t) - F(h_2(t))| + \gamma \tag{3.10}$$

Since the inequalty (3.10) holds for any $x \in A$ and $\gamma > 0$, we have that

$$d_\Delta(h_2, Fh_1) \le \theta d_\Delta(h_1, h_2)$$

which implies

$$d_\Delta(Fh_1, Fh_2) \leq \theta d_\Delta(h_1, h_2) + L d_\Delta(h_1, Th_2).$$

Hence, all condition of Corollary 2.6 are satisfied for the mapping F and therefore the proof is finished.

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