Fixed Point Theory, 16(2015), No. 2, 215-224
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

TREND CONSTANTS FOR LIPSCHITZ MAPPINGS

KRZYSZTOF BOLIBOK AND KAZIMIERZ GOEBEL

Institute of Mathematics, Maria Curie-Sklodowska University
20-031 Lublin, Poland
E-mails: bolibok@hektor.umcs.lublin.pl, goebel@hektor.umcs.lublin.pl

Abstract. Lipschitz mappings are naturally classified by the value of their Lipschitz constant. Here we propose to use two other *trend constants* which allow a more subtle classification of such mappings. Basic facts, examples and applications to metric fixed point theory are shown.
Key Words and Phrases: Lipschitz mappings, nonexpansive mappings, fixed points.
2010 Mathematics Subject Classification: 47H10, 54H25.

1. Introduction

Investigations of nonexpansive mappings on convex sets in Banach spaces is the core subject of metric fixed point theory. In spite of the fact that a big number of researchers devoted themselves to this field and worked for many years, there are still open old and new problems and directions to study. Except of the class defined by the classical definition of nonexpansiveness there are other classes which, in better or similar way, behave like nonexpansive ones. It leads to new fixed point theorems, new results and applications.

Our aim is to present some examples of such situation. We propose two new constants for Lipschitz mappings which characterize in a more subtle way behavior of their convex combinations with identity. We hope the presented formalism have chances for applications.

Let $(X, \|\circ\|)$ be a Banach space with the closed unit ball B and the unit sphere S. Let $C \subset X$ be a closed convex set. Depending on the situation we shall assume that C is bounded or not. Let $T : C \to X$ be a mapping. The mapping T is said to be *Lipschitz (or lipschitzian) with constant* k if for all $x, y \in C$

$$\|Tx - Ty\| \leq k \|x - y\| . \tag{1.1}$$

We also refer to such mappings as being of class $\mathcal{L}(k)$. The smallest k for which $T \in \mathcal{L}(k)$ is called the Lipschitz constant for T and is denoted by $k(T)$. Mappings with $k(T) < 1$ are called *contractions* and those with $k(T) \leq 1$ are said to be *nonexpansive.* The mapping T is said to be *firmly nonexpansive* if for all $x, y \in C$ the function

$$\varphi_{x,y}(t) = \|(1-t)(x-y) + t(Tx - Ty)\| \tag{1.2}$$

is nonincreasing. Firmly nonexpansive mappings form a proper subclass of nonexpansive ones. For example if $X = H$ is a Hilbert space then T is firmly nonexpansive if and only if it is of the form $T = \frac{1}{2}(I + F)$, where F is nonexpansive.

There is a class "in between" contractions and nonexpansive mappings. A mapping $T : C \to X$ is said to be contractive if for any $x, y \in C, x \neq y$

$$\|Tx - Ty\| < \|x - y\|.$$

If T, S are two lipschitzian self mappings on C, composing them we get

$$k(T \circ S) \leq k(T)\, k(S),$$

and iterating T we have

$$k(T^n) \leq k(T)^n.$$

The *fixed point set* for T is defined as

$$FixT = [x : x = Tx].$$

For any $t \in (0, 1]$, the convex combination of T with identity $T_t = (1 - t)I + tT$ has the same fixed point set as T,

$$FixT = Fix(T_t).$$

2. Initial and terminal trends

If $T : C \to X$ belongs to $\mathcal{L}(k)$ the convex combination with identity $T_t = (1 - t)I + tT$, belongs to $\mathcal{L}((1 - t) + tk)$. More precisely

$$k(T_t) = k((1 - t)I + tT) \leq (1 - t) + tk = 1 + t(k - 1).$$

However the last Lipschitz constant can be much smaller. The simple example is $Tx = -kx$.

For any $x, y \in C, x \neq y$, and $t \in [0, 1]$ consider the function $\varphi_{x,y}(t)$ and modify it defining $\psi_{x,y}(t)$ on $[0, 1]$ as $\psi_{x,y}(t) = \varphi_{x,y}(t)\|x - y\|^{-1}$. Thus we have,

$$\begin{aligned} \varphi_{x,y}(t) &= \|(1 - t)(x - y) + t(Tx - Ty)\| && (2.1) \\ &= \left\|(1 - t)\frac{x - y}{\|x - y\|} + t\frac{Tx - Ty}{\|x - y\|}\right\| \|x - y\| \\ &= \psi_{x,y}(t)\|x - y\|. && (2.2) \end{aligned}$$

Each of the functions $\varphi_{x,y}(t)$ and $\psi_{x,y}$ is convex with $\psi_{x,y} = \psi_{y,x}$, $\psi_{x,y}(0) = 1$ and $\psi_{x,y}(1) \leq \max[1, k]$. In view of convexity each function $\psi_{x,y}$ has, at each point, left and right derivative. Both derivatives are nondecreasing and right dominates the left. Put

$$\psi(t) = \sup[\psi_{x,y}(t) : x, y \in C, x \neq y]. \tag{2.3}$$

Two coefficients seem to be of interest:

Definition 2.1. *The initial trend coefficient* of T on the set C is defined as

$$\iota(T) = \sup\left[\psi'_{x,y}(0) : x, y \in C, x \neq y\right] \tag{2.4}$$

and *terminal trend coefficient as*

$$\tau(T) = \sup\left[\psi'_{x,y}(1) : x, y \in C, x \neq y\right]. \tag{2.5}$$

Remark 2.1. Observe that both coefficients are related to a given set C. If necessary it will be indicated by subscript: $\iota_C(T)$, $\tau_C(T)$. If $D \subset C$ than, $\iota_D(T) \le \iota_C(T)$ and $\tau_D(T) \le \tau_C(T)$.

Remark 2.2. Observe also that initial and final trend coefficients can be defined for any continuous mapping. However, for our purpose we shall consider only lipschitzian ones, assuming that $T \in \mathcal{L}(k)$ for some $k \ge 0$.

Remark 2.3. In the special case, if $A : X \to X$ is a linear mapping corresponding definitions can be simplified. We can consider only $x \in S, \|x\| = 1$ and simplified functions

$$\psi_x(t) = \|(1-t)x + tAx\|$$

and

$$\psi(t) = \sup[\psi_x(t) : x \in S].$$

Initial and terminal trend coefficients are defined analogously.

By analogy to the classical case, let us accept the following terminology:

Definition 2.2. The mapping T is said to be *the initial contraction* if $\iota(T) < 0$, *initially nonexpansive* if $\iota(T) \le 0$ and initially contractive if for all $x, y,\ x \ne y$, $\psi'_{x,y}(0) < 0$.

3. Basics

Let us list some elementary basic properties of initial and terminal trends. For simplicity assume that always $k(T) = k$. We skip most of elementary proofs leaving them to the reader.

3.1. **Facts.** The following facts come straight from the definition:

- If there exists $t_0 > 0$ such that for all $0 < t \le t_0$and all $x, y \in C, \psi_{x,y}(t) \le c < 1$, then for all $0 < t \le t_0$, the mapping $(1-t)I + tT$ is a contraction;
- The terminal trend coefficient $\tau(T) \le 0$, if and only if T is firmly nonexpansive;
- For all $x, y \in C$ and $t, s \in [0,1]$
$$|\psi_{x,y}(t) - \psi_{x,y}(s)| \le (k+1)|t-s|;$$
- In view of the above and the convexity of $\psi_{x,y}(t)$, if $k(T) = k$, for initial and terminal trend we have
$$\iota(T) \le \sup\left[\frac{\psi_{x,y}(1) - \psi_{x,y}(0)}{1-0} : x, y \in C\right] = k - 1 \le \tau(T) \le k+1;$$
- Since $\psi_{x,z}(t)$ can be written and estimated in the form
$$\begin{aligned}\psi_{x,y}(t) &= \left\|\frac{x-y}{\|x-y\|} + t\left(\frac{Tx-Ty}{\|x-y\|} - \frac{x-y}{\|x-y\|}\right)\right\| \\ &\le 1 + t\left\|\frac{Tx-Ty}{\|x-y\|} - \frac{x-y}{\|x-y\|}\right\| \le 1 + tk(T-I)\end{aligned} \qquad (3.1)$$

we get more than above,

$$\iota(T) \leq k(T-I).$$

In some cases $k(T-I) < k-1$.

- The above inequalities are sharp. For $T : X \to X$, defined as $Tx = kx$, $k(T) = k$. However, for $k \geq 0$, we have

$$\iota(T) = \tau(T) = k-1,$$

while for $k < 0$.

$$\iota(T) = k-1 \text{ and } \tau(T) = |k| + 1 = 1-k = -\iota(T),$$

- The same holds for any mapping of the form $Tx = a+k(x-a)$. In particular, for $k = 0$ any constant mapping $Tx = z = const, \iota(T) = -1 = \tau(T)$, Also for $k = 1, \iota(I) = 0$.
- More, if $T : C \to C$ is a Lipschitz mapping for which there exists $z \in C$ and two points $x, y, x \neq y$ such that $Tx = z + k(x-z)$ and $Ty = z + k(y-z)$ then $\iota(T) \geq k-1$ and $\tau(T) \geq k$.
- Both coefficients are convex in the following sense. If F, T are two mappings of class $\mathcal{L}(k)$ and $\alpha \in [0,1]$ then

$$\iota((1-\alpha)F + \alpha T) \leq (1-\alpha)\iota(F) + \alpha\iota(T)$$

and

$$\tau((1-\alpha)F + \alpha T) \leq (1-\alpha)\tau(F) + \alpha\tau(T)$$

- Especially,

$$\iota((1-\alpha)I + \alpha T) \leq \alpha\iota(T)$$

and for constant mapping $Fx = z = const$

$$\iota((1-\alpha)z + \alpha T) \leq -(1-\alpha) + \alpha\iota(T).$$

3.2. **Examples.** Let us complete this section with some elementary examples.

Example 3.1. Let $C = [a,b]$ and $f : [a,b] \to [a,b]$, $f \in \mathcal{L}(k)$. Take any $x, y \in [a,b]$, $x < y$. Now, (3.1) leads to

$$\psi_{x,y}(t) = \left| 1 + t\left(\frac{f(y) - f(x)}{y-x} - 1\right)\right|,$$

which for small $t > 0$ reads

$$\frac{\psi_{x,y}(t) - \psi_{x,y}(0)}{t} = \frac{\psi_{x,y}(t) - 1}{t} = \frac{f(y) - f(x)}{y-x} - 1,$$

implying

$$\iota(f) = \sup\left[\frac{f(x) - f(y)}{x-y} : x, y \in [a,b], x \neq y\right] - 1.$$

In particular if f is differentiable then $\iota(f) = \sup[f'(x) : x \in [a,b]] - 1$. If f is not differentiable we can also write

$$\iota(f) = \sup\left[f'_+(x) : x \in [a,b]\right] - 1,$$

where f'_+ denotes the upper derivative of f.

Since the Lipschitz constant

$$k(f) = \sup \left[\left| \frac{f(x) - f(y)}{x - y} \right| : x, y \in [a, b], x \neq y \right],$$

the above shows that the initial trend constant $\iota(f)$ can be for some functions f properly smaller then $k(f) - 1$.

Similar estimate holds for composition operators in the space of continuous functions.

Example 3.2. Let $X = C[a, b]$ with the standard norm

$$\|x\| = \max [|x(s)| : s \in [a, b]].$$

Take $M > 0$ and consider the set $B(0, M) = [x : \|x\| \leq M]$. Assume the function $f : [-M, M] \to [-M, M]$ is of class $\mathcal{L}(k)$. Consider the composition operation $F : B(0, M) \to B(0, M)$ defined by

$$Fx(s) = f(x(s)).$$

Certainly $F \in \mathcal{L}(k)$ on $B(0, M)$. Let $x, y \in B(0, M)$ and $s \in [a, b]$. Now we have,

$$\psi_{x,y}(t) = \max \left[\left| \frac{x(s) - y(s)}{\|x - y\|} + t \left(\frac{Fx(s) - Fy(s)}{\|x - y\|} - \frac{x(s) - y(s)}{\|x - y\|} \right) \right| : s \in [a, b] \right].$$

Now for all s such that $x(s) \neq y(s)$ and sufficiently small t

$$\begin{aligned} & \left| \frac{x(s) - y(s)}{\|x - y\|} + t \left(\frac{Fx(s) - Fy(s)}{\|x - y\|} - \frac{x(s) - y(s)}{\|x - y\|} \right) \right| \\ = & \left| 1 + t \left(\frac{f(x(s)) - f(y(s))}{x(s) - y(s)} - 1 \right) \right| \left| \frac{x(s) - y(s)}{\|x - y\|} \right| \\ \leq & \ 1 + t \left(\frac{f(x(s)) - f(y(s))}{x(s) - y(s)} - 1 \right) \\ \leq & \ 1 + t \left(\sup \left[\frac{f(x) - f(y)}{x - y} : x, y \in [-M, M] \right] - 1 \right) \\ = & \ 1 + t \left(\sup \left[f'_+(x) : x \in [-M, M] \right] - 1 \right). \end{aligned}$$

This implies the inequality

$$\begin{aligned} \iota(F) & \leq \sup \left[\frac{f(x) - f(y)}{x - y} : x, y \in [a, b], x \neq y \right] - 1 \\ & = \sup \left[f'_+(x) : x \in [a, b] \right] - 1. \end{aligned}$$

However the set $B(0, M)$ contains constant functions on which equality is justified as in previous example. Thus, again

$$\iota(F) = \sup \left[f'_+(x) : x \in [a, b] \right] - 1.$$

We propose the reader to extend the above for the more general case of the composition operator in $C[0, 1]$ defined by $T(x)(s) = f(s, x(s))$.

Example 3.3. Let X be the Euclidian plane R^2. For any $k \geq 1, 0 \leq \alpha \leq \pi$ consider the mapping

$$T(x,y) = kO_\alpha(x,y) = k(x\cos\alpha - y\sin\alpha, x\sin\alpha + y\cos\alpha).$$

In other words consider complex plane $\mathbb{C}$ and $Tz = ke^{i\alpha}z$. We have $k(T) = k$. we leave to the reader finding that $\iota(T) = k\cos\alpha - 1$. It follows that, for $0 < \alpha < \frac{\pi}{2}, \iota(T) < k(T)$ the linear mapping T is an initial contraction for $k < \frac{1}{\cos\alpha}$ and initially nonexpansive if $k = \frac{1}{\cos\alpha}$. If $\frac{\pi}{2} \leq \alpha \leq \pi, T$ is the initial contraction for all $k > 1$.

Example 3.4. Consider the space $X = l^2$ with the classic norm. Let $\alpha \in [0, \frac{\pi}{2})$ and $\{k_n\}$ be a sequence satisfying $1 < k_1 < k_2 < k_3 < ...$ with $\lim_{n\to\infty} k_n = \frac{1}{\cos\alpha}$. Define the linear mapping $A : l^2 \to l^2$as

$$Ax = A(x_1, x_2, x_3, x_4, ...) = (y_1, y_2, y_3, y_4, ...),$$

where

$$\begin{array}{lll} y_n = k_i(x_n\cos\alpha - x_{n+1}\sin\alpha) & \text{if} & n = 2i-1, i = 1,2,3,... \\ y_n = k_i(x_n\sin\alpha + x_{n+1}\cos\alpha) & \text{if} & n = 2i, i = 1,2,3,... \end{array}.$$

The norm, the Lipschitz constant, of A is greater than $1, \|A\| = \frac{1}{\cos\alpha}$.However, A is initially nonexpansive $\iota(A) = 0$, and more A is initially contractive.

Example 3.5. Let H be a Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and $C \subset H$ be an arbitrary set. For any $x, y \in C, x \neq y$,

$$\varphi'_{x,y}(t) = \frac{\left(\varphi^2_{x,y}(t)\right)'}{\varphi_{x,y}(t)} = \frac{\langle(1-t)(x-y) + t(Tx-Ty), (Tx-Ty)-(x-y)\rangle}{\|(1-t)(x-y) + t(Tx-Ty)\|}.$$

For $t = 0$, we have

$$\varphi'_{x,y}(0) = \frac{\langle Tx-Ty, x-y\rangle - \|x-y\|^2}{\|x-y\|} = \left\langle Tx-Ty, \frac{x-y}{\|x-y\|}\right\rangle - \|x-y\|,$$

$$\psi'_{x,y}(0) = \frac{\varphi'_{x,y}(0)}{\|x-y\|}.$$

Consequently

$$\iota_C(T) = \sup\left[\left\langle\frac{Tx-Ty}{\|x-y\|}, \frac{x-y}{\|x-y\|}\right\rangle - 1 : x, y \in C, x \neq y\right].$$

Thus T is initially nonexpansive on C if for all $x, y \in C$

$$\langle Tx-Ty, x-y\rangle \leq \|x-y\|^2.$$

4. Some applications to fixed point theory

It seems that the notions of initial and terminal trends can have some applications in metric fixed point theory. Here are some illustrations.

Suppose $C \subset X$ is closed convex and bounded. If $T : C \to C$ is nonexpansive

$$\|Tx - Ty\| \leq \|x-y\|$$

for all $x, y \in C$, then the minimal displacement $d(T)$ for T defined as

$$d(T) = \inf [\|x - Tx\| : x \in C] = 0.$$

It follows from the fact that T can be uniformly approximated by contractions of the form $T_\varepsilon x = \varepsilon z + (1-\varepsilon) Tx$ with arbitrarily fixed $z \in C$. The set C is said to have *fixed point property (shortly fpp)* for nonexpansive mappings if for all nonexpansive $T : C \to C$ the above infimum is achieved. It means, if the set of fixed points for T

$$FixT = [x : x = Tx]$$

is nonempty. The space X has fpp if all the convex, closed and bounded $C \subset X$ have it.

The fixed point property depends very much on the internal geometry of the underlying space X or the set C itself. The classical independent results due to D. Göhde [6] and F.E. Browder [1] states that it holds if the space X is uniformly convex. The stronger theorem by W.A. Kirk [7] says the same for weakly compact sets C having so called normal structure. Nowadays, the theory is very much developed and details can be found in the cited books ([3],[5],[8]).

4.1. **Technical observation.** Let $F : C \to C$ and $T : C \to C$ be two mappings of class $\mathcal{L}(k)$. Suppose that for certain $\alpha \in (0,1)$ there are two points $x, y \in C$ such that

$$Fx = (1-\alpha)x + \alpha TFx \text{ and } Fy = (1-\alpha)y + \alpha TFy.$$

Consider three functions

$$\begin{aligned} {}^{TF}\psi_{x,y}(t) &= \left\| (1-t)\frac{x-y}{\|x-y\|} + t\frac{TFx - TFy}{\|x-y\|} \right\|, \\ {}^{F}\psi_{x,y}(t) &= \left\| (1-t)\frac{x-y}{\|x-y\|} + t\frac{Fx - Fy}{\|x-y\|} \right\| \end{aligned}$$

and

$${}^{T}\psi_{Fx,Fy}(t) = \left\| (1-t)\frac{Fx-Fy}{\|Fx-Fy\|} + t\frac{TFx - TFy}{\|Fx-Fy\|} \right\|.$$

Put $t_0 = \alpha$ and evaluate ${}^{TF}\psi'_{x,y}(\alpha^+)$ and ${}^{TF}\psi'_{x,y}(\alpha^-)$. For $h > 0$ we have

$$\begin{aligned} & {}^{TF}\psi_{x,y}(\alpha+h) - {}^{TF}\psi_{x,y}(\alpha) \\ = & \left\| (1-\alpha-h)\frac{x-y}{\|x-y\|} + (\alpha+h)\frac{TFx - TFy}{\|x-y\|} \right\| - \frac{\|Fx-Fy\|}{\|x-y\|} \\ = & \frac{1}{\|x-y\|}\left(\|(Fx-Fy) - h(x-y) + h(TFx-TFy)\| - \|Fx-Fy\| \right). \end{aligned}$$

Since $x = \frac{Fx - \alpha TFx}{1-\alpha}$ and $y = \frac{Fy-\alpha TFy}{1-\alpha}$, substituting we get

$$\begin{aligned} & {}^{TF}\psi_{x,y}(\alpha+h) - {}^{TF}\psi_{x,y}(\alpha) = \\ = & \frac{\left\| \left(1 - \frac{h}{1-\alpha}\right)(Fx-Fy) + \frac{h}{1-\alpha}(TFx-TFy) \right\| - \|Fx-Fy\|}{\|x-y\|} \\ = & \frac{\|Fx-Fy\|}{\|x-y\|}\left(\left\| \left(1 - \frac{h}{1-\alpha}\right)\frac{Fx-Fy}{\|Fx-Fy\|} + \frac{h}{1-\alpha}\frac{TFx-TFy}{\|Fx-Fy\|} \right\| - 1 \right) \end{aligned}$$

and consequently

$$^{TF}\psi'_{x,y}\left(\alpha^{+}\right)=\frac{^{F}\psi_{x,y}\left(1\right)}{1-\alpha}\ {}^{T}\psi'_{Fx,Fy}\left(0\right).$$

By similar calculation in view that $TFx=\frac{1}{\alpha}Fx-\left(\frac{1}{\alpha}-1\right)x$

$$\begin{aligned}
&{}^{TF}\psi_{x,y}\left(\alpha\right)-{}^{TF}\psi_{x,y}\left(\alpha-h\right)\\
=\ &\frac{1}{\|x-y\|}\left(\|Fx-Fy\|-\|(1-\alpha-h)(x-y)+(\alpha-h)(TFx-TFy)\|\right)\\
=\ &\frac{1}{\|x-y\|}\left(\|Fx-Fy\|-\left\|\frac{h}{\alpha}(x-y)+\left(1-\frac{h}{\alpha}\right)(Fx-Fy)\right\|\right)\\
=\ &{}^{F}\psi_{x,y}\left(1\right)-{}^{F}\psi_{x,y}\left(1-\frac{h}{\alpha}\right).
\end{aligned}$$

This implies

$$^{TF}\psi'_{x,y}\left(\alpha^{-}\right)=\frac{1}{\alpha}\ {}^{F}\psi'_{x,y}\left(1\right).$$

Finally,

$$\frac{1}{\alpha}\ {}^{F}\psi'_{x,y}\left(1\right)={}^{TF}\psi'_{x,y}\left(\alpha^{-}\right)\leq{}^{TF}\psi'_{x,y}\left(\alpha^{+}\right)=\frac{^{F}\psi_{x,y}\left(1\right)}{1-\alpha}\ {}^{T}\psi'_{Fx,Fy}\left(0\right). \tag{4.1}$$

4.2. **Regularization of mappings.** There is a standard procedure used for Lipschitz mappings which is sometimes called *regularization*. Let $T:C\to C$ be of class $\mathcal{L}\left(k\right)$. Take $A>k$ and for any $x\in C$ consider the equation

$$y=\left(1-\frac{1}{A}\right)x+\frac{1}{A}Ty.$$

Since the right hand side is a contraction with Lipschitz constant $\frac{k}{A}<1$, for any $x\in C$ the equation has exactly one solution depending on x. Assigning this solution to each x we define the mapping $F:C\to C$ satisfying the implicit formula

$$Fx=\left(1-\frac{1}{A}\right)x+\frac{1}{A}TFx. \tag{4.2}$$

Since

$$\begin{aligned}
\|Fx-Fy\| &\leq \left(1-\frac{1}{A}\right)\|x-y\|+\frac{1}{A}\|TFx-TFy\|\\
&\leq \left(1-\frac{1}{A}\right)\|x-y\|+\frac{k}{A}\|Fx-Fy\|,
\end{aligned}$$

F is lipschitzian and $F\in\mathcal{L}\left(\frac{A-1}{A-k}\right)$

$$\|Fx-Fy\|\leq\frac{A-1}{A-k}\|x-y\|.$$

Selecting A large we can make the Lipschitz constant $k\left(F\right)$ as close to 1 as we wish. In special case when $A=k+1$, $F\in\mathcal{L}\left(k\right)$ as well as T.

Also for any $x \in C$ we have

$$\|x - Fx\| = \frac{1}{A-1}\|Fx - TFx\| = \frac{1}{A}\|x - TFx\|. \tag{4.3}$$

All the above implies $FixF = FixT$ and if $FixT = \emptyset$,

$$d(F) \geq \frac{1}{A-1} d(T).$$

Moreover, for any $x, y \in C$, F, T are subject to the conditions (4.1) with $\alpha = \frac{1}{A}$. From this, taking suprema of both sides and observing that ${}^{F}\psi_{x,y}(1) \leq \frac{A-1}{A-k}$ we get

$$\tau(F) \leq \frac{1}{A-k}\iota(T).$$

Again, if $A = k+1$, we just get $\tau(F) \leq \iota(T)$. The most important consequence is that $\iota(T) \leq 0$, implies $\tau(F) \leq 0$. This justifies in a stronger sense the mentioned term, regularization.

4.3. **Main corollaries.** The most important fact can be formulated as

Theorem 4.1. *If $T : C \to C$ is initially nonexpansive then the regularized mapping $F : C \to C$ defined by (4.2) is firmly nonexpansive.*

This has a number of corollaries showing that initially nonexpansive mappings share several properties of nonexpansive ones.

Corollary 4.1. *If $T : C \to C$ is initially nonexpansive, then*

$$d(T) = \inf[\|x - Tx\| : x \in C] = 0.$$

This follows straight from (4.3). Also we have the next:

Corollary 4.2. *If C has fixed point property for nonexpansive mappings, then it has fixed point property for the larger class of initially nonexpansive ones.*

Initial nonexpansiveness has also impact on the quality of the fixed point sets.

Corollary 4.3. *It is known that if X is a strictly convex space, then the fixed point set of a nonexpansive mapping $T : C \to C$ is convex. In view of $FixT = FixF$, the same holds for any T initially nonexpansive.*

Extention of the above is due to R. Bruck whose result states that if C is weakly compact and nonexpansive $T : C \to C$ has a fixed point in and convex, closed T invariant, subset $D \subset C$ then $FixT$ is a nonexpansive retract of C. This means that there exists a nonexpansive mapping $R : C \to FixT$ such that $FixR = FixT$. Nonexpansive retracts do not need to be convex (see [4]. We have next:

Corollary 4.4. *If C is weakly compact and such that all the closed and convex subsets $D \subset C$ have fixed point property for nonexpansive mappings, then fixed point set $FixT$ for any initially nonexpansive T is a nonexpansive retract of C.*

In a special case:

Corollary 4.5. *If the space X is reflexive and has fixed point property of nonexpansive mappings, then for any convex, closed and bounded set $C \subset X$ and any initially nonexpansive mapping $T : C \to C$ the fixed point set $FixT$ is a nonexpansive retract of C.*

5. Conclusion

The aim of this article was to turn the reader attention to the fact that the class of nonexpansive mapping has certain, in some way, natural extention. The initially nonexpansive mappings form a wider class but imitate in certain manner the behaviour of nonexpansive ones. We presented here only basic facts and properties. It is our feeling that the subject can be developed and may have some applications in general theory of equations. It looks challenging and interesting to find ways to estimate initial trend constants for various linear and nonlinear operators. In particular we think that the above considerations may have applications in the theory of nonlinear integral equations.

References

[1] F.E. Browder, *Nonexpansive nonlinear operators in a Banach space*, Proc. Nat. Acad. Sci. USA, **56**(1965), 1041-1044.
[2] R.E. Bruck, *Properties of fixed point sets of nonexpansive mappings*, Trans. Amer. Math. Soc., **179**(1973), 251-262.
[3] K. Goebel, *Concise Course on Fixed Point Theorems*, Yokohama Publishers, 2002.
[4] K. Goebel, S. Prus, *Shapes ans sizes of fixed point sets of nonexpansive mappings*, Nonlinear Analysis and Convex Analysis I, Yokohama Publishers (2013) 75-98.
[5] K. Goebel, W.A. Kirk, *Topics in Metric Fixed Point Theory,* Cambridge University Press, 1990.
[6] D. Göhde, *Zum princip der kontrakiven Abbildung*, Math. Nachr., **30**(1965), 251-258.
[7] W.A. Kirk, *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Monthly, **72**(1965), 1004-1006.
[8] W.A. Kirk, B. Sims, *Handbook on Metric Fixed Point Theory,* Kluwer Academic Publ., 2001.

Received: January 30, 2013; Accepted: March 22, 2014.