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WEAK ORTHOGONALITY AND SUZUKI NONEXPANSIVE-TYPE MAPPINGS

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Abstract. It is shown that if X is a weakly orthogonal Banach lattice, K is a nonempty weakly compact and convex subset of X and $T : K \to K$ satisfies condition (C) or is continuous and satisfies condition (C_λ) for some $\lambda \in (0, 1)$, then T has a fixed point. This generalizes Sims's result from [11].
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1. Introduction

Let K be a nonempty subset of a Banach space X. A mapping $T : K \to K$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for $x, y \in K$. We say that a Banach space X has the weak fixed point property if every nonexpansive mapping defined on a nonempty weakly compact convex subset of X has a fixed point. There is a large literature concerning fixed point theory of nonexpansive mappings and their generalizations (see [9] and references therein). Recently, Suzuki [13] defined a class of generalized nonexpansive mappings as follows.

Definition 1.1. A mapping $T : K \to K$ is said to satisfy condition (C) if for all $x, y \in K$,

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \quad \text{implies} \quad \|Tx - Ty\| \leq \|x - y\|.$$

Subsequently, the above definition has been extended in [6].

Definition 1.2. Let $\lambda \in (0, 1)$. A mapping $T : K \to K$ is said to satisfy condition (C_λ) if for all $x, y \in K$,

$$\lambda\|x - Tx\| \leq \|x - y\| \quad \text{implies} \quad \|Tx - Ty\| \leq \|x - y\|.$$

We say that X has the weak fixed point property for continuous mappings satisfying condition (C_λ) if every such mapping defined on a nonempty weakly compact convex subset of X has a fixed point.

It is not difficult to see that if $\lambda_1 < \lambda_2$, then condition (C_{λ_1}) implies condition (C_{λ_2}). Several examples of mappings satisfying condition (C_λ) are given in [6, 13].

Moreover, if K is convex and $T : K \to K$ satisfies condition (C_λ) for some $\lambda \in (0,1)$, then for every $\gamma \in [\lambda, 1)$ the mapping $T_\gamma : K \to K$ defined by $T_\gamma x = \gamma T x + (1-\gamma)x$ satisfies condition $(C_{\frac{\lambda}{\gamma}})$.

Recall that (x_n) is an approximate fixed point sequence for T (in short afps) if $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

2. Basic lemmas

Recall that a mapping $T : M \to M$ acting on a metric space (M, d) is said to be asymptotically regular if

$$\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0$$

for all $x \in M$.

Lemma 2.1. [6, Theorem 4] *Let K be a bounded convex subset of a Banach space X. Assume that $T : K \to K$ satisfies condition (C_λ) for some $\lambda \in (0,1)$. For $\gamma \in [\lambda, 1)$ define a sequence (x_n) in K by taking $x_1 \in K$ and*

$$x_{n+1} = \gamma T x_n + (1-\gamma)x_n$$

for $n \geq 1$. Then (x_n) is an approximate fixed point sequence for T, that is T_γ is asymptotically regular. In [1] the following theorem was proven which is the uniform version of the above theorem.

Theorem 2.2. *Let K be a bounded convex subset of a Banach space X. Fix $\lambda \in (0,1), \gamma \in [\lambda, 1)$ and let $\mathcal{F}$ denote the collection of all mappings which satisfy condition (C_λ). Let $T_\gamma = (1-\gamma)I + \gamma T$ for $T \in \mathcal{F}$. Then for every $\varepsilon > 0$, there exists a positive integer n_0 such that $\left\|T_\gamma^{n+1}x - T_\gamma^n x\right\| < \varepsilon$ for every $n \geq n_0, x \in K$ and $T \in \mathcal{F}$.*

Let D be a nonempty weakly compact convex subset of a Banach space X and $T : D \to D$. It follows from the Kuratowski-Zorn lemma that there exists a minimal (in the sense of inclusion) convex and weakly compact set $K \subset D$ which is invariant under T. The next lemma below is a counterpart of the Goebel-Karlovitz lemma (see [7, 8]). It was proved by Dhompongsa and Kaewcharoen [4, Theorem 4.14] in the case of mappings which satisfy condition (C), and from Butsan, Dhompongsa and Takahashi result in[2, Lemma 3.2] and Lloréns Fuster and Moreno Gálvez result in [10, Th. 4.7] we have the same conclusion in the case of continuous mappings satisfying condition (C_λ) for some $\lambda \in (0,1)$. Denote by

$$r(K, (x_n)) = \inf\{\limsup_{n\to\infty} \|x_n - x\| : x \in K\}$$

the asymptotic radius of a sequence (x_n) relative to K.

Lemma 2.3. *Let K be a nonempty convex weakly compact subset of a Banach space X which is minimal invariant under $T : K \to K$. If T is continuous and satisfies condition (C_λ) for some $\lambda \in (0,1)$, then there exists an approximate fixed point sequence (x_n) for T such that*

$$\lim_{n\to\infty} \|x_n - x\| = \inf\{r(K, (y_n)) : (y_n) \text{ is an afps in } K\}$$

for every $x \in K$. In the case $\lambda = \frac{1}{2}$ continuity assumption can be dropped.

Now let (x_n) and (x'_n) be sequences in K. Fix $t \in (\frac{2}{3}, 1)$ and put $v_n = tx_n + (1-t)x'_n$. The following technical lemma deals with the behaviour of sequences $(T^k_\gamma v_n)_{n\in\mathbb{N}}$, $k = 1, 2, ...$

Lemma 2.4. *Let K be a nonempty convex bounded subset of Banach lattice X such that $0 \in K$, $\operatorname{diam}K \geq 1$ and let $T : K \to K$ satisfy condition (C_λ) for some $\lambda \in (0, 1)$. Fix $\gamma \in [\lambda, 1)$, a positive integer $N > 1$, $0 < \varepsilon < \min\{\frac{2}{3(N+2)}, \frac{1}{12N}\}$ and $\frac{2}{3} + 2N\varepsilon < t < 1 - 2N\varepsilon$. Suppose that (x_n), (x'_n) are sequences in K such that $\operatorname{diam}((x_n) \cup (x'_n)) = 1$, $\lim_{n\to\infty} \||x_n| \wedge |x'_n|\| = 0$ and the following conditions are satisfied for every $n \in \mathbb{N}$ and $k = 1, ..., N$:*

(i) $\min\{\|x_n\|, \|x_n - T^k_\gamma 0\|, \|x'_n\|, \|x'_n - T^k_\gamma 0\|, \|x_n - x'_n\|\} > 1 - \varepsilon$,

(ii) $\|Tx_n - x_n\| < \varepsilon, \|Tx'_n - x'_n\| < \varepsilon$.

Let $v_n = tx_n + (1-t)x'_n$. Then, there exists $n_0 \in \mathbb{N}$ such that for every $n \in \mathbb{N}, n \geq n_0$ and $k = 1, ..., N$,

$$t - (k+1)\varepsilon < \left\|T^k_\gamma v_n - T^k_\gamma 0\right\| \leq t + \varepsilon, \tag{2.1}$$

$$1 - t - (k+1)\varepsilon < \left\|T^k_\gamma v_n - x_n\right\| < 1 - t + k\varepsilon. \tag{2.2}$$

$$t - (k+1)\varepsilon < \left\|T^k_\gamma v_n - x'_n\right\| < t + k\varepsilon.$$

Proof. Notice that $\lim_{n\to\infty} \|(t|x_n|) \wedge ((1-t)|x'_n|)\| = 0$ and hence

$$\begin{aligned}\lim_{n\to\infty} \|v_n\| = \lim_{n\to\infty} \|tx_n + (1-t)x'_n\| &= \lim_{n\to\infty} \|t|x_n| + (1-t)|x'_n|\| \\ \leq t \lim_{n\to\infty} \||x_n| + |x'_n|\| &= t \lim_{n\to\infty} \|x_n - x'_n\| \leq t.\end{aligned}$$

On the other hand

$$\begin{aligned}\|v_n\| =& \|(1-t)(x'_n - x_n) + x_n\| \geq \|x_n\| - (1-t)\|x'_n - x_n\| \\ >& 1 - \varepsilon - (1-t) = t - \varepsilon.\end{aligned}$$

Hence there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$

$$t - \varepsilon < \|v_n\| \leq t + \varepsilon.$$

Fix $n \geq n_0$ and note that

$$1 - t - \varepsilon < \|x_n - v_n\| = (1-t)\|x_n - x'_n\| \leq 1 - t,$$

$$t - \varepsilon < \|x'_n - v_n\| = t\|x_n - x'_n\| \leq t$$

Since

$$\lambda\|Tx_n - x_n\| < \|Tx_n - x_n\| < \varepsilon < 1 - t - \varepsilon < \|x_n - v_n\|, \ (t < 1 - 2\varepsilon),$$

it follows from condition (C_λ) that

$$\|Tx_n - Tv_n\| \leq \|x_n - v_n\|.$$

Hence

$$\|T_\gamma x_n - T_\gamma v_n\| \leq \gamma\|Tx_n - Tv_n\| + (1-\gamma)\|x_n - v_n\| \leq \|x_n - v_n\| \leq 1 - t \tag{2.3}$$

and, similarly,

$$\lambda\|Tx'_n - x'_n\| < \|Tx'_n - x'_n\| < \varepsilon < t - \varepsilon < \|x'_n - v_n\|, \ (\varepsilon < \frac{t}{2}),$$

$$\|Tx'_n - Tv_n\| \le \|x'_n - v_n\|,$$

so

$$\|T_\gamma x'_n - T_\gamma v_n\| \le \gamma\|Tx'_n - Tv_n\| + (1-\gamma)\|x'_n - v_n\| \le \|x'_n - v_n\| \le t. \tag{2.4}$$

Furthermore,

$$\begin{aligned}\|T_\gamma v_n - v_n\| = \gamma\|Tv_n - v_n\| &\le \|Tv_n - Tx_n\| + \|Tx_n - x_n\| + \|x_n - v_n\| \\ &< 2\|x_n - v_n\| + \varepsilon \le 2(1-t) + \varepsilon.\end{aligned}$$

Now we proceed by induction on k.

For $k = 1$, notice that

$$\frac{\lambda}{\gamma}\|T_\gamma v_n - v_n\| \le \|T_\gamma v_n - v_n\| < 2(1-t) + \varepsilon < t - 2\varepsilon < \|v_n\|,\ (t > \frac{2}{3} + \varepsilon),$$

and it follows from condition $(C_{\frac{\lambda}{\gamma}})$ that

$$\|T_\gamma v_n - T_\gamma 0\| \le \|v_n\| \le t + \varepsilon,$$

Furthermore,

$$\|T_\gamma v_n - x_n\| \le \|T_\gamma v_n - T_\gamma x_n\| + \|T_\gamma x_n - x_n\| < 1 - t + \varepsilon \tag{2.5}$$

by (2.3) and, similarly, by (2.4)

$$\|T_\gamma v_n - x'_n\| \le \|T_\gamma v_n - T_\gamma x'_n\| + \|T_\gamma x'_n - x'_n\| < t + \varepsilon. \tag{2.6}$$

To prove the reverse inequalities, notice that by assumption, (2.5) and (2.6),

$$\|T_\gamma v_n - T_\gamma 0\| \ge \|x_n - T_\gamma 0\| - \|T_\gamma v_n - x_n\| > t - 2\varepsilon,$$

$$\|T_\gamma v_n - x_n\| \ge \|x_n - x'_n\| - \|T_\gamma v_n - x'_n\| > 1 - t - 2\varepsilon,$$

$$\|T_\gamma v_n - x'_n\| \ge \|x_n - x'_n\| - \|T_\gamma v_n - x_n\| > t - 2\varepsilon.$$

Now suppose the lemma is true for a fixed $k < N$. Then

$$\left\|T_\gamma^{k+1} v_n - T_\gamma^{k+1} 0\right\| \le \left\|T_\gamma^k v_n - T_\gamma^k 0\right\| \le t + \varepsilon, \tag{2.7}$$

since for every $m \in \mathbb{N}$

$$\frac{\lambda}{\gamma}\|T_\gamma^m v_n - T_\gamma^{m-1} v_n\| \le \|T_\gamma^{m-1} v_n - T_\gamma^m v_n\|$$

and it follows from the fact that T_γ satisfies condition $(C_{\frac{\lambda}{\gamma}})$ that for every $m \in \mathbb{N}$

$$\|T_\gamma^{m+1} v_n - T_\gamma^m v_n\| \le \|T_\gamma^m v_n - T_\gamma^{m-1} v_n\|,$$

so

$$\begin{aligned}\left\|T_\gamma T_\gamma^k v_n - T_\gamma^k v_n\right\| &\le \left\|T_\gamma^k v_n - T_\gamma^{k-1} v_n\right\| \le ... \le \|T_\gamma v_n - v_n\| \\ &< 2(1-t) + \varepsilon < t - (k+1)\varepsilon < \left\|T_\gamma^k v_n - T_\gamma^k 0\right\|,\end{aligned}$$

(notice that $t > \frac{2}{3} + \frac{(k+2)\varepsilon}{3}$). Furthermore, by induction assumption,

$$\frac{\lambda}{\gamma}\|T_\gamma x_n - x_n\| \le \|T_\gamma x_n - x_n\| < \varepsilon < 1 - t - (k+1)\varepsilon < \|x_n - T_\gamma^k v_n\|,$$

$(t < 1 - (k+2)\varepsilon)$, and hence

$$\|T_\gamma^{k+1} v_n - T_\gamma x_n\| \le \|T_\gamma^k v_n - x_n\|.$$

We thus get

$$\begin{aligned} \left\|T_\gamma^{k+1} v_n - x_n\right\| &\le \|T_\gamma^{k+1} v_n - T_\gamma x_n\| + \|T_\gamma x_n - x_n\| \\ &< \|T_\gamma^k v_n - x_n\| + \varepsilon < 1 - t + (k+1)\varepsilon. \end{aligned} \tag{2.8}$$

Similarly,

$$\frac{\lambda}{\gamma}\|T_\gamma x_n' - x_n'\| \le \|T_\gamma x_n' - x_n'\| < \varepsilon < t - (k+1)\varepsilon < \|x_n' - T_\gamma^k v_n\|,$$

$(\varepsilon < \frac{2}{3(N+2)} < \frac{t}{k+2})$, hence

$$\|T_\gamma^{k+1} v_n - T_\gamma x_n'\| \le \|T_\gamma^k v_n - x_n'\|,$$

and

$$\begin{aligned} \left\|T_\gamma^{k+1} v_n - x_n'\right\| &\le \|T_\gamma^{k+1} v_n - T_\gamma x_n'\| + \|T_\gamma x_n - x_n'\| \\ &< \|T_\gamma^k v_n - x_n'\| + \varepsilon < t + (k+1)\varepsilon. \end{aligned}$$

Now we prove the reverse inequalities

$$\begin{aligned} \left\|T_\gamma^{k+1} v_n - T_\gamma^{k+1} 0\right\| &\ge \|x_n - T_\gamma^{k+1} 0\| - \|T_\gamma^{k+1} v_n - x_n\| > t - (k+2)\varepsilon, \\ \left\|T_\gamma^{k+1} v_n - x_n\right\| &\ge \|x_n - x_n'\| - \|T_\gamma^{k+1} v_n - x_n'\| > 1 - t - (k+2)\varepsilon, \\ \left\|T_\gamma^{k+1} v_n - x_n'\right\| &\ge \|x_n - x_n'\| - \|T_\gamma^{k+1} v_n - x_n\| > t - (k+2)\varepsilon. \end{aligned}$$

In the sequel we will need the following lemma.

Lemma 2.5. *Let K be a convex weakly compact subset of a Banach lattice X such that $0 \in K$. Suppose that a mapping $T : K \to K$ satisfies condition (C_λ) for some $\lambda \in (0,1)$, (x_n) and (x_n') are weakly null, approximate fixed point sequences for T such that*

$$r = \lim_{n\to\infty} \|x_n - x\| = \lim_{n\to\infty} \|x_n' - x\| = \inf\{r(K,(y_n)) : (y_n) \text{ is an afps in } K\} \tag{2.9}$$

$$\lim_{n\to\infty} \||x_n| \wedge |x_n'|\| = 0 \tag{2.10}$$

for every $x \in K$. Then, for every $\varepsilon > 0$ and $t \in (\frac{2}{3}, 1)$, there exist subsequences of (x_n) and (x_n'), denoted again (x_n) and (x_n'), sequence (z_n) in K and element $z \in K$ such that

(i) $\|z_n\| > r(1-\varepsilon)$,
(ii) $\|z_n - x_n\| \le r(1 - t + \varepsilon)$,
(iii) $\|z_n - x_n'\| \le r(t+\varepsilon)$,
(iv) $\|z_n - z\| \le r(t+\varepsilon)$.

Proof. Let us first notice that if $S : \frac{1}{r}K \to \frac{1}{r}K$ is defined by $Sy = \frac{1}{r}T(ry)$, then

$$\|Sy - y\| = \frac{1}{r}\|T(ry) - ry\|$$

and S satisfies condition (C_λ). It follows that if the sequences $(x_n), (x'_n)$ satisfy the assumptions of Lemma 2, then the sequences $(\frac{x_n}{r}), (\frac{x'_n}{r})$ satisfy these assumptions with S and $r = 1$. Therefore it suffices to prove the lemma for $r = 1$.

We claim that for every $\varepsilon > 0$ there exists $\delta(\varepsilon)$ such that if $x \in K$ and $\|Tx - x\| < \delta(\varepsilon)$ then $\|x\| > 1-\varepsilon$. Indeed, otherwise, arguing as in [5], there exists ε_0 such that we can find $w_n \in K$ with $\|Tw_n - w_n\| < \frac{1}{n}$ and $\|w_n\| \le 1-\varepsilon_0$ for every $n \in \mathbb{N}$. Then the sequence (w_n) is an approximate fixed point sequence in K, but $\limsup_{n\to\infty} \|w_n\| \le 1 - \varepsilon_0$, which contradicts our assumption that $\limsup_{n\to\infty} \|w_n\| \ge 1$.

Fix $\varepsilon > 0$, $t \in \left(\frac{2}{3}, 1\right)$ and $\gamma \in [\lambda, 1)$. From Theorem 2, there exists $N > 1$ such that

$$\|T_\gamma^{N+1}x - T_\gamma^N x\| < \gamma\delta(\varepsilon) \tag{2.11}$$

for every $x \in K$. Choose $\eta > 0$ so small that $0 < \eta < \min\left\{\frac{2}{3(N+2)}, \frac{\varepsilon}{N}, \frac{1}{12N}\right\}$ and $\frac{2}{3} + 2N\eta < t < 1 - 2N\eta$. Put $v_n = tx_n + (1-t)x'_n$ and consider sequences $(T_\gamma^k v_n)_{n\in\mathbb{N}}$ for $k = 1, ..., N$. Applying (2.9) (with $r = 1$) and passing to subsequences, we can assume that the assumptions of Lemma 2 are satisfied i.e., $\operatorname{diam}((x_n) \cup (x'_n)) = 1$, $\lim_{n\to\infty} \||x_n| \wedge |x'_n|\| = 0$ and for every $n \in \mathbb{N}$ and $k = 1, ..., N$,

(i) $\min\{\|x_n\|, \|x_n - T_\gamma^k 0\|, \|x'_n\|, \|x'_n - T_\gamma^k 0\|, \|x_n - x'_n\|\} > 1 - \eta$,
(ii) $\|Tx_n - x_n\| < \eta, \|Tx'_n - x'_n\| < \eta$.

Denote $z_n = T_\gamma^N v_n$ and $z = T_\gamma^N 0$. It follows from Lemma 2 that that there exists $n_0 \in \mathbb{N}$ such that for every $n \ge n_0, n \in \mathbb{N}$, we have

$$\|z_n - z\| = \|T_\gamma^N v_n - T_\gamma^N 0\| \le t + \eta < t + \varepsilon,$$

$$\|z_n - x'_n\| = \|T_\gamma^N v_n - x'_n\| < t + N\eta < t + \varepsilon.$$

$$\|z_n - x_n\| = \|T_\gamma^N v_n - x_n\| < 1 - t + N\eta < 1 - t + \varepsilon.$$

Furthermore, by (2.11),

$$\|Tz_n - z_n\| = \frac{1}{\gamma}\|T_\gamma^{N+1} v_n - T_\gamma^N v_n\| < \delta(\varepsilon)$$

and consequently, $\|z_n\| > 1 - \varepsilon$, which completes the proof.

3. Fixed point theorem

In [3] J. Borwein and B. Sims introduced the notation of weakly orthogonal Banach lattice.

Definition 3.1. We will say that a Banach lattice X is weakly orthogonal if whenever (x_n) converges weakly to 0 we have

$$\lim_{n\to\infty} \||x_n| \wedge |x|\| = 0, \text{ for all } x \in X.$$

The proof of the following inequality we can find in [12].

Lemma 3.2. *Let X be a weakly orthogonal Banach lattice and let (u_n),(v_n) be weakly null sequences in X such that*

$$\lim_{n\to\infty} \||u_n| \wedge |v_n|\| = 0.$$

Then for every sequence (w_n) *in* X *and for every* x *in* X

$$2\limsup_{n\to\infty}\|w_n\| \le \limsup_{n\to\infty}\|w_n - x\| + \limsup_{n\to\infty}\|w_n - u_n\| + \limsup_{n\to\infty}\|w_n - v_n\|.$$

B. Sims in [11] proved that every weakly orthogonal Banach lattice has the weak fixed point property. Now we generalize his result.

Theorem 3.3. *Every weakly orthogonal Banach lattice has the weak fixed point property for continuous mappings satisfying condition* (C_λ) *for some* $\lambda \in (0,1)$. *In the case* $\lambda = \frac{1}{2}$ *continuity assumption can be dropped.*

Proof. Assume that theorem is false. Then there exists a nonempty weakly compact convex subset K of X and a mapping $T : K \to K$ satisfying condition (C) or a continuous mapping satisfying condition (C_λ) for some $\lambda \in (0,1)$ which has no fixed point. We can assume that K is minimal and T-invariant. By Lemma 2 there exists an approximate fixed point sequence (x_n) for T in K such that

$$r = \lim_{n\to\infty}\|x_n - x\| = \inf\{r(K,(y_n)) : (y_n) \text{is an afps in } K\}$$

for every $x \in K$. There is no loss of generality in assuming that $r = 1$ and (x_n) converges weakly to $0 \in K$.

We can find subsequences (u_n) and (u'_n) of (x_n) such that

$$\lim_{n\to\infty}\|u_n - u'_n\| = 1 \text{ and } \lim_{n\to\infty}\||u_n| \wedge |u'_n|\| = 0.$$

Fix $\varepsilon > 0$ and $t \in (\frac{2}{3}, 1)$ such that $5\varepsilon + t < 1$. Then from Lemma 2 there exist subsequences of (u_n) and (u'_n), denoted again (u_n) and (u'_n), sequence (z_n) in K and $z \in K$ such that for large n

(i) $\|z_n\| > 1 - \varepsilon$,
(ii) $\|z_n - x_n\| \le 1 - t + \varepsilon$,
(iii) $\|z_n - x'_n\| \le t + \varepsilon$,
(iv) $\|z_n - z\| \le t + \varepsilon$.

By Lemma 3

$$2\limsup_{n\to\infty}\|z_n\| \le \limsup_{n\to\infty}\|z_n - x_n\| + \limsup_{n\to\infty}\|z_n - x'_n\| + \limsup_{n\to\infty}\|z_n - z\|.$$

This, in turn, implies that

$$2(1-\varepsilon) \le 1 + t + 3\varepsilon,$$

which contradicts $5\varepsilon + t < 1$.

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