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FIXED POINT ON A CLOSED BALL IN ORDERED DISLOCATED QUASI METRIC SPACES

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Abstract. Sufficient conditions for the existence of fixed point for mappings satisfying locally contractive conditions on a closed ball in an ordered left K-sequentially as well as right K-sequentially complete dislocated quasi metric space have been obtained. The notion of dominated mappings is applied to approximate the unique solution to non linear functional equations. Our results improve several well known existing results.
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1. Introduction

Recently, many results appeared related to fixed point theorem in complete metric spaces endowed with a partial ordering in literature. Ran and Reurings [24] proved an analogue of Banach's fixed point theorem in metric space endowed with a partial order and gave applications to matrix equations. In this way, they weakened the usual contractive condition. Subsequently, Nieto et al. [21] extended this result in [24] for nondecreasing mappings and applied it to obtain a unique solution for a 1st order ordinary differential equation with periodic boundary conditions. Thereafter, many work related to fixed point problems have also been considered in partially ordered metric spaces (see [4, 8, 9, 10, 11, 13, 20]).

On the other hand notion of a partial metric space was introduced by Matthews [19]. To generalize partial metric, Hitzler and Seda [15] introduce the concept of a dislocated topology and its corresponding generalized metric named as dislocated metric (metric-like space [3]). The notion of dislocated topology has useful applications in the context of logic programming semantics (see [14, 16]). Further useful results can be seen in [1, 3, 17, 26, 27]. Zeyada et. al. [28] introduced the concept of dislocated quasi metric space.

From the application point of view the situation is not yet completely satisfactory because it frequently happens that a mapping T is a contraction not on the entire space X but merely on a subset Y of X. However, if Y is closed and a Picard iterative sequence $\{x_n\}$ in X converges to some x in X, then, by imposing a subtle restriction on the choice of x_0, one may force Picard iterative sequence to stay eventually in Y. In this case, closedness of Y coupled with some suitable contractive condition establish the existence of a fixed point of T. Arshad et al. [5] obtained a significant result concerning the existence of fixed points of a mapping satisfying a contractive conditions on a closed ball of a complete dislocated metric space. Other results can also be seen in [6, 7]. The dominated mapping [2], which satisfies the condition $fx \preceq x$ occurs very naturally in several practical problems. For example if x denotes the total quantity of food produced over a certain period of time and $f(x)$ gives the quantity of food consumed over the same period in a certain town, then we must have $fx \preceq x$. In this paper, we have obtained fixed point theorems on a closed ball for a contractive dominated self-mapping in an ordered left K-sequentially as well as right K-sequentially complete dislocated quasi metric space. Our results generalize, extend and improve a classical fixed point result on a closed ball (see [18]). We have used weaker contractive conditions and weaker restrictions to obtain a unique fixed point. We have given examples which show how these results can be used for some mappings, when the corresponding results in quasi-metric spaces can not hold .

We give the following definitions and results which will be needed in the sequel.

Definition 1.1. [28] Let X be a nonempty set and let $d_q : X \times X \to [0, \infty)$ be a function, called a dislocated quasi metric (or simply d_q-metric) if the following conditions hold for any $x, y, z \in X$:

(i) If $d_q(x, y) = d_q(y, x) = 0$, then $x = y$,

(ii) $d_q(x, y) \leq d_q(x, z) + d_q(z, y)$.

The pair (X, d_q) is called a dislocated quasi metric space.

It is clear that if $d_q(x, y) = d_q(y, x) = 0$, then from (i), $x = y$. But if $x = y$, $d_q(x, y)$ may not be 0. It is observed that if $d_q(x, y) = d_q(y, x)$ for all $x, y \in X$, then (X, d_q) becomes a dislocated metric space (metric-like space). We will denote (X, d_l) a dislocated metric space. For $x \in X$ and $\varepsilon > 0$, $\overline{B}(x, \varepsilon) = \{y \in X : d_q(x, y) \leq \varepsilon\}$ is a closed ball in X.

Example 1.2. If $X = R^+ \cup \{0\}$ then $d_q(x, y) = x + \max\{x, y\}$ defines a dislocated quasi metric d_q on X.

Reilly et al. [25] introduced the notion of left (right) K-Cauchy sequence and left (right) K-sequentially complete spaces(see also [7, 12]). We use this concept to introduce the following definition.

Definition 1.3. Let (X, d_q) be a dislocated quasi metric space. A sequence $\{x_n\}$ in (X, d_q) is called

(a) left (right) K-Cauchy if $\forall\ \varepsilon > 0$, $\exists\ n_0 \in N$ such that $\forall\ n > m \geq n_0$, $d_q(x_m, x_n) < \varepsilon$ (respectively $d_q(x_n, x_m) < \varepsilon$);

(b) dislocated quasi-convergent (for short d_q-convergent) [28] to x if

$$\lim_{n\to\infty} d_q(x_n, x) = \lim_{n\to\infty} d_q(x, x_n) = 0.$$

In this case x is called a d_q-limit of $\{x_n\}$.

A dislocated quasi metric space (X, d_q) is called left (right) K-sequentially complete if every left (right) K-Cauchy sequence in it is d_q -convergent.

Definition 1.4. [23] Let $(X, \preceq)$ be a partially ordered set. Then $x, y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$ holds.

Definition 1.5. [5] Let $(X, \preceq)$ be a partially ordered set. A self mapping f on X is called dominated if $fx \preceq x$ for each x in X.

Example 1.6. [5] Let $X = [0, 1]$ be endowed with the usual ordering and $f : X \to X$ be defined by $fx = x^n$ for some $n \in \mathbb{N}$. Since $fx = x^n \leq x$ for all $x \in X$, therefore f is a dominated map.

Definition 1.7. Let X be a nonempty set, then $(X, \preceq, d_q)$ is called an ordered dislocated quasi metric space if:

(i) d_q is a dislocated quasi metric on X , and

(ii) $\preceq$ is a partial order on X.

2. Main results

Theorem 2.1. *Let $(X, \preceq, d_q)$ be an ordered left K-sequentially complete dislocated quasi metric space, $S : X \to X$ be a dominated map and x_0 be an arbitrary point in X. Suppose there exists $k \in [0, 1)$ such that*

$$d_q(Sx, Sy) \leq kd_q(x, y), \text{ for all comparable elements } x, y \text{ in } \overline{B}(x_0, r), \tag{2.1}$$

and

$$d_q(x_0, Sx_0) \leq (1 - k)r \ . \tag{2.2}$$

If, for every nonincreasing sequence $\{x_n\} \to u$ implies that $u \preceq x_n$. Then there exists a point x^* in $\overline{B}(x_0, r)$ such that $x^* = Sx^*$ and $d_q(x^*, x^*) = 0$.

Moreover, if for any two points x, y in $\overline{B}(x_0, r)$ there exists a point $z \in \overline{B}(x_0, r)$ such that $z \preceq x$ and $z \preceq y$, that is, every pair of elements in $\overline{B}(x_0, r)$ has a lower bound, then, the point x^* is the unique fixed point of S.

Proof. Consider a Picard sequence $x_{n+1} = Sx_n$ with initial guess x_0 satisfying (2.2). Then $x_{n+1} = Sx_n \preceq x_n$ for all $n \in \{0\} \cup N$. Now by the inequality (2.2),

$$d_q(x_0, Sx_0) \leq (1 - k)r \leq r,$$

so that $x_1 \in \overline{B}(x_0, r)$. Let $x_2, \cdots, x_j \in \overline{B}(x_0, r)$ for some $j \in N$. Using the inequality (2.1), we obtain,

$$\begin{aligned} d_q(x_j, x_{j+1}) &= d_q(Sx_{j-1}, Sx_j) \leq kd_q(x_{j-1}, x_j) \\ &\leq k^2 d_q(x_{j-2}, x_{j-1}) \leq \cdots \leq k^j d_q(x_0, x_1). \end{aligned} \tag{2.3}$$

Now by using the inequalities (2.2) and (2.3) we obtain,

$$\begin{aligned} d_q(x_0, x_{j+1}) &\leq d_q(x_0, x_1) + d_q(x_1, x_2) + \cdots + d_q(x_j, x_{j+1}) \\ &\leq (1 - k)r \frac{(1 - k^{j+1})}{1 - k} \leq r. \end{aligned}$$

Thus $x_{j+1} \in \overline{B}(x_0, r)$. Hence, $x_n \in \overline{B}(x_0, r)$, for all $n \in N$. Now the inequality (2.3) can be written as,

$$d_q(x_n, x_{n+1}) \leq k^n d_q(x_0, x_1), \text{ for all } n \in N. \tag{2.4}$$

By the inequality (2.4) we get,

$$\begin{aligned} d_q(x_n, x_{n+i}) &\leq d_q(x_n, x_{n+1}) + ... + d_q(x_{n+i-1}, x_{n+i}) \\ &\leq \frac{k^n(1-k^i)}{1-k} d_q(x_0, x_1) \longrightarrow 0 \text{ as } n \to \infty. \end{aligned}$$

Therefore the sequence $\{x_n\}$ is a left K-Cauchy sequence in $(\overline{B}(x_0, r), d_q)$. As $\overline{B}(x_0, r)$ is closed, it is left K-sequentially complete. Therefore, there exists a point $x^* \in \overline{B}(x_0, r)$ with

$$\lim_{n\to\infty} d_q(x_n, x^*) = \lim_{n\to\infty} d_q(x^*, x_n) = 0. \tag{2.5}$$

Now,

$$d_q(x^*, Sx^*) \leq d_q(x^*, x_n) + d_q(x_n, Sx^*).$$

By assumptions $x^* \preceq x_n \preceq x_{n-1}$, therefore,

$$d_q(x^*, Sx^*) \leq \lim_{n\to\infty} [d_q(x^*, x_n) + k d_q(x_{n-1}, x^*)].$$

Thus,

$$d_q(x^*, Sx^*) \leq 0.$$

Similarly, $d_q(Sx^*, x^*) \leq 0$. Hence $x^* = Sx^*$. Now,

$$d_q(x^*, x^*) = d_q(Sx^*, Sx^*) \leq k d_q(x^*, x^*).$$

This implies that

$$d_q(x^*, x^*) = 0.$$

Uniqueness: Let y be another point in $\overline{B}(x_0, r)$ such that, $y = Sy$. If x^* and y are comparable then,

$$d_q(x^*, y) = d_q(Sx^*, Sy) \leq k d_q(x^*, y).$$

Similarly, $d_q(y, x^*) \leq 0$. This shows that $x^* = y$. Now if x^* and y are not comparable then there exists a point $z \in \overline{B}(x_0, r)$ which is lower bound of both x^* and y that is $z \preceq x^*$ and $z \preceq y$. Moreover by assumptions $x^* \preceq x_n$ as $x_n \to x^*$. Therefore $z \preceq x^* \preceq x_n \preceq ... \preceq x_0$.

$$\begin{aligned} d_q(x_0, Sz) &\leq d_q(x_0, x_1) + d_q(x_1, Sz) \\ &\leq (1-k)r + k d_q(x_0, z), \text{ (by (2.1) and (2.2))} \\ d_q(x_0, Sz) &\leq (1-k)r + kr \leq r. \end{aligned}$$

It follows that $Sz \in \overline{B}(x_0, r)$. Now we will prove that $S^n z \in \overline{B}(x_0, r)$, by using mathematical induction. Let $S^2 z, ..., S^j z \in \overline{B}(x_0, r)$ for some $j \in N$. As $S^j z \preceq S^{j-1} z \preceq ... \preceq z \preceq x^* \preceq x_n ... \preceq x_0$, then,

$$\begin{aligned} d_q(x_{j+1}, S^{j+1} z) &= d_q(Sx_j, S(S^j z)) \leq k d_q(x_j, S^j z) \\ &\leq \ldots \leq k^{j+1} d_q(x_0, z_0). \end{aligned} \tag{2.6}$$

Now,

$$\begin{aligned} d_q(x_0,S^{j+1}z) &\le d_q(x_0,x_1)+d_q(x_1,x_2)+...+d_q(x_j,x_{j+1})+d_q(x_{j+1},S^{j+1}z)\\ &\le d_q(x_0,x_1)+kd_q(x_0,x_1)+...+k^{j+1}d_q(x_0,z_0),\ \text{(by (2.6))}\\ d_q(x_0,S^{j+1}z) &\le d_q(x_0,x_1)[1+k+...k^j]+k^{j+1}r,\ (\text{as } z_0\in\overline{B}(x_0,r))\\ d_q(x_0,S^{j+1}z) &\le (1-k)r\frac{(1-k^{j+1})}{1-k}+k^{j+1}r=r. \end{aligned}$$

It follows that $S^{j+1}z\in\overline{B}(x_0,r)$ and thus $S^nz\in\overline{B}(x_0,r)$ for all n. Now,

$$\begin{aligned} d_q(x^*,y) &= d_q(S^nx^*,S^ny)\\ &\le d_q(S^nx^*,S^{n-1}z)+d_q(S^{n-1}z,S^ny). \end{aligned}$$

As $S^{n-1}z\preceq S^{n-2}z\preceq ...\preceq z\preceq x^*$ and $S^{n-1}z\preceq y$ for all $n\in N$. It further implies that $S^{n-1}z\preceq S^nx^*$ and $S^{n-1}z\preceq S^ny$ for all $n\in N$ as $S^nx^*=x^*$ and $S^ny=y$ for all $n\in N$. Thus,

$$\begin{aligned} d_q(x^*,y) &\le kd_q(S^{n-1}x^*,S^{n-2}z)+kd_q(S^{n-2}z,S^{n-1}y)\ \text{(by (2.1))}\\ &\ \vdots\\ &\le k^{n-2}d_q(x^*,Sz)+k^{n-2}d_q(Sz,y)\longrightarrow 0 \text{ as } n\to\infty \end{aligned}$$

Similarly $d_q(y,x^*)\le 0$. Hence $x^*=y$.

Example 2.2. Let $X=[0,+\infty)\cap Q$ with the dislocated quasi-metric d_q given by $d_q(x,y)=2x+y$ and the order $x\preceq y$ iff $d_q(x,x)\le d_q(y,y)$. Then $(X,\preceq,d_q)$ be an ordered complete dislocated quasi metric space. Let $S:X\to X$ be defined by

$$Sx=\begin{cases} \dfrac{x}{7} \text{ if } x\in[0,1]\cap X\\ x-\frac{1}{3} \text{ if } x\in(1,\infty)\cap X \end{cases}$$

Clearly, S is a dominated mapping. Then, if $x_0=1$, $r=3$, we have $\overline{B}(x_0,r)=[0,1]\cap X$ and for $k=\frac{1}{5}$,

$$(1-k)r=(1-\frac{1}{5})3=\frac{12}{5}.$$

Also

$$d_q(x_0,Sx_0)=d_q(1,S1)=d_q(1,\frac{1}{7})=2+\frac{1}{7}=\frac{15}{7}<\frac{12}{5}$$

Now if $x,y\in(1,\infty)\cap X$, then,

$$\begin{aligned} d_q(Sx,Sy) &= 2x-\frac{2}{3}+y-\frac{1}{3}\\ &\ge \frac{1}{5}\{2x+y\}\\ d_q(Sx,Sy) &\ge kd_q(x,y). \end{aligned}$$

So the contractive condition does not hold on X. Now if $x,y\in\overline{B}(x_0,r)$, then

$$\begin{aligned} d_q(Sx,Sy) &= \frac{2x}{7}+\frac{y}{7}=\frac{1}{7}\{2x+y\}\\ &\le \frac{1}{5}\{2x+y\}=kd_q(x,y). \end{aligned}$$

Therefore, all the conditions of Theorem 2.1 are satisfied. Moreover, 0 is the unique fixed point of S.

In Theorem 2.1, the condition (2.2) is imposed to restrict the condition (2.1) only for x, y in $\overline{B}(x_0, r)$. Example 2.2 explains the utility of this restriction. The following result relax the conditions (2.2) but impose the condition (2.1) for all comparable elements in the whole space X.

Corollary 2.3. *Let $(X, \preceq, d_q)$ be an ordered left K-sequentially complete dislocated quasi metric space, $S : X \to X$ be a dominated map and x_0 be an arbitrary point in X. Suppose there exists $k \in [0, 1)$ with*

$$d_q(Sx, Sy) \leq kd_q(x, y), \text{ for all comparable elements } x, y \text{ in } X$$

If, for every nonincreasing sequence $\{x_n\}$in X, $\{x_n\} \to u$ implies that $u \preceq x_n$ and every pair of elements in X has a lower bound, then there exists a unique point x^* in X such that $x^* = Sx^*$. Further $d_q(x^*, x^*) = 0$.

In Theorem 2.1, the existence of a lower bound and for every nonincreasing sequence $\{x_n\}$in X, $\{x_n\} \to u$ implies that $u \preceq x_n$ are imposed to restrict the condition (2.1) only for comparable elements. However, the following result relaxes these conditions but imposes the condition (2.1) for all elements in $\overline{B}(x_0, r)$.

Corollary 2.4. *Let (X, d_q) be a left K-sequentially complete dislocated quasi metric space, $S : X \to X$ be a map and x_0 be an arbitrary point in X. Suppose there exists $k \in [0, 1)$ with*

$$d_q(Sx, Sy) \leq kd_q(x, y), \text{ for all elements } x, y \text{ in } \overline{B}(x_0, r)$$

and

$$d_q(x_0, Sx_0) \leq (1 - k)r \ .$$

then there exists a unique point x^* in $\overline{B}(x_0, r)$ such that $x^* = Sx^*$. Further $d_q(x^*, x^*) = 0$.

In the following we present some results for the Kannan mappings and obtain a unique fixed point on a closed ball in an ordered dislocated quasi metric space.

Theorem 2.5. *Let $(X, \preceq, d_q)$ be an ordered left K-sequentially complete dislocated quasi metric space, $S : X \to X$ be a dominated map and x_0 be an arbitrary point in X. Suppose there exists $k \in [0, \frac{1}{2})$ with*

$$d_q(Sx, Sy) \leq k[d_q(x, Sx) + d_q(y, Sy)], \tag{2.7}$$

for all comparable elements x, y in $\overline{B}(x_0, r)$ and

$$d_q(x_0, Sx_0) \leq (1 - \theta)r, \tag{2.8}$$

where $\theta = \frac{k}{1-k}$. If for every nonincreasing sequence $\{x_n\} \to u$ implies that $u \preceq x_n$. Then there exists a point x^* in $\overline{B}(x_0, r)$ such that $x^* = Sx^*$ and $d_q(x^*, x^*) = 0$. Moreover, if for any two points x, y in $\overline{B}(x_0, r)$ there exists a point $z \in \overline{B}(x_0, r)$ such that $z \preceq x$ and $z \preceq y$,and

$$d_q(x_0, Sx_0) + d_q(z, Sz) \leq d_q(x_0, z) + d_q(Sx_0, Sz) \text{ for all } z \preceq x_0. \tag{2.9}$$

then, the point x^* is unique.

Proof. Consider a Picard sequence $x_{n+1} = Sx_n$ with initial guess x_0 satisfying (2.8). Then $x_{n+1} = Sx_n \preceq x_n$ for all $n \in \{0\} \cup N$ and by the inequality (2.8), we have

$$d_q(x_0, Sx_0) \leq (1-\theta)r \leq r.$$

Therefore, $x_1 \in \overline{B}(x_0, r)$. Let $x_2, \cdots, x_j \in \overline{B}(x_0, r)$ for some $j \in N$. Thus, by the inequality (2.7), we have

$$d_q(x_j, x_{j+1}) = d_q(Sx_{j-1}, Sx_j) \leq k[d_q(x_{j-1}, Sx_{j-1}) + d_q(x_j, Sx_j)].$$

It implies that

$$\begin{aligned} d_q(x_j, x_{j+1}) &\leq \theta d_q(x_{j-1}, x_j) \\ &\leq \theta^2 d_q(x_{j-2}, x_{j-1}) \leq ... \leq \theta^j d_q(x_0, x_1). \end{aligned} \tag{2.10}$$

Now by the inequalities (2.8) and (2.10) we get,

$$\begin{aligned} d_q(x_0, x_{j+1}) &\leq d_q(x_0, x_1) + d_q(x_1, x_2) + \cdots + d_q(x_j, x_{j+1}) \\ &\leq (1-\theta)r \frac{(1-\theta^{j+1})}{(1-\theta)} \leq r. \end{aligned}$$

It gives that $x_{j+1} \in \overline{B}(x_0, r)$. Hence $x_n \in \overline{B}(x_0, r)$ for all $n \in N$. It further implies that the inequality (2.10) can be written as,

$$d_q(x_n, x_{n+1}) \leq \theta^n d_q(x_0, x_1), \text{ for all } n \in N. \tag{2.11}$$

By the inequality (2.11), we have,

$$\begin{aligned} d_q(x_n, x_{n+i}) &\leq d_q(x_n, x_{n+1}) + ... + d_q(x_{n+i-1}, x_{n+i}) \\ &\leq \frac{\theta^n(1-\theta^i)}{1-\theta} d_q(x_0, x_1) \longrightarrow 0 \text{ as } n \to \infty. \end{aligned}$$

Thus the sequence $\{x_n\}$ is a left K-Cauchy sequence in $(\overline{B}(x_0, r), d_q)$. Therefore there exists a point $x^* \in \overline{B}(x_0, r)$ with $\lim_{n\to\infty} x_n = x^*$. Also,

$$\lim_{n\to\infty} d_q(x_n, x^*) = \lim_{n\to\infty} d_q(x^*, x_n) = 0. \tag{2.12}$$

Now,

$$d_q(x^*, Sx^*) \leq d_q(x^*, x_n) + d_q(x_n, Sx^*),$$

by assumptions, $x^* \preceq x_n \preceq x_{n-1}$, therefore,

$$d_q(x^*, Sx^*) \leq \lim_{n\to\infty} [d_q(x^*, x_n) + k\{d_q(x_{n-1}, Sx_{n-1}) + d_q(x^*, Sx^*)\}].$$

By the inequality (2.12) we obtain

$$(1-k)d_q(x^*, Sx^*) \leq k \lim_{n\to\infty} d_q(x_{n-1}, x_n),$$

and by the inequality (2.11)

$$(1-k)d_q(x^*, Sx^*) \leq 0.$$

Similarly, $d_q(Sx^*, x^*) \leq 0$ and hence, $x^* = Sx^*$. Now,

$$\begin{aligned} d_q(x^*, x^*) &= d_q(Sx^*, Sx^*) \\ &\leq k\{d_q(x^*, Sx^*) + d_q(x^*, Sx^*)\}. \end{aligned}$$

Thus

$$(1-2k)d_q(x^*,x^*) \leq 0,$$

which implies

$$d_q(x^*,x^*) = 0. \tag{2.13}$$

Uniqueness: Now we show that x^* is unique. Let y be another point in $\overline{B}(x_0,r)$ such that $y = Sy$. Then following similar arguments as we have used to prove the inequality (2.12), we obtain,

$$d_q(y,y) = 0. \tag{2.14}$$

Now if x^* and y are comparable, then,

$$\begin{aligned} d_q(x^*,y) &= d_q(Sx^*,Sy) \\ &\leq k[d_q(x^*,Sx^*) + d_q(y,Sy)] \\ &= 0. \ \text{(by (2.13) and (2.14))} \end{aligned}$$

Similarly,

$$d_q(y,x^*) = 0.$$

Hence we have $x^* = y$. Now if x^* and y are not comparable then there exists a point $z \in \overline{B}(x_0,r)$ which is a lower bound of both x^* and y. Now we will prove that $S^n z \in \overline{B}(x_0,r)$. Moreover by assumptions $z \preceq x^* \preceq x_n ... \preceq x_0$. Now by the inequality (2.7) , we have,

$$\begin{aligned} d_q(Sx_0,Sz) &\leq k[d_q(x_0,x_1) + d_q(z,Sz)] \\ &\leq k[d_q(x_0,z) + d_q(x_1,Sz)], \quad \text{by using (2.9)} \\ d_q(x_1,Sz) &\leq \theta d_q(x_0,z). \end{aligned} \tag{2.15}$$

Now,

$$\begin{aligned} d_q(x_0,Sz) &\leq d_q(x_0,x_1) + d_q(x_1,Sz) \\ &\leq d_q(x_0,x_1) + \theta d_q(x_0,z), \quad \text{by using (2.15)} \\ d_q(x_0,Sz) &\leq (1-\theta)r + \theta r = r. \end{aligned}$$

It follows that $Sz \in \overline{B}(x_0,r)$. Next, we show that $S^n z \in \overline{B}(x_0,r)$, by using mathematical induction to apply the inequality (2.7). Let $S^2 z, ..., S^j z \in \overline{B}(x_0,r)$ for some $j \in N$. As $S^j z \preceq S^{j-1} z \preceq ... \preceq z \preceq x^* \preceq x_n ... \preceq x_0$, then,

$$d_q(S^j z, S^{j+1} z) = d_q(S(S^{j-1}z), S(S^j z)) \leq k[d_q(S^{j-1}z, S^j z) + d_q(S^j z, S^{j+1} z)].$$

It further implies that,

$$\begin{aligned} d_q(S^j z, S^{j+1} z) &\leq \theta d_q(S^{j-1} z, S^j z) \\ &\leq \theta^2 d_q(S^{j-2} z, S^{j-1} z) \leq ... \leq \theta^j d_q(z, Sz). \end{aligned} \tag{2.16}$$

Now,

$$\begin{aligned} d_q(x_{j+1}, S^{j+1} z) &= d_q(Sx_j, S(S^j z)) \leq k[d_q(x_j, Sx_j) + d_q(S^j z, S^{j+1} z)] \\ &\leq k[\theta^j d_q(x_0,x_1) + \theta^j d_q(z,Sz)] \quad \text{(by 2.11 and 2.16)} \\ &\leq k\theta^j [d_q(x_0,z) + d_q(x_1,Sz)] \quad \text{(by 2.9)} \\ &\leq k\theta^j [d_q(x_0,z) + \theta d_q(x_0,z)] = \theta^{j+1} d_q(x_0,z_0). \end{aligned} \tag{2.17}$$

Thus,

$$\begin{aligned} d_q(x_0, S^{j+1}z) &\le d_q(x_0, x_1) + d_q(x_1, x_2) + ... + d_q(x_j, x_{j+1}) + d_q(x_{j+1}, S^{j+1}z) \\ &\le d_q(x_0, x_1) + \theta d_q(x_0, x_1) + ... + \theta^{j+1} d_q(x_0, z), \text{ (by (2.11) and (2.17))} \\ d_q(x_0, S^{j+1}z) &\le d_q(x_0, x_1)[1 + \theta + ...\theta^j] + \theta^{j+1} r, \quad (\text{as } z \in \overline{B}(x_0, r)) \\ d_q(x_0, S^{j+1}z) &\le (1-\theta) r \frac{(1-\theta^{j+1})}{1-\theta} + \theta^{j+1} r = r. \end{aligned}$$

It follows that $S^{j+1}z \in \overline{B}(x_0, r)$ and hence $S^n z \in \overline{B}(x_0, r)$. Now the inequality (2.16) can be written as,

$$d_q(S^n z, S^{n+1}z) \le \theta^n d_q(z, Sz) \longrightarrow 0 \text{ as } n \to \infty. \tag{2.18}$$

Therefore,

$$\begin{aligned} d_q(x^*, y) &= d_q(Sx^*, Sy) \\ &\le d_q(Sx^*, S^{n+1}z) + d_q(S^{n+1}z, Sy) \\ &\le k[d_q(x^*, Sx^*) + d_q(S^n z, S^{n+1}z)] + k[d_q(S^n z, S^{n+1}z) + d_q(y, Sy)] \\ &\le k d_q(x^*, x^*) + 2k d_q(S^n z, S^{n+1}z) + k d_q(y, y) \\ &\le 0 \quad (\text{by 2.13,2.14 and 2.18}) \end{aligned}$$

Similarly, $d_q(y, x^*) = 0$. Hence $x^* = y$.

Example 2.6. Let $X = R^+ \cup \{0\}$ be endowed with usual order and let $d_q : X \times X \to X$ be defined by $d_q(x, y) = \dfrac{x}{2} + y$. Let $S : X \to X$ be defined by

$$Sx = \begin{cases} \dfrac{x}{7} \text{ if } x \in [0, 1] \\ x - \frac{1}{2} \text{ if } x \in (1, \infty) \end{cases}$$

Clearly, S is a dominated mapping. Then for $x_0 = 1$, $r = \frac{3}{2}$, $\theta = \frac{3}{7}$, $\overline{B}(x_0, r) = [0, 1]$ and for $k = \frac{3}{10}$,

$$(1-\theta) r = \left(1 - \frac{3}{7}\right) \frac{3}{2} = \frac{6}{7},$$

and

$$d_q(x_0, Sx_0) = d_q(1, S1) = d_q(1, \frac{1}{7}) = \frac{1}{2} + \frac{1}{7} = \frac{9}{14} < \frac{6}{7}.$$

Also if $x, y \in (1, \infty)$, then

$$\begin{aligned} 5x + 10y &\ge \frac{9}{2}x + \frac{9}{2}y + \frac{9}{2} \\ &\Rightarrow 5x - \frac{5}{2} + 10y - 5 \ge 3\left[\frac{3}{2}x + \frac{3}{2}y - 1\right] \\ &\Rightarrow 10(\frac{x}{2} - \frac{1}{4} + y - \frac{1}{2}) \ge 3\left[\frac{x}{2} + x - \frac{1}{2} + \frac{y}{2} + y - \frac{1}{2}\right] \\ &\Rightarrow d_q(Sx, Sy) \ge k[d_q(x, Sx) + d_q(y, Sy)]. \end{aligned}$$

So the contractive condition does not hold on X. Now if $x, y \in \overline{B}(x_0, r)$, then

$$\begin{aligned} d_q(Sx, Sy) &= \frac{x}{14} + \frac{y}{7} = \frac{1}{7}\left\{\frac{x}{2} + y\right\} \\ &\leq \frac{3}{10}\left\{\frac{x}{2} + \frac{y}{2}\right\} \leq \frac{3}{10}\left\{\frac{x}{2} + \frac{x}{7} + \frac{y}{2} + \frac{y}{7}\right\} \\ &= k[d_q(x, Sx) + d_q(y, Sy)]. \end{aligned}$$

Also,

$$d_q(x_0, Sx_0) + d_q(z, Sz) = d_q(x_0, z) + d_q(Sx_0, Sz) \text{ for all } z \preceq x_0.$$

Therefore, all the conditions of Theorem 2.5 are satisfied. Moreover, 0 is the fixed point of S.

In Theorem 2.5, the conditions (2.8) and (2.9) are imposed to restrict the condition (2.7) only for x, y in $\overline{B}(x_0, r)$. Example 2.6 explains the utility of these restrictions. The following result relax the conditions (2.8) and (2.9) but impose the condition (2.7) for all comparable elements in the whole space X.

Theorem 2.7. *Let $(X, \preceq, d_q)$ be an ordered left K-sequentially complete dislocated quasi metric space, $S : X \to X$ be a dominated map and x_0 be an arbitrary point in X. Suppose there exists $k \in [0, \frac{1}{2})$ with*

$$d_q(Sx, Sy) \leq k[d_q(x, Sx) + d_q(y, Sy)],$$

for all comparable elements x, y in X. If, for every nonincreasing sequence $\{x_n\}$ in X, $\{x_n\} \to u$ implies that $u \preceq x_n$ and every pair of elements in X has a lower bound, then there exists a unique point x^* in X such that $x^* = Sx^*$ and $d_q(x^*, x^*) = 0$.

In Theorem 2.5, the condition (2.9), the existence of a lower bound and for every nonincreasing sequence $\{x_n\}$in X, $\{x_n\} \to u$ implies that $u \preceq x_n$ are imposed to restrict the condition (2.7) only for comparable elements. However, the following result relaxes these conditions but imposes the condition (2.7) for all elements in $\overline{B}(x_0, r)$.

Theorem 2.8. *Let (X, d_q) be a complete left K-sequentially dislocated quasi metric space, $S : X \to X$ be a map and x_0 be an arbitrary point in X. Suppose there exists $k \in [0, \frac{1}{2})$ such that*

$$d_q(Sx, Sy) \leq k[d_q(x, Sx) + d_q(y, Sy)],$$

for all $x, y \in \overline{B}(x_0, r)$; where x_0 is a point in X satisfying the condition

$$d_q(x_0, Sx_0) \leq (1 - \theta)r$$

with $\theta = \frac{k}{1-k}$. Then there exists a unique point x^* in $\overline{B}(x_0, r)$ such that $x^* = Sx^*$ and $d_q(x^*, x^*) = 0$.

Remark 2.9. The above results can easily be proved in right K-sequentially dislocated quasi metric space.

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