# A STUDY OF A NONLINEAR INTEGRAL EQUATION VIA WEAKLY PICARD OPERATORS 

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Abstract. The purpose of this paper is to study on gauge spaces the following nonlinear integral equation:

$$
x(t)=g(t, x(t))+h(t, x(t)) \cdot \int_{0}^{t} K(t, s, x(s)) d s, t \geq 0 .
$$

Our results are connected with some results by K. Balachandran and M. Diana Julie (Asymptotic stability of solutions of nonlinear integral equations, Nonlinear Functional Analysis and Applications, Vol.13, No.2(2008), pp 311-322). Also, we given an example which show us that the results from the above paper can not be applied, but our results are fulfilled.
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## 1. Introduction

The theory of integral equations has many applications in describing numerous events and problems of real world. For example, integral equations are often applicable in mathematical physics, engineering, economics and biology (see [2], [3], [4], [5], [6] and their references). In [1] the authors investigate the solvability and asymptotic stability of solutions for the following nonlinear integral equation

$$
\begin{equation*}
x(t)=g(t, x(t))+h(t, x(t)) \cdot \int_{0}^{t} K(t, s, x(s)) d s, t \geq 0 \tag{1.1}
\end{equation*}
$$

The study was made in certain classes of bounded and continuous functions. The central tool in the analysis is the fixed point theorem of Darbo. In this paper we shall study, on gauge spaces, the equation (1.1). More exactly we shall prove that the equation (1.1) has, in $C\left(\mathbb{R}_{+}, \mathbb{R}\right)$, a unique solution. Also, we given an example which show us that the results from [1] can not be applied, but our results are fulfilled(see Example 3.1). Next we are going to study data dependence(continuity and smooth dependence on parameter) for the solution of equation (1.1). Our study was made
in the gauge space $\left(C\left(\mathbb{R}_{+}, \mathbb{R}\right),\left(d_{m}\right)_{m \in \mathbb{N}^{\star}}\right)$, where $d_{m}(x, y):=\max _{t \in[0, m]}|x(t)-y(t)|$ by using the fixed point theorem of Colojoară(see [8]).

## 2. Basic notions and results of the weakly Picard operators theory

Throughout this paper we shall follow the standard terminologies and notations in nonlinear analysis. For the convenience of the reader we shall recall some of them.

Let $X$ be a nonempty set and $A: X \rightarrow X$ an operator. We denote by $A^{0}:=1_{X}$, $A^{1}:=A, A^{n+1}:=A \circ A^{n}, n \in \mathbb{N}$ the iterate operators of the operator $A$. We also have

$$
\begin{gathered}
P(X):=\{Y \subset X \mid Y \neq \emptyset\} \\
F_{A}:=\{x \in X \mid A(x)=x\} \\
I(A):=\{Y \in P(X) \mid A(Y) \subset Y\}
\end{gathered}
$$

By $(X, \rightarrow)$ we will denote an L-space. For example, Hausdorff topological spaces, metric spaces, generalized metric spaces (in Perov's sense: $d(x, y) \in \mathbb{R}_{+}^{n}$, Luxemburg sense: $d(x, y) \in \mathbb{R}_{+} \cup\{\infty\}, d(x, y) \in K, \mathrm{~K}$ a cone in an ordered Banach space), gauge spaces, probabilistic metric spaces, have a natural structure of L-spaces(see [9], [10]).

In this paper, we need the notations(I.A. Rus [11] and [12]).
Definition 2.1. Let $(X, \rightarrow)$ be an L-space. An operator $A: X \rightarrow X$ is weakly Picard operator (briefly WPO) if the sequence $\left(A^{n}(x)\right)_{n \in N}$ converges, for all $x \in X$ and the limit (which may depend on $x$ ) is a fixed point of $A$

Definition 2.2. Let $(X, \rightarrow)$ be an L-space. An operator $A: X \rightarrow X$ is Picard operator(briefly PO) if:
(i) $F_{A}=\left\{x^{\star}\right\}$;
(ii) $A^{n}(x) \rightarrow x^{\star}$ as $n \rightarrow \infty$, for all $x \in X$.

If $A: X \rightarrow X$ is weakly Picard operator, then we may define the operator $A^{\infty}$ : $X \rightarrow X$ by $A^{\infty}(x)=\lim _{n \rightarrow \infty} A^{n}(x)$. Moreover, if $A$ is PO and we denote by $x^{\star}$ its unique fixed point, then $A^{\infty}(x)=x^{\star}$, for each $x \in X$.

We have (see [7], [11],[12] and [13]):
Theorem 2.1. (existence and uniqueness) Let $\left(X,\left(d_{i}\right)_{i \in I}\right)$ be a sequentially complete Hausdorff gauge space and let $T: X \rightarrow X$ be such that for every $i \in I$ there exists $\alpha_{i} \in I$ such that

$$
d_{i}(T(x), T(y)) \leq \alpha_{i} \cdot d_{i}(x, y),
$$

for each $x, y \in X$. Then $T$ is $P O$.
Theorem 2.2. (data dependence) Let $\left(X,\left(d_{\lambda}\right)_{\lambda \in \Lambda}\right)$ be a gauge space and $A, B: X \rightarrow$ $X$ be two $c_{\lambda}-W P O s$. We suppose that, for each $\lambda \in \Lambda$ there exists $\eta_{\lambda}>0$ such that

$$
d_{\lambda}(A(x)), B(x) \leq \eta_{\lambda}, \text { for all } x \in X
$$

Then

$$
H_{d_{\lambda}}\left(F_{A}, F_{B}\right) \leq c_{\lambda} \cdot \eta_{\lambda}, \text { for all } \lambda \in \Lambda .
$$

In order to study smooth dependence of parameter we shall use the following result(see [7])

Theorem 2.3. Let $(X, \rightarrow)$ be an L-space and $\left(Y,\left(d_{i}\right)_{i \in I}\right)$ be a sequentially complete Hausdorff gauge space. Let $B: X \rightarrow X$ and $C: X \times Y \rightarrow Y$ be two operators.

We suppose that:
(i) $B$ is a Picard operator (PO) (we denote by $x^{\star}$ its unique fixed point);
(ii) for every $i \in I$ there exists $\alpha_{i} \in(0,1)$ such that

$$
d_{i}\left(C\left(x, y_{1}\right)\right), C\left(x, y_{2}\right) \leq \alpha_{i} d_{i}\left(y_{1}, y_{2}\right)
$$

for all $x \in X$ and $y_{1}, y_{2} \in Y$ (we denote by $y^{\star}$ the unique fixed point of the operator $\left.C\left(x^{\star}, \cdot\right)\right)$;
(iii) the operator $C\left(\cdot, y^{\star}\right)$ is continuous in $x^{\star}$.

Then, the operator $A: X \times Y \rightarrow X \times Y, A(x, y):=(B(x), C(x, y))$ is Picard operator. Moreover, $F_{A}=\left\{\left(x^{\star}, y^{\star}\right)\right\}$.

## 3. Existence and uniqueness

In this section we shall prove that the equation (1.1) has a unique solution in $C\left(\mathbb{R}_{+}, \mathbb{R}\right)$. For this, in what follows we consider the gauge space $X:=$ $\left(C([0, \infty), \mathbb{R}),\left(d_{m}\right)_{m \in \mathbb{N}^{\star}}\right)$, where

$$
d_{m}(x, y):=\max _{t \in[0, m]}|x(t)-y(t)|
$$

Our first result is the following
Theorem 3.1. We consider equation (1.1) under following hypothesis:
$\left(C_{1}\right)$ there exists $l \in C\left(\mathbb{R}_{+} \times \mathbb{R}_{+}, \mathbb{R}_{+}\right)$and $n_{1}, n_{2} \in C\left(\mathbb{R}_{+}, \mathbb{R}_{+}\right)$such that

$$
\begin{gathered}
|K(t, s, u)-K(t, s, v)| \leq l(t, s)|u-v|,(\forall) t, s \in \mathbb{R}_{+}, u, v \in \mathbb{R} \\
|g(t, u)-g(t, v)| \leq n_{1}(t)|u-v|,(\forall) t \in \mathbb{R}_{+}, u, v \in \mathbb{R} \\
|h(t, u)-h(t, v)| \leq n_{2}(t)|u-v|,(\forall) t \in \mathbb{R}_{+}, u, v \in \mathbb{R}
\end{gathered}
$$

$\left(C_{2}\right)$ for each $m \in \mathbb{N}^{\star}$ there exists $\alpha(m)>0$ such that

$$
|h(t, x(t))| \cdot \int_{0}^{t} l(t, s) d s+n_{2}(t) \cdot \int_{0}^{t}|K(t, s, y(s))| d s \leq \alpha(m)
$$

for all $t \in[0, m]$ and $x, y \in C\left(\mathbb{R}_{+}, \mathbb{R}\right)$.
$\left(C_{3}\right) M_{g}(m)+\alpha(m)<1$, for each $m \in \mathbb{N}^{\star}$. Here we denoted $M_{g}(m)=\max _{t \in[0, m]} n_{1}(t)$.
Then the equation (1.1) has, in $C\left(\mathbb{R}_{+}, \mathbb{R}\right)$, a unique solution $x^{\star}$.
Proof. We consider the operator $A: X \rightarrow X$ defined by:

$$
A(x)(t)=g(t, x(t))+h(t, x(t)) \cdot \int_{0}^{t} K(t, s, x(s)) d s
$$

We remark that for all $x, y \in X$, and $t \in[0, m]$ we have that

$$
|A(x)(t)-A(y)(t)|
$$

$$
\begin{gathered}
\leq|g(t, x(t))-g(t, y(t))|+\left|h(t, x(t)) \int_{0}^{t} K(t, s, x(s)) d s-h(t, y(t)) \int_{0}^{t} K(t, s, y(s)) d s\right| \\
\leq|g(t, x(t))-g(t, y(t))| \\
+|h(t, x(t))| \int_{0}^{t}|K(t, s, x(s))-K(t, s, y(s))| d s+|h(t, x(t))-h(t, y(t))| \cdot\left|\int_{0}^{t} K(t, s, y(s)) d s\right| \\
\leq n_{1}(t)|x(t)-y(t)|+|h(t, x(t))| \int_{0}^{t} l(t, s)|x(s)-y(s)| d s \\
+n_{2}(t) \cdot|x(t)-y(t)| \int_{0}^{t}|K(t, s, y(s))| d s \\
\leq\left(M_{g}(m)+\alpha(m)\right) \cdot d_{m}(x, y) .
\end{gathered}
$$

It follows that for each $m \in \mathbb{N}^{\star}$ we have

$$
d_{m}(A x, A y) \leq\left(M_{g}(m)+\alpha(m)\right) \cdot d_{m}(x, y),
$$

for all $x, y \in X$. From Theorem 2.1, we obtain the conclusion.
Example 3.1. Let us consider the following nonlinear integral equation
$x(t)=\frac{t}{1+t} \cdot \sin \left(a_{1} x(t)\right)+\frac{t \cos \left(a_{2} x(t)\right)}{\left(1+t+t^{3}\right)\left(1+t^{2}\right)\left(2+t^{2}\right)} \cdot \int_{0}^{t} a_{3} \cdot t \cdot \ln \left(1+t s+\frac{1}{1+x^{2}(s)}\right) d s$,
$a_{1}, a_{2}, a_{3} \in \mathbb{R}$ with $\left|a_{1}\right|+\left|a_{2}\right|+\left|a_{2} \cdot a_{3}\right| \leq 1$. Then equation (3.1) has, in $C\left(\mathbb{R}_{+}, \mathbb{R}\right)$, a unique solution.

Proof. Firstly, we remark that the maps

$$
\begin{gathered}
(t, u) \in \mathbb{R}_{+} \times \mathbb{R} \xrightarrow{g} \frac{t}{1+t} \cdot \sin \left(a_{1} u\right) \in \mathbb{R} \\
(t, s, u) \in \mathbb{R}_{+} \times \mathbb{R}_{+} \times \mathbb{R} \xrightarrow{K} a_{3} \cdot t \cdot \ln \left(1+t s+\frac{1}{1+u^{2}}\right) \in \mathbb{R} \\
(t, u) \in \mathbb{R}_{+} \times \mathbb{R} \xrightarrow{h} \frac{t \cos \left(a_{2} u\right)}{\left(1+t+t^{3}\right)\left(1+t^{2}\right)\left(2+t^{2}\right)} \in \mathbb{R}
\end{gathered}
$$

verify the hypothesis $\left(C_{1}\right)$ of Theorem 3.1 with

$$
\begin{gathered}
n_{1}, n_{2}: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}, n_{1}(t)=\left|a_{1}\right| \frac{t}{1+t}, n_{2}(t)=\frac{\left|a_{2}\right| t}{\left.\left(1+t+t^{3}\right)\left(1+t^{2}\right)\left(2+t^{2}\right)\right)} \\
l: \mathbb{R}_{+} \times \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}, l(t, s)=\frac{\left|a_{3}\right| t}{1+t s}
\end{gathered}
$$

So, the condition $\left(C_{1}\right)$ hold.

Let be $m \in \mathbb{N}^{*}$. Then for all $t \in[0, m]$ and $x, y \in C\left(\mathbb{R}_{+}, \mathbb{R}\right)$ we get

$$
\begin{gathered}
|h(t, x(t))| \cdot \int_{0}^{t} l(t, s) d s+n_{2}(t) \cdot \int_{0}^{t}|K(t, s, y(s))| d s \\
=\frac{t\left|\cos \left(a_{2} x(t)\right)\right|}{\left(1+t+t^{3}\right)\left(1+t^{2}\right)\left(2+t^{2}\right)} \cdot \int_{0}^{t} \frac{\left|a_{3}\right| t}{1+t s} d s \\
+\frac{\left|a_{2} \cdot a_{3}\right| t}{\left.\left(1+t+t^{3}\right)\left(1+t^{2}\right)\left(2+t^{2}\right)\right)} \cdot \int_{0}^{t} t \cdot \ln \left(1+t s+\frac{1}{1+x^{2}(s)}\right) d s \\
\leq \frac{\left|a_{3}\right| t}{\left(1+t+t^{3}\right)\left(1+t^{2}\right)\left(2+t^{2}\right)} \cdot t^{2}+\frac{\left|a_{2} \cdot a_{3}\right| t}{\left.\left(1+t+t^{3}\right)\left(1+t^{2}\right)\left(2+t^{2}\right)\right)} \cdot\left(1+t^{2}\right)\left(2+t^{2}\right) \\
\leq\left|a_{3}\right| \frac{t^{3}}{1+t^{3}}+\left|a_{2} \cdot a_{3}\right| \frac{t}{t+1} \leq\left|a_{3}\right| \frac{m^{3}}{1+m^{3}}+\left|a_{2} \cdot a_{3}\right| \frac{m}{m+1}
\end{gathered}
$$

Now we apply Theorem 3.1 for the operator

$$
A: C\left(\mathbb{R}_{+}, \mathbb{R}\right) \rightarrow C\left(\mathbb{R}_{+}, \mathbb{R}\right)
$$

$A x(t)=\frac{t}{1+t} \cdot \sin \left(a_{1} x(t)\right)+\frac{t \cos \left(a_{2} x(t)\right)}{\left(1+t+t^{3}\right)\left(1+t^{2}\right)\left(2+t^{2}\right)} \cdot \int_{0}^{t} a_{3} \cdot t \cdot \ln \left(1+t s+\frac{1}{1+x^{2}(s)}\right) d s$
and

$$
\begin{gathered}
M_{g}(m)=\left|a_{1}\right| \frac{m}{1+m} \\
\alpha(m)=\left|a_{3}\right| \frac{m^{3}}{1+m^{3}}+\left|a_{2} \cdot a_{3}\right| \frac{m}{m+1}
\end{gathered}
$$

Remark 3.1. From the above example it follows that for all $t \in \mathbb{R}_{+}$and $x \in$ $C\left(\mathbb{R}_{+}, \mathbb{R}\right)$, we have

$$
\begin{aligned}
\left|a_{3}\right|\left(\left(t^{2}+1\right) \cdot \ln \left(t^{2}+1\right)-t^{2}\right) & \leq \int_{0}^{t}|K(t, s, x(s))| d s \leq\left|a_{3}\right|\left(\left(t^{2}+2\right) \cdot \ln \left(t^{2}+2\right)-t^{2}\right) \\
& \leq\left|a_{3}\right|\left(t^{2}+1\right) \cdot\left(t^{2}+2\right)
\end{aligned}
$$

Then,

$$
\lim _{t \rightarrow \infty} \int_{0}^{t}|K(t, s, x(s))| d s=\infty
$$

which means that hypothesis $\left(H_{5}\right)$, from [1] pp 314, is not fulfilled.
Remark 3.2. For $a_{1}=a_{3}=\frac{1}{3}$ and $a_{2}=1$ we get that $M_{g}(m)+\alpha(m)<1$ even if $\left|a_{1}\right|+\left|a_{3}\right|+\left|a_{2} \cdot a_{3}\right|=1$.

Finally by using weakly Picard operators technique, we remark that we can study the equation (1.1) in $C\left(\mathbb{R}_{+}, X\right)$ where $(X,\|\cdot\|)$ will be a Banach space. Similar studies we found in [14].

## 4. Data dependence

We consider the following equations

$$
\begin{equation*}
x(t)=g_{i}(t, x(t))+h_{i}(t, x(t)) \cdot \int_{0}^{t} K_{i}(t, s, x(s)) d s, t \geq 0 \tag{4.1}
\end{equation*}
$$

under conditions of Theorem 3.1. Let $x_{i}, i=1,2$ be the unique solution of equation (4.1). Then we have

Theorem 4.1. Let $g_{i}, h_{i}, K_{i}$ be as in the Theorem 3.1. We suppose that
(i) there exists $\eta_{1}, \eta_{2}, \eta_{3}>0$ such that

$$
\begin{aligned}
\left|g_{1}(t, u)-g_{2}(t, u)\right| \leq \eta_{1},(\forall)(t, u) & \in \mathbb{R}_{+} \times \mathbb{R} \\
\left|h_{1}(t, u)-h_{2}(t, u)\right| \leq \eta_{2},(\forall)(t, u) & \in \mathbb{R}_{+} \times \mathbb{R} \\
\left|K_{1}(t, s, v)-K_{2}(t, s, v)\right| \leq \eta_{3},(\forall)(t, s, v) & \in \mathbb{R}_{+} \times \mathbb{R}_{+} \times \mathbb{R} ;
\end{aligned}
$$

(ii) for each $m \in \mathbb{N}^{\star}$ there exists $C_{1}\left(K_{2}, m\right)>0$ and $C_{2}\left(h_{1}, m\right)$ such that

$$
\left|K_{2}(t, s, u)\right| \leq C_{1}\left(K_{2}, m\right), \forall t, s \in[0, m], \forall u \in \mathbb{R}
$$

and

$$
\left|h_{1}(t, u)\right| \leq C_{2}\left(h_{1}, m\right), \forall t \in[0, m], \forall u \in \mathbb{R}
$$

Then for each $m \in \mathbb{N}^{*}$ we have
$d_{m}\left(x_{1}, x_{2}\right) \leq\left(\eta_{1}+\eta_{2} \cdot m \cdot C_{1}\left(K_{2}, m\right)+\eta_{3} \cdot m \cdot C_{2}\left(h_{1}, m\right)\right) \max _{i=1,2}\left\{\frac{1}{1-M_{g_{i}}(m)-\alpha_{i}(m)}\right\}$,
where $M_{g_{i}}(m)$ and $\alpha_{i}(m)$ are as in hypothesis $\left(C_{2}\right)$ and $\left(C_{3}\right)$ of Theorem 3.1.
Proof. Under conditions of Theorem 3.1 the operators $A_{i}: X \rightarrow X$ defined by:

$$
A_{i}(x)(t)=g_{i}(t, x(t))+h_{i}(t, x(t)) \cdot \int_{0}^{t} K_{i}(t, s, x(s)) d s
$$

are $\frac{1}{1-M_{g_{i}}(m)-\alpha_{i}(m)}$ - Picard operators. On the other hand, for all $x \in C\left(\mathbb{R}_{+}, \mathbb{R}\right)$ and $t \in[0, m]$ we have

$$
\begin{aligned}
& \left|A_{1}(x)(t)-A_{2}(x)(t)\right| \leq\left|g_{1}(t, x(t))-g_{2}(t, x(t))\right| \\
& +\left|h_{1}(t, x(t)) \int_{0}^{t} K_{1}(t, s, x(s)) d s-h_{2}(t, x(t)) \int_{0}^{t} K_{2}(t, s, x(s)) d s\right| \\
& +\left|h_{1}(t, x(t)) \int_{0}^{t} K_{1}(t, s, x(s)) d s-h_{1}(t, x(t)) \int_{0}^{t} K_{2}(t, s, x(s)) d s\right| \\
& +\left|h_{1}(t, x(t)) \int_{0}^{t} K_{2}(t, s, x(s)) d s-h_{2}(t, x(t)) \int_{0}^{t} K_{2}(t, s, x(s)) d s\right|
\end{aligned}
$$

$$
\begin{aligned}
\leq \mid g_{1}(t, x(t))- & g_{2}(t, x(t))\left|+\left|h_{1}(t, x(t))\right| \int_{0}^{t}\right| K_{1}(t, s, x(s))-K_{2}(t, s, x(s)) \mid d s \\
& +\left|h_{1}(t, x(t))-h_{2}(t, x(t))\right| \int_{0}^{t}\left|K_{2}(t, s, x(s))\right| d s \\
\leq & \eta_{1}+\eta_{2} \cdot m \cdot C_{1}\left(K_{2}, m\right)+\eta_{3} \cdot m \cdot C_{2}\left(h_{1}, m\right)=: \eta_{m} .
\end{aligned}
$$

The conclusion follow from Theorem 2.2.
Example 4.1. Let us consider the following nonlinear integral equations

$$
\begin{gather*}
x(t)=\frac{t}{1+\beta_{i} t} \cdot \sin \left(a_{i} \cos x(t)\right) \\
+\frac{t \cos \left(b_{i} \sin x(t)\right)}{\left(1+t+t^{3}\right)\left(2+t^{2}\right)^{2}} \cdot \int_{0}^{t} c_{i} \cdot \frac{t^{2}}{(1+t+t s)^{2}} \cdot \ln \left(1+t s+\frac{1}{1+x^{2}(s)}\right) d s \tag{4.2}
\end{gather*}
$$

where
(i) $a_{i}, b_{i}, c_{i} \in \mathbb{R}$ with $\left|a_{i}\right|+\left|b_{i}\right|+\left|b_{i} \cdot c_{i}\right|<1$ for all $i=1,2$;
(ii) $\beta_{1}, \beta_{2} \geq 1$.

Then Theorem 4.1 hold.
Proof. First of all we remark that for each $i=1,2$ the operators

$$
\begin{gathered}
(t, u) \in \mathbb{R}_{+} \times \mathbb{R}^{g_{i}} \frac{t}{1+\beta_{i} t} \cdot \sin \left(a_{i} \cos u\right), \\
(t, u) \in \mathbb{R}_{+} \times \mathbb{R}^{h_{i}} \frac{t}{\left(1+t+t^{3}\right)\left(2+t^{2}\right)^{2}} \cdot \cos \left(b_{i} \sin u\right), \\
(t, s, u) \in \mathbb{R}_{+} \times \mathbb{R}_{+} \times \mathbb{R}^{\frac{K_{i}}{\rightarrow} c_{i} \cdot \frac{t^{2}}{(1+t+t s)^{2}} \cdot \ln \left(1+t s+\frac{1}{1+u^{2}}\right)}
\end{gathered}
$$

verify the hypothesis of Theorem 3.1 with

$$
M_{g_{i}}(m)=\frac{m}{1+\beta_{i} m} \quad \text { and } \alpha_{i}(m)=\left|c_{i}\right| \frac{m^{3}}{1+m^{3}}+\left|b_{i} \cdot c_{i}\right| \frac{m}{m+1} .
$$

On the other hand we observe that

$$
\begin{aligned}
&\left|g_{1}(t, u)-g_{2}(t, u)\right| \leq\left|\frac{t}{1+\beta_{1} t} \cdot \sin \left(a_{1} \cos u\right)-\frac{t}{1+\beta_{2} t} \cdot \sin \left(a_{2} \cos u\right)\right| \\
& \leq \frac{t}{1+\beta_{1} t} \cdot\left|\sin \left(a_{1} \cos u\right)-\sin \left(a_{2} \cos u\right)\right|+\left|\frac{t}{1+\beta_{1} t}-\frac{t}{1+\beta_{2} t}\right| \cdot\left|\sin \left(a_{2} \cos u\right)\right| \\
& \leq\left|a_{1}-a_{2}\right|+\left|\beta_{1}-\beta_{2}\right|=: \eta_{1}
\end{aligned}
$$

and

$$
\left|h_{1}(t, u)-h_{2}(t, u)\right| \leq \frac{t}{\left(1+t+t^{3}\right)\left(2+t^{2}\right)^{2}} \cdot\left|b_{1}-b_{2}\right| \leq\left|b_{1}-b_{2}\right|=: \eta_{2}
$$

Also, for all $t, s \in \mathbb{R}_{+}$and $u \in \mathbb{R}$ we have

$$
\left|K_{1}(t, s, u)-K_{2}(t, s, u)\right|
$$

$$
\begin{gathered}
=\left|c_{1} \cdot \frac{t^{2}}{(1+t+t s)^{2}} \cdot \ln \left(1+t s+\frac{1}{1+u^{2}}\right)-c_{2} \cdot \frac{t^{2}}{(1+t+t s)^{2}} \cdot \ln \left(1+t s+\frac{1}{1+u^{2}}\right)\right| \\
=\left|c_{1}-c_{2}\right| \cdot \frac{t^{2}}{(1+t+t s)^{2}} \cdot \ln \left(1+t s+\frac{1}{1+u^{2}}\right) \leq\left|c_{1}-c_{2}\right|:=\eta_{3}
\end{gathered}
$$

Since

$$
\begin{gathered}
\int_{0}^{t}\left|K_{i}(t, s, u)\right| d s \geq \int_{0}^{t} \frac{t^{2}}{(1+t+t s)^{2}} \cdot \ln (1+t s) d s \\
=-\frac{t}{1+t+t^{2}} \ln \left(1+t^{2}\right)+\ln \frac{1+t^{2}}{1+t+t^{2}}+\ln (1+t) \rightarrow \infty
\end{gathered}
$$

it follows that hypothesis $\left(H_{5}\right)$, from [1] pp 314, is not fulfilled.
Next we remark that for each $m \in \mathbb{N}^{*}$ we have

$$
\left|K_{2}(t, s, u)\right| \leq c_{2} \frac{m}{1+m}:=C_{1}\left(K_{2}, m\right), \forall t, s \in[0, m], \text { and } u \in \mathbb{R}
$$

and

$$
|h(t, u)| \leq \frac{m}{1+m}:=C_{2}\left(h_{1}, m\right), \forall t \in[0, m], \text { and } u \in \mathbb{R} .
$$

Now we apply Theorem 4.1.

## 5. Smooth dependence on parameter

Throughout of this section we consider gauge space $X:=\left(C\left(\mathbb{R}_{+} \times J, \mathbb{R}\right), d_{m}\right)$, where $J \subset \mathbb{R}$ is a compact interval and

$$
d_{m}(x, y)=\max _{(t, \lambda) \in[0, m] \times J}|x(t, \lambda)-y(t, \lambda)|, m \in \mathbb{N}^{\star} .
$$

Let us consider the integral equation

$$
\begin{equation*}
x(t, \lambda)=g(t, x(t, \lambda), \lambda)+h(t, x(t, \lambda), \lambda) \cdot \int_{0}^{t} K(t, s, x(s, \lambda), \lambda) d s, t \geq 0, \lambda \in J \tag{5.1}
\end{equation*}
$$

We assume that:
(H1) $g, h \in C^{1}\left(\mathbb{R}_{+} \times \mathbb{R} \times J, \mathbb{R}\right), K \in C^{1}\left(\mathbb{R}_{+} \times \mathbb{R}_{+} \times \mathbb{R} \times J, \mathbb{R}\right)$;
(H2) there exists $l \in C\left(\mathbb{R}_{+} \times \mathbb{R}_{+} \times J, \mathbb{R}_{+}\right)$and $n_{1}, n_{2} \in C\left(\mathbb{R}_{+} \times J, \mathbb{R}_{+}\right)$such that

$$
\begin{aligned}
\left|\frac{\partial K}{\partial u}(t, s, u, \lambda)\right| & \leq l(t, s, \lambda) \\
\left|\frac{\partial g}{\partial u}(t, u, \lambda)\right| & \leq n_{1}(t, \lambda) \\
\left|\frac{\partial h}{\partial u}(t, u, \lambda)\right| & \leq n_{2}(t, \lambda)
\end{aligned}
$$

for all $t, s \in \mathbb{R}_{+}$and $u \in \mathbb{R}$;
$\left(H_{3}\right)$ for each $m \in \mathbb{N}^{*}$ there exists $\alpha(m) \in(0,1)$ such that

$$
|h(t, x(s, \lambda), \lambda)| \int_{0}^{t} l(t, s, \lambda) d s+n_{2}(t, \lambda) \int_{0}^{t}|K(t, s, y(s, \lambda), \lambda)| d s \leq \alpha(m)
$$

for all $t \in[0, m], \lambda \in J$ and $x, y \in C\left(\mathbb{R}_{+} \times J\right)$.
$\left(H_{4}\right) \quad M_{g}(m)+\alpha(m)<1$, for each $m \in \mathbb{N}^{\star}$.
We define the operator

$$
B: X \rightarrow X
$$

$$
B(x)(t, \lambda)=g(t, x(t, \lambda), \lambda)+h(t, x(t, \lambda), \lambda) \cdot \int_{0}^{t} K(t, s, x(s, \lambda), \lambda) d s
$$

It is clear that, in the conditions $\left(H_{1}\right)-\left(H_{4}\right), B$ is Picard operator. Let $x^{\star}(\cdot, \lambda)$ the unique fixed point of operator $B$. Then

$$
\begin{equation*}
x^{\star}(t, \lambda)=g\left(t, x^{\star}(t, \lambda), \lambda\right)+h\left(t, x^{\star}(t, \lambda), \lambda\right) \cdot \int_{0}^{t} K\left(t, s, x^{\star}(s, \lambda), \lambda\right) d s \tag{5.2}
\end{equation*}
$$

for all $t \in \mathbb{R}_{+}$and $\lambda \in J$. We suppose that there exists $\frac{\partial x^{\star}}{\partial \lambda}$. Then from relation (5.2) we obtain that

$$
\begin{gathered}
\frac{\partial x^{\star}}{\partial \lambda}(t, \lambda)=\frac{\partial g}{\partial u}\left(t, x^{\star}(t, \lambda), \lambda\right) \cdot \frac{\partial x^{\star}}{\partial \lambda}(t, \lambda)+\frac{\partial g}{\partial \lambda}\left(t, x^{\star}(t, \lambda), \lambda\right) \\
+\frac{\partial h}{\partial u}\left(t, x^{\star}(t, \lambda), \lambda\right) \cdot \frac{\partial x^{\star}}{\partial \lambda}(t, \lambda) \cdot \int_{0}^{t} K\left(t, s, x^{\star}(s, \lambda), \lambda\right) d s \\
\quad+\frac{\partial h}{\partial \lambda}\left(t, x^{\star}(t, \lambda), \lambda\right) \cdot \int_{0}^{t} K\left(t, s, x^{\star}(s, \lambda), \lambda\right) d s \\
+h\left(t, x^{\star}(t, \lambda), \lambda\right) \cdot \int_{0}^{t} \frac{\partial K}{\partial u}\left(t, s, x^{\star}(s, \lambda), \lambda\right) \cdot \frac{\partial x^{\star}}{\partial \lambda}(s, \lambda) d s \\
\quad+h\left(t, x^{\star}(t, \lambda), \lambda\right) \cdot \int_{0}^{t} \frac{\partial K}{\partial \lambda}\left(t, s, x^{\star}(s, \lambda), \lambda\right) d s
\end{gathered}
$$

This relation suggest us to consider the following operator

$$
\begin{gathered}
C: X \times X \rightarrow X, \\
C(x, y)(t, \lambda)=\frac{\partial g}{\partial u}(t, x(t, \lambda), \lambda) \cdot y(t, \lambda)+\frac{\partial g}{\partial \lambda}(t, x(t, \lambda), \lambda) \\
+\frac{\partial h}{\partial u}(t, x(t, \lambda), \lambda) y(t, \lambda) \int_{0}^{t} K(t, s, x(s, \lambda), \lambda) d s
\end{gathered}
$$

$$
\begin{gathered}
+\frac{\partial h}{\partial \lambda}(t, x(t, \lambda), \lambda) \int_{0}^{t} K(t, s, x(s, \lambda), \lambda) d s \\
+h(t, x(t, \lambda), \lambda) \int_{0}^{t} \frac{\partial K}{\partial u}(t, s, x(s, \lambda), \lambda) y(s, \lambda) d s \\
+h(t, x(t, \lambda), \lambda) \int_{0}^{t} \frac{\partial K}{\partial \lambda}(t, s, x(s, \lambda), \lambda) d s
\end{gathered}
$$

Let be $m \in \mathbb{N}$ and $x \in X$. Then for all $y, z \in X$ we have

$$
\begin{gathered}
|C(x, y)(t, \lambda)-C(x, z)(t, \lambda)| \leq\left|\frac{\partial g}{\partial u}(t, x(t, \lambda), \lambda)\right| \cdot|y(t, \lambda)-z(t, \lambda)| \\
+\left|\frac{\partial h}{\partial u}(t, x(t, \lambda), \lambda)\right| \cdot \int_{0}^{t}|K(t, s, x(s, \lambda), \lambda)| d s \cdot|y(t, \lambda)-z(t, \lambda)| \\
+|h(t, x(t, \lambda), \lambda)| \cdot \int_{0}^{t}\left|\frac{\partial K}{\partial u}(t, s, x(s, \lambda), \lambda)\right| \cdot|y(s, \lambda)-z(s, \lambda)| d s \\
\leq\left(M_{g}(m)+\alpha(m)\right) d_{m}(y, z) .
\end{gathered}
$$

It follows that

$$
d_{m}(C(x, y), C(x, z)) \leq\left(M_{g}(m)+\alpha(m)\right) d_{m}(y, z)
$$

In this way we have the triangular operator

$$
\begin{gathered}
A: X \times X \rightarrow X \times X \\
A(x, y)(t, \lambda)=(B(x)(t, \lambda), C(x, y)(t, \lambda)) .
\end{gathered}
$$

Using Theorem 2.3 we conclude that $A$ is a Picard operator. So, the sequences

$$
x_{n+1}=B\left(x_{n}\right), n \in \mathbb{N}, \quad y_{n+1}=C\left(x_{n}, y_{n}\right)
$$

converges uniformly on each compact of $\mathbb{R}_{+} \times J$ to $\left(x^{\star}, y^{\star}\right) \in F_{A}$, for all $x_{0}, y_{0} \in X$.
If we take $x_{0}=0, y_{0}=\frac{\partial x_{0}}{\partial \lambda}=0$ then $y_{1}=\frac{\partial x_{1}}{\partial \lambda}$ and thus by induction we can prove that $y_{n}=\frac{\partial x_{n}}{\partial \lambda}$, for all $n \in \mathbb{N}^{\star}$.

Hence

$$
\begin{gathered}
x_{n} \rightarrow x^{\star}, \text { uniform as } n \rightarrow \infty \\
\frac{\partial x_{n}}{\partial \lambda} \rightarrow y^{\star}, \text { uniform as } n \rightarrow \infty
\end{gathered}
$$

These imply that there exists $\frac{\partial x^{\star}}{\partial \lambda}$ and $\frac{\partial x^{\star}}{\partial \lambda}=y^{\star}$
From the above considerations, we have that
Theorem 5.1. We consider the integral equation (5.2) in the hypothesis $\left(H_{1}\right)-\left(H_{5}\right)$. Then
(i) the equation (5.2) has a unique solution $x^{\star}(t, \cdot) \in X$;
(ii) $x^{\star}(t, \cdot) \in C^{1}(J)$, for all $t \in[a, b]$.

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