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FIXED POINTS OF INFINITELY CONNECTED DOMAIN CONTINUOUS MAPPINGS

ALEXANDER KIRILLOV* AND VICTOR STARKOV**

*Institute of Applied Mathematical Research
Karelian Research Centre
Russian Academy of Sciences
11, Pushkinskaya str., Petrozavodsk
Karelia 185910
Russia
E-mail: kirillov@krc.karelia.ru

** Department of Mathematics
Petrozavodsk State University
33, Lenin str., Petrozavodsk
Karelia 185910
Russia
E-mail:vstar@psu.karelia.ru

Abstract. We consider infinitely connected domain continuous mappings, satisfying the condition of interior boundary components interchange. The fixed point theorem is proved.
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1. Introduction

Let X be a topological space with the fixed point property (TSFPP), e.g. every continuous mapping $f : X \to X$ has at least one fixed point in X. Brouwer proved [3] that any compact $X \subset \mathbb{R}^n$, homeomorphic to a closed ball, is a TSFPP. The condition "X is a topological ball" is crucial here: it is sufficient to remind the example of a planar circular annulus rotation.

The Poincaré – Birkhoff theorem [9], [1], [2], and a number of following results dealing with it ([4], [5], [6], [8], etc.) propose various sufficient conditions under which a continuous map f(satisfying some additional restraints) of a closed topological annulus X has a fixed point.

For a finitely connected compact X in $\mathbb{R}^n$ Lefschetz [7] obtained the sufficient condition for X to be the TSFPP. But his arguments are based on the assumption of X triangulability. Hence the Lefschetz results, in general, are inapplicable to infinitely connected compacts. Bellow we give the example of such compact.

Example 1.1. Let us denote by R the planar circle annulus, $R = \{z \in \mathbb{R}^2 : 1 \leq ||z|| \leq 3\}$. For $\forall n \in \mathbb{N}$ their centres are on the the circle with center at origin and radius $1 + \frac{1}{10^{n-1}}$, the arguments of their centres equal $\frac{\pi k}{2^n},\ k = 1, 2, ..., 2^{n+1}$. Let K be the compact which is obtained by moving off all constructed circles. Then $K$is not triangulable.

This article deals with a continuous mapping f of infinitely-connected compacts $X \subset \mathbb{R}^n$, where f interchanges, in some sense, the inner boundary components of images and preimages. Under some assumptions we give simple sufficient condition of a fixed point existence in X.

2. Main results

Definition 2.1. Let $B_0,\ B_k,\ k \in M \subset N$, be topological balls from $\mathbb{R}^n,\ n \geq 2$, $B_k \subset B_0,\ intB_k \bigcap intB_j = \emptyset,\ \forall k, j \in M,\ k \neq j$. We say that a compact $X = B_0 \setminus (\bigcup_{k \in M} intB_k)$ is *a globular compact.*

The globular compact is constructed in Example 1.1.

Denote $J_0 = \partial B_0,\ J_k = \partial B_k,\ k \in M$. Let us note that in general for globular compact $\partial X \neq \bigcup_{k \in M \cup 0} J_k$, because ∂X contains the limit points c of the set $\bigcup_{k \in M \cup 0} J_k$, but it may occurs that $c \notin J_k,\ \forall k$.

The next theorem uses the designations from the above definition.

Theorem 2.2. Let X be infinitely connected ($M = \mathbb{N}$) globular compact, $B_0 \supset \hat{B}_0$ a topological ball. Let $f : X \to \hat{B}_0$ be a continuous mapping such that for all $J_k,\ k \in \mathbb{N}$, exists a topological ball $\hat{B}_k \subset \hat{B}_0 :\ f(J_k) \subset \hat{B}_k,\ intB_k \bigcap \hat{B}_k = \emptyset$. Then f has a fixed point in X.

Remark 2.3. In this theorem $f(X)$ is not necessary a subset of X.

Remark 2.4. The theorem is also true for finitely connected compacts.

Proof. Denote by D^n a closed unit ball from $\mathbb{R}^n$ and for some fixed $k \in \mathbb{N}$ let $h : D^n \to B_k,\ g : D^n \to \hat{B}_k$ be homeomorphisms, which are guaranteed by the theorem assumption. Any continuous mapping φ of the sphere $S^{n-1} = \partial D^n$ into B_k is homotopic to constant map. We can take $\Phi(x, r)$ as corresponding homotopy, where $\Phi(x, r) = h \circ [r(h^{-1} \circ \varphi)] : S^{n-1} \times [0, 1] \to B_k,\ \Phi(x, 1) = \varphi(x)$. Hence the mapping $f|_{J_k} \circ h : S^{n-1} \to \hat{B}_k$ can be extended to a continuous mapping $F_k : D^n \to \hat{B}_k$ [10]. Therefore $f_k = F_k \circ h^{-1}$ is the continuous extension of $f|_{J_k}$ from J_k to B_k, and moreover $f_k(B_k) \subset \hat{B}_k$.

Thus we continuously extend f from X to B_0. Let us denote this extension by $F : B_0 \to B_0$. According to the Brouwer fixed point theorem there exists a fixed point $x_0 \in B_0$ of the map F. It is left to note that $x_0 \notin intB_k,\ \forall k \in M$, since $F(intB_k) = f_k(intB_k) \subset \hat{B}_k,\ intB_k \bigcap \hat{B}_k = \emptyset$. Hence $x_0 \in X$.

Remark 2.5. If $n = 2$ the globular compact X will be the Jordan compact, i.e. in definition 2.1 $\partial B_k,\ k \in M \bigcup \{0\}$, will be the Jordan curves.

Example 2.6. Let $\Pi = [-8; 8] \times [-4; 4] := \Pi_- \bigcup \Pi_+$ be the rectangular, where Π_-, Π_+ are closed squares symmetric with respect to ordinate axis. Denote $K_\pm := K + (\pm 4; 0)$, where K is a globular compact from Example 1.1. Let $f : \Pi \to \Pi$ be any continuous mapping such that $f(K_\pm) = K_\mp$. Then f, according to Theorem 2.2, has a fixed point in the infinitely connected not triangulable globular compact Π.

Remark 2.7. The proof of the theorem is still be true for the finitely connected compacts X as well.

Corollary 2.8. Let X be a globular compact from definition 2.1. A homeomorphism $f : X \to X$ is such that $f(J_0) = J_0,\ f(J_k) = J_{j(k)},\ j(k) \neq k, \forall k \in M$. Then f has a fixed point in X.

When we apply the theorem 2.2 it is not always convenient to operate with topological balls from definition 2.1 of a globular compact. Hence we introduce the following notion.

Definition 2.9. Let $M \subset \mathbb{N}$ be a finite or an infinite subset; $J_k, k \in M \cup \{0\}$, are the topological spheres in $\mathbb{R}^n$ such that the closing $\bar{G}_k$ of the bounded component $G_k \subset \mathbb{R}^n \setminus J_k$ is the n–dimensional manifold with boundary, $G_k \subset G_0,\ G_k \bigcap G_j = \emptyset,\ \forall j, k \in M,\ j \neq k$. Then the compact $X = \bar{G}_0 \setminus \bigcup_{j \in M} G_j$ will be called a *spherical compact*.

According to the definition of a spherical compact, for all $j \in M \cup \{0\}$ there exists a homeomorphism $h_j : S^{n-1} \to J_j$. Therefore the properties of J_j, included in definition 2.5, and the generalized Shoenflis theorem [11],[12], imply the extensions of homeomorphisms h_j to homeomorphisms of a ball $D^n \to \bar{G}_j$. Thus any spherical compact is a globular compact as well (however the class of spherical compacts does not coincide with the class of globular compacts).

Using the designations from definition 2.10 we can formulate the following result.

Corollary 2.10. Let X be an infinitely connected ($M = \mathbb{N}$) spherical compact, $G_0 \supset \hat{G}_0$ is a spherical compact. Let $f : X \to \hat{G}_0$ be a continuous map such that for all $J_k,\ k \in \mathbb{N}$, there exists a topological sphere $\hat{J}_k$ in $\mathbb{R}^n$ such that the closing $\bar{\hat{G}}_k$ of a bounded component $G_k \subset \mathbb{R}^n \setminus \hat{J}_k$ is a $n-$ dimensional manifold with the bound, and moreover $f(J_k) \subset \bar{\hat{G}}_k,\ G_k \bigcap \hat{G}_k = \emptyset$. Then f has a fixed point in X.

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References

[1] G.D. Birkhoff, *Proof of Poincaré's geometric theorem*, Trans. Amer. Math. Soc., **14**(1913), 14-22.

[2] G.D. Birkhoff, *An extension of Poincare's last geometric theorem*, Acta Math., **47**(1925), 297-311.

[3] L.E.J. Brouwer, *Ueber Abbildungen von Mannigfaltigkeiten*, Math. Ann., **71**(1912), 97-115.

[4] T. Ding, *Approaches to the qualitative theory of ordinary differential equations*, Peking University Series in Mathematics, vol. 3, World Scientific, 2007.

[5] W.-Y. Ding, *A generalization of the Poincaré - Birkhoff theorem*, Proc. Amer. Math. Soc., **88**(1983), 341-346.

[6] J. Franks, *Generalizations of the Poincaré - Birkhoff theorem*, Ann. of Math., **128**(1988), 139-151.

[7] S. Lefschetz, *Intersections and transformations of complexes and manifolds*, Trans. Amer. Math. Soc., **28**(1926), no. 1, 1-49.

[8] A. Pascoletti, F. Zanolin, *A topological approach to bend-twist maps with applications*, Int. J. Differ. Eq., **2011**(2011), 20 p.

[9] H. Poincare, *Sur un théorème de geométrie*, Rendiconti del Circolo Mat. Palermo, **33**(1912), 375-507.

[10] V. A. Rokhlin, D.B. Fuks, *The First Course of Topology*, Nauka, (1912) (in Russian).
[11] A. Schoenflies, *BeitrZage zur theorie der punktenmenge*, Math. Ann. Bd., **62**(1906), 86-328.
[12] R. Schultz, *Some recent results on topological manifolds*, Amer. Math. Monthly, **78**(1971), no. 9, 941–952.

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