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THE COMMON FIXED POINT SET OF COMMUTING HOLOMORPHIC MAPPINGS IN CARTESIAN PRODUCTS OF BANACH SPACES

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Abstract. Using the metric approach to holomorphic fixed point theory, we establish a theorem regarding the structure of the common fixed point set of commuting holomorphic self-mappings of domains in Cartesian products of complex Banach spaces.
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1. Introduction

The main result of this paper (see Theorem 8.1 below) is concerned with families of commuting holomorphic self-mappings of domains in Cartesian products of complex Banach spaces. It declares that under certain appropriate conditions, the (nonempty) common fixed point set of such a family is a holomorphic retract. Our proof uses the metric approach to holomorphic fixed point theory (see [18] and [31]) and is based on a method due to R. E. Bruck (see [2] and [3]).

Our paper is organized as follows. In Section 2 we recall basic properties of the Kobayashi distance k_D on a bounded and convex domain D in a complex Banach space. In addition, we also prove a modification of the Harris theorem characterizing k_D-bounded sets. This modified theorem is applied in our constructions of families of equi-bounded and locally equi-uniformly linearly convex domains (see Section 4). We also recall connections between the Kobayashi distance and holomorphic mappings. In Section 3 we deal with locally uniform convexity of a bounded and convex domain D with respect to the Kobayashi distance k_D. This is a basic notion which appears in the assumptions of our main theorems in Sections 7 and 8 (Theorems 7.1 and 8.1). In this section we also investigate a modulus of linear convexity for the Kobayashi distance k_D. In Section 4 we first establish a theorem describing the behavior of the

Kobayashi distances corresponding to a decreasing sequence of bounded and convex domains in $\mathbb{C}^n$ (Theorem 4.4). This theorem and the known theorem regarding the behavior of the Kobayashi distances corresponding to an increasing sequence of bounded and convex domains in $\mathbb{C}^n$ (Theorem 4.1) are the basic tools in the constructions of families of equi-bounded and equi-uniformly linearly convex domains presented in this section. Section 5 is devoted to the Khamsi conditions which guarantee existence of a common fixed point of a commuting family of nonexpansive mappings. In Section 6 we introduce a bounded and convex domain D in $\ell^\infty(X_j, J)$ generated by the Cartesian product of equi-bounded and convex domains D_j in $(X_j, \|\cdot\|)$, $j \in J$. Under some additional assumptions, this domain D is locally admissible in the Khamsi sense as we prove in Section 7. The last section is devoted to proving the main theorem (Theorem 8.1) of our paper. Namely, we prove that if D in $\ell^\infty(X_j, J)$ is generated by the Cartesian product of equi-bounded, convex and locally equi-uniformly linearly convex domains D_j in $(X_j, \|\cdot\|)$, $j \in J$, then for any family $\mathcal{F}$ of commuting holomorphic (k_D-nonexpansive) self-mappings of D with a nonempty common fixed point set $\mathrm{Fix}(\mathcal{F})$, the set $\mathrm{Fix}(\mathcal{F})$ is a holomorphic (k_D-nonexpansive) retract of D.

2. The Kobayashi distance and holomorphic mappings

Throughout this paper all Banach spaces X are complex and all domains $D \subset X$ are bounded and convex.

Let Δ be the open unit disc in the complex plane $\mathbb{C}$. Recall that the Poincaré distance $k_\Delta = \omega$ on Δ is given by

$$\begin{aligned} k_\Delta(z, w) = \omega(z, w) &:= \arg\tanh \left| \frac{z - w}{1 - z\overline{w}} \right| \\ &= \arg\tanh\left(1 - \sigma(z, w)\right)^{\frac{1}{2}}, \end{aligned}$$

where

$$\sigma(z, w) := \frac{\left(1 - |z|^2\right)\left(1 - |w|^2\right)}{|1 - z\overline{w}|^2}, \qquad z, w \in \Delta$$

[18], [31].

Next, let D be a bounded and convex domain in a complex Banach space $(X, \|\cdot\|)$. We use the following definition of the Lempert function δ_D, which, in this case, is equal to the Kobayashi distance k_D:

$$k_D(x, y) = \delta_D(x, y)$$

$$:= \inf\left\{\omega(0, \lambda) : \lambda \in [0, 1) \ \&\ \text{there exists } f \in H(\Delta, D) \text{ so that } f(0) = x, f(\lambda) = y\right\},$$

where $x, y \in D$ [34], [15] (see also [17], [21], [25], [26] and [37]). To arrive at an equivalent definition of the Kobayashi distance we first define the infinitesimal Kobayashi pseudometric $\kappa_D : D \times X \to \mathbb{R}^+$ for D by

$$\kappa_D(x, v) := \inf\left\{\eta > 0 : \ \exists\, g \in H(\Delta, D) \ \text{ with } g(0) = x,\ g'(0)\,\eta = v\right\}$$

for all $x \in D$ and $v \in X$. Now for a given pair of points $x, y \in D$, we consider the family of all curves $\tilde{\gamma} : [0, 1] \to D$ that join x and y and have piecewise continuous derivatives. We call such curves admissible and define

$$L(\tilde{\gamma}) := \int_0^1 \kappa_D(\tilde{\gamma}(t), \tilde{\gamma}'(t))\, dt.$$

We now recall the following theorem.

Theorem 2.1. [38] *Let D be a bounded and convex domain in a complex Banach space $(X, \|\cdot\|)$. The Kobayashi distance k_D is the integrated form of κ_D, that is,*

$$k_D(x, y) = \inf\{L(\tilde{\gamma}) : \tilde{\gamma} \text{ is admissible with } \tilde{\gamma}(0) = x \text{ and } \tilde{\gamma}(1) = y\}$$

for all $x, y \in D$.

It is known that the Kobayashi distance k_D is locally equivalent to the norm $\|\cdot\|$ in X [19]. Indeed, if

$$\operatorname{dist}_{\|\cdot\|}(x, \partial D) := \inf\{\|x - y\| : y \in \partial D\}$$

denotes the distance in $(X, \|\cdot\|)$ between a point x and the boundary ∂D of the domain D and

$$\operatorname{diam}_{\|\cdot\|} D := \sup\{\|x - y\| : x, y \in D\}$$

is the diameter of D in $(X, \|\cdot\|)$, then the following theorem holds.

Theorem 2.2. *If D is a bounded and convex domain in a complex Banach space $(X, \|\cdot\|)$, then*

$$\arg\tanh\left(\frac{\|x - y\|}{\operatorname{diam}_{\|\cdot\|} D}\right) \le k_D(x, y)$$

for all $x, y \in D$ and

$$k_D(x, y) \le \arg\tanh\left(\frac{\|x - y\|}{\operatorname{dist}_{\|\cdot\|}(x, \partial D)}\right)$$

whenever $\|x - y\| < \operatorname{dist}_{\|\cdot\|}(x, \partial D)$.

We also use the following result. Let D be a bounded and convex domain in a complex Banach space $(X, \|\cdot\|)$. If $x, y, w, z \in D$ and $s \in [0, 1]$, then

$$\begin{aligned} &k_D(sx + (1 - s)y, sw + (1 - s)z) \\ &\le \max\{k_D(x, w), k_D(y, z)\}. \end{aligned}$$

Hence each open (closed) k_D-ball in the metric space (D, k_D) is convex [33].

We now turn to a characterization of k_D-bounded sets due to L. A. Harris ([19]). Let D be a bounded and convex domain in a complex Banach space $(X, \|\cdot\|)$.

A nonempty subset C of D is said to lie *strictly inside* D if

$$\operatorname{dist}_{\|\cdot\|}(C, \partial D) := \inf\{\|x - y\| : x \in C \;\&\; y \in \partial D\} > 0.$$

L. A. Harris proved the following theorem for bounded and convex domains (see Proposition 23 in [19]).

Theorem 2.3. *Let D be a bounded and convex domain in a complex Banach space $(X, \|\cdot\|)$. A nonempty subset C of D is k_D-bounded if and only if C lies strictly inside D.*

We now observe that a modification of the proof of Theorem 2.3 can be used to prove the following theorem, which we later apply in constructions of families of equi-bounded and equi-uniformly linearly convex domains in Section 5.

Theorem 2.4. *Let $(X, \|\cdot\|)$ be a complex Banach space and let $B_{\|\cdot\|}(x, R)$ denote the open ball in $(X, \|\cdot\|)$ of center x and radius R.*

(i) *Given $R > 0$ and $\epsilon > 0$, there exists $r_1 > 0$ such that for each $\tilde{x} \in X$, each bounded and convex domain $D \subset B_{\|\cdot\|}(\tilde{x}, R)$ with $\tilde{x} \in D$, and for each nonempty subset $C \subset D$ with $C + B_{\|\cdot\|}(0, \epsilon) \subset D$, we have $C \subset B_{k_D}(\tilde{x}, r_1)$, where $B_{k_D}(\tilde{x}, r_1)$ denotes an open ball in (D, k_D) of center $\tilde{x}$ and radius r_1. .*

(ii) *Given $r > 0$ and $r_2 > 0$, there exists $\epsilon > 0$ such that for each $\tilde{x} \in X$, each bounded and convex domain D containing the open ball $B_{\|\cdot\|}(\tilde{x}, r)$, and for the closed ball $\overline{B_{k_D}(\tilde{x}, r_2)} \subset D$ in (D, k_D), we have $\overline{B_{k_D}(\tilde{x}, r_2)} + B_{\|\cdot\|}(0, \epsilon) \subset D$.*

Proof. As we have already mentioned above, the proof of this theorem is a slight modification of the Harris proof of Theorem 2.3 (see [19]). We provide this modified proof here for the convenience of the reader.

(i) Assume that $R > 0$, $D \subset B_{\|\cdot\|}(\tilde{x}, R) \subset X$ is a convex domain with $\tilde{x} \in D$, and C is a nonempty subset of D such that $C + B_{\|\cdot\|}(0, \epsilon) \subset D$ for some $\epsilon > 0$. Without loss of generality we may assume that $\tilde{x} \in C$ replacing, if need be, ϵ by $\min\{\epsilon, \mathrm{dist}_{\|\cdot\|}(\tilde{x}, \partial D)\}$. Observe that for each $x \in D$ such that $B_{\|\cdot\|}(x, \epsilon) \subset D$ and for each $v \in X \setminus \{0\}$, the function $f : \Delta \to X$ given by $f(z) = x + \frac{zv}{\eta}$, $z \in \Delta$, where $\eta = \frac{\|v\|}{\epsilon}$, maps the unit disc Δ holomorphically into D and satisfies $f(0) = x$ and $f'(0)\eta = v$. Therefore for the infinitesimal Kobayashi pseudometric κ_D we get

$$\kappa_D(x, v) \leq \frac{\|v\|}{\epsilon}.$$

Next, for $x_1, x_2 \in C$, $0 \leq t \leq 1$ and $\gamma(t) = (1-t)x_1 + tx_2$, we have

$$B_{\|\cdot\|}(\gamma(t), \epsilon) \subset (1-t)B_{\|\cdot\|}(x_1, \epsilon) + tB_{\|\cdot\|}(x_2, \epsilon) \subset D,$$

which implies that

$$\kappa_D(\gamma(t), \gamma'(t)) \leq \frac{\|x_2 - x_1\|}{\epsilon}.$$

Hence by Theorem 2.1, we obtain $k_D(x_1, x_2) \leq \frac{2R}{\epsilon}$. Since $\tilde{x} \in C$, we may set $r_1 = \frac{2R+1}{\epsilon}$ to get $C \subset B_{k_D}(\tilde{x}, r_1)$.

(ii) Without any loss of generality we may assume that $\tilde{x} = 0 \in D$ and let $0 < r_0 < 1$ be such that

$$r_2 = k_\Delta(0, r) = \omega(0, r) = \arg\tanh(r_0).$$

If $f : \mathbb{C} \setminus \{2\} \to \mathbb{C}$ is defined by $f(z) = z(2-z)^{-1}$, then we have

$$(1 - |f(z)|^2)|2 - z|^2 = 4(1 - \mathrm{Re}\, z).$$

Hence there exists $0 < t < 1$ such that $\mathrm{Re}\, z \leq t$ whenever $|f(z)| \leq r_0$ and $\mathrm{Re}\, z \leq 1$. By the above equality, we also get that the holomorphic mapping $g = f \circ l$ transforms D into Δ for each $l \in X^*$ such that $\mathrm{Re}\, l(x) < 1$ for each $x \in D$. Next, we have

$$\arg\tanh(|g(x)|) = k_\Delta(g(x), g(0)) \leq k_D(x, 0) \leq r_2 = \arg\tanh(r_0)$$

for each $x \in \overline{B_{k_D}(0,r_2)}$. This means that $|g(x)| \leq r_0$ and therefore Re $l(x) \leq t$ or equivalently, Re $l(\frac{x}{t}) \leq 1$ for each $x \in \overline{B_{k_D}(0,r_2)}$. Applying the Hahn-Banach separation theorem ([39]), we get $\frac{1}{t}\overline{B_{k_D}(0,r_2)} \subset \overline{D}$ and hence we have

$$\overline{B_{k_D}(0,r_2)} + (1-t)B_{\|\cdot\|}(0,r) \subset t\overline{D} + (1-t)D = D.$$

We now complete the proof by setting $\epsilon = (1-t)r$. □

Complex geodesics are one of the basic tools in our proofs. Recall the following definition [13].

Definition 2.1. Let D be a bounded and convex domain in a complex Banach space X. A mapping $f \in H(\Delta, X)$ is a *complex geodesic* if $f(\Delta) \subset D$ and there exist $z_1, z_2 \in \Delta$ such that

$$k_\Delta(z_1,z_2) = k_D(f(z_1), f(z_2)).$$

If, moreover, $z_1 = 0$ and $z_2 \in \mathbb{R}^+$ we call f a *normalized* complex k_D-geodesic joining $f(z_1)$ and $f(z_2)$.

Theorem 2.5. [13] *Every complex geodesic is an isometric k_D-embedding.*

The next observation regarding complex geodesics is due to S. Dineen and R. M. Timoney [14].

Theorem 2.6. *If X is a complex reflexive Banach space and $D \subset X$ is a bounded and strictly convex domain in X (that is, $\overline{D}$ is strictly convex in X), then any two points in D can be joined by a unique normalized complex k_D-geodesic.*

Now, let D_1 and D_2 be two bounded and convex domains in two complex Banach spaces $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$, respectively. A mapping $f: D_1 \to D_2$ is said to be nonexpansive with respect to the Kobayashi distance if

$$k_{D_2}(f(x), f(y)) \leq k_{D_1}(x,y)$$

for all $x, y \in D_1$. If $D_1 = D_2 = D$, then we say that f is k_D-nonexpansive. Every holomorphic mapping $f: D_1 \to D_2$ is nonexpansive with respect to the Kobayashi distance [18].

In order to formulate a characterization of holomorphic mappings, we need the concept of a norming set.

Let $(X, \|\cdot\|)$ be a complex Banach space and let $\mathcal{N}$ be a nonempty subset of its dual X^*. If there exist positive constants c and C such that

$$\sup\{|l(x)| : l \in \mathcal{N}, \|l\| \leq C\} \geq c\|x\|$$

for each $x \in X$, then we say that $\mathcal{N}$ is a norming set for X [16].

We now recall several known results concerning the Hausdorff linear topology $\sigma(X, \mathcal{N})$ on X.

Theorem 2.7. [8] *If X is a complex Banach space, $\mathcal{N}$ is a norming set for X, and $D \subset X$ is a bounded and convex domain such that its norm closure $\overline{D}$ is compact in $\sigma(X,\mathcal{N})$, then any two points in D can be joined by at least one normalized complex k_D-geodesic.*

Theorem 2.8. [22] *(see also* [28] *and* [11]*) Let X be a complex Banach space, $\mathcal{N}$ a norming set for X, and let $D \subset X$ be a bounded and convex domain such that its norm closure $\overline{D}$ is compact in $\sigma(X, \mathcal{N})$. If $\{x_\beta\}_{\beta \in J}$ and $\{y_\beta\}_{\beta \in J}$ are nets in D which are convergent in $\sigma(X, \mathcal{N})$ to x and y, respectively, and $x, y \in D$, then*

$$k_D(x, y) \leq \liminf_{\beta} k_D(x_\beta, y_\beta).$$

Theorem 2.9. [8] *Let D_1, D_2 be two bounded and convex domains in complex Banach spaces $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$, respectively, and let $\mathcal{N}$ be a norming set in $(X_2, \|\cdot\|_2)$. If $\{f_\lambda\}_{\lambda \in J}$ is a net of holomorphic mappings $f_\lambda : D_1 \to D_2$ which is pointwise convergent in the topology $\sigma(X_2, \mathcal{N})$ to a function $f : D_1 \to \overline{D_2}$ and there exists a point $z_0 \in D_1$ such that $f(z_0) \in D_2$, then $f : D_1 \to D_2$ and f is holomorphic.*

Finally, we mention an analogous property of k_D-nonexpansive mappings.

Theorem 2.10. [22] *Let D_1, D_2 be two bounded and convex domains in two complex Banach spaces $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$, respectively, and let $\mathcal{N}$ be a norming set in $(X_2, \|\cdot\|_2)$. If $\overline{D}_2$ is compact in $\sigma(X_2, \mathcal{N})$, $\{f_\lambda\}_{\lambda \in J}$ is a net of nonexpansive (with respect to the Kobayashi distance) mappings $f_\lambda : D_1 \to D_2$, which is pointwise convergent in the topology $\sigma(X_2, \mathcal{N})$ to a function $f : D_1 \to \overline{D_2}$, and there exists a point $z_0 \in D_1$ such that $f(z_0) \in D_2$, then f also maps D_1 into D_2 and is nonexpansive with respect to the Kobayashi distance.*

3. Locally uniform linear convexity of a domain D with respect to the Kobayashi distance

In 2003 the first author introduced the notion of locally uniform linear convexity of a domain D with respect to the Kobayashi distance [5].

Definition 3.1. Let D be bounded and convex domain in a complex Banach space X. The metric space (D, k_D) is said to be *locally uniformly linearly convex* if there exist a point $w \in D$ and a real function

$$\tilde{\delta}_D(w, \cdot, \cdot, \cdot, \cdot)$$

such that for all $R_1 > 0$, $0 < R_2 \leq R_3$ and $0 < \epsilon \leq 2$, we have

$$\tilde{\delta}_D(w, R_1, R_2, R_3, \epsilon) > 0,$$

and the implication

$$\left.\begin{array}{l} k_D(z, x) \leq R \\ k_D(z, y) \leq R \\ k_D(x, y) \geq \epsilon R \end{array}\right\} \Rightarrow k_D\left(z, \frac{1}{2}x + \frac{1}{2}y\right) \leq \left(1 - \tilde{\delta}_D(w, R_1, R_2, R_3, \epsilon)\right) R$$

is valid for each $z \in D$ with $k_D(w, z) \leq R_1$, each $0 < R_2 \leq R \leq R_3$ and for all $x, y \in D$. The function $\tilde{\delta}_D(w, \cdot, \cdot, \cdot, \cdot,)$ is called a *modulus of linear convexity* for the Kobayashi distance k_D.

It is easy to observe that the point w in the above definition of locally uniform linear convexity may be replaced by any other point $w' \in D$.

Examples of such domains and their applications to holomorphic fixed point theory can be found in [6] (see also [27], [4] and Section 4 of the present paper).

In [6] it was proved that if the metric space (B_X, k_{B_X}) is locally uniformly linearly convex, then the Banach space $(X, \|\cdot\|)$ is uniformly convex and therefore $(X, \|\cdot\|)$ is reflexive. We also have a general result which shows that the locally uniform linear convexity of (D, k_D) implies the reflexivity of the Banach space X.

Theorem 3.1. [10] *Let X be a complex Banach space and let $D \subset X$ be a bounded and convex domain in X. If the metric space (D, k_D) is locally uniformly linearly convex, then the Banach space X is reflexive.*

Now we formulate the following equivalent definition of a locally uniformly linearly convex bounded domain.

Theorem 3.2. *Let D be bounded and convex domain in a complex Banach space $(X, \|\cdot\|)$. The following conditions are equivalent:*

(i) *the metric space (D, k_D) is locally uniformly linearly convex, that is, there exist a point $w \in D$ and a function*
$$\tilde{\delta}_D(w, \cdot, \cdot, \cdot, \cdot)$$
such that for all $R_1 > 0$, $0 < R_2 \le R_3$ and $0 < \epsilon \le 2$, we have
$$\tilde{\delta}_D\left(w, R_1, R_2, R_3, \epsilon\right) > 0,$$
and the implication
$$\left.\begin{array}{l} k_D\left(z, x\right) \le R \\ k_D\left(z, y\right) \le R \\ k_D\left(x, y\right) \ge \epsilon R \end{array}\right\} \Rightarrow k_D\left(z, \frac{1}{2}x + \frac{1}{2}y\right) \le \left(1 - \tilde{\delta}_D\left(w, R_1, R_2, R_3, \epsilon\right)\right) R$$
is valid for each $z \in D$ with $k_D(w, z) \le R_1$, each $0 < R_2 \le R \le R_3$ and for all $x, y \in D$;

(ii) *there exist a point $w \in D$ and a function*
$$\tilde{\delta}_{D,1}(w, \cdot, \cdot, \cdot, \cdot)$$
such that for all $R_1 > 0$, $0 < R_2 \le R_3$ and $0 < \epsilon \le 2$, we have
$$\tilde{\delta}_{D,1}\left(w, R_1, R_2, R_3, \epsilon\right) > 0,$$
and the implication
$$\left.\begin{array}{l} k_D\left(z, x\right) \le R \\ k_D\left(z, y\right) \le R \\ k_D\left(x, y\right) = \epsilon R \end{array}\right\} \Rightarrow k_D\left(z, \frac{1}{2}x + \frac{1}{2}y\right) \le \left(1 - \tilde{\delta}_{D,1}\left(w, R_1, R_2, R_3, \epsilon\right)\right) R$$
is valid for each $z \in D$ with $k_D(w, z) \le R_1$, each $0 < R_2 \le R \le R_3$ and for all $x, y \in D$.

Proof. (i) $\Rightarrow$ (ii) We simply can take $\tilde{\delta}_{D,1} := \tilde{\delta}_D$.

(ii) $\Rightarrow$ (i) We first observe that if $k_D\left(z, x\right) \le R$, $k_D\left(z, y\right) \le R$ and $k_D\left(x, y\right) \ge \epsilon R$, then there exist points x_1 and y_1 on the linear segment $[x, y]$ such that $k_D\left(x_1, y_1\right) = \epsilon R$ and $\frac{x_1 + y_1}{2} = \frac{x+y}{2}$. We also have $k_D\left(z, x_1\right) \le R$ and $k_D\left(z, y_1\right) \le R$. Therefore we may set $\tilde{\delta}_D := \tilde{\delta}_{D,1}$. □

Next we recall that if the bounded domain D is strictly convex, that is, $\overline{D}$ is strictly convex in a complex reflexive Banach space $(X, \|\cdot\|)$, then we have more information regarding the linear convexity of balls in (D, k_D).

Theorem 3.3. *([8], [9], [36], [40], [41]) Let D be a bounded and convex domain in a complex reflexive Banach space $(X, \|\cdot\|)$. If D is strictly convex, then each k_D-ball is also strictly convex in the linear sense.*

Remark 3.1. *Directly from the above theorem we get that for each bounded and strictly convex domain D in $\mathbb{C}^n$, the metric space (D, k_D) is locally uniformly linearly convex.*

Finally, we introduce the notion of a common modulus of linear convexity.

Definition 3.2. Let $\{D_j\}_{j\in J}$ be a family of equi-bounded and convex domains D_j in reflexive Banach spaces $(X_j, \|\cdot\|)$, $j \in J$, such that there exist $0 < r_1 < r_2$ and a point $\tilde{x} = \{\tilde{x}_j\}_{j\in J} \in \ell^\infty(X_j, J)$ with

$$B_j(\tilde{x}_j, r_1) \subset D_j \subset B_j(\tilde{x}_j, r_2)$$

for each $j \in J$. Also, for each $j \in J$, let the function $\tilde{\delta}_{D_j}(x_j, \cdot, \cdot, \cdot, \cdot)$ be a modulus of linear convexity for the Kobayashi distance k_{D_j}. If

$$\tilde{\delta}_{\{D_j\}_{j\in J}}(\tilde{x}, R_1, R_2, R_3, \epsilon) = \inf_{j\in J} \tilde{\delta}_{D_j}(x_j, R_1, R_2, R_3, \epsilon) > 0$$

for all $0 < \epsilon \leq 2$, $R_1 > 0$ and $0 < R_2 \leq R_3$, then we say that the family $\{D_j\}_{j\in J}$ of equi-bounded and convex domains is locally equi-uniformly linearly convex, or that this family has a common modulus of linear convexity $\tilde{\delta}_{\{D_j\}_{j\in J}}(\tilde{x}, \cdot, \cdot, \cdot, \cdot)$.

Remark 3.2. *It is easy to observe that in the above definition we may replace the modulus $\tilde{\delta}_{D_j}(x_j, \cdot, \cdot, \cdot, \cdot)$ with the modulus $\tilde{\delta}_{D_j,1}(x_j, \cdot, \cdot, \cdot, \cdot)$.*

4. Constructions of examples of locally equi-uniformly linearly convex domains

In this section we consider domains which are generated by either decreasing or increasing sequences of domains. In the case of bounded and convex domains in $\mathbb{C}^n$, we have the following results on the behavior of the Kobayashi distance on a bounded and convex domain D generated by a monotone sequence of bounded and convex domains.

Theorem 4.1. [20], [21], [26] *Let D be a bounded and convex domain in $\mathbb{C}^n$ and let $D = \bigcup_{m=1}^\infty D_m$, where $\{D_m\}_{m=1}^\infty$ be an increasing sequence of bounded and convex domains. Then the sequence $\{k_{D_m}\}_{m=1}^\infty$ of the corresponding Kobayashi distances converges as m tends to infinity to k_D, uniformly on compact sets.*

Before we state the second result we recall the following definitions.

Definition 4.1. We call $\{D_m\}_{m=1}^\infty$ a decreasing sequence of bounded and convex domains in $\mathbb{C}^n$ converging to a bounded and convex domain D if the sequence $\{D_m\}_{m=1}^\infty$ satisfies the following two conditions:

(i) $D_m \supset\supset D_{m+1}$ for all $m \in \mathbb{N}$;

(ii) $\bigcap_{m=1}^{\infty} D_m = \overline{D}$,

where $D_m \supset\supset D_{m+1}$ means that D_{m+1} is a relatively compact subset of D_m and $\overline{D}$ denotes the closure of D in $\mathbb{C}^n$.

Definition 4.2. Let D be a bounded domain in $\mathbb{C}^n$. A holomorphic function P defined in a neighborhood of the closure $\overline{D}$ of D is called a weak peak function for D at a boundary point $\xi \in \partial D$ if $\|P(\xi)\| = 1$ and $\|P(x)\| < 1$ for all $x \in D$.

Theorem 4.2. [24] *Let D be a bounded domain in $\mathbb{C}^n$ such that there exists a weak peak function for each point of ∂D and let $\{D_m\}_{m=1}^{\infty}$ be a decreasing sequence of bounded and convex domains converging to D. Then the sequence $\{k_{D_m}\}_{m=1}^{\infty}$ of the corresponding Kobayashi distances converges as m tends to infinity to k_D, uniformly on compact sets.*

It turns out that in the case of bounded and convex domains, the assumption regarding weak peak functions (see Corollary 2.1.11 in [1]) and the assumption that $D_m \supset\supset D_{m+1}$ for all $m \in \mathbb{N}$ are not necessary. We may simply assume that $D_m \supset D_{m+1}$ for all $m \in \mathbb{N}$ and that $\overline{\bigcap_{m=1}^{\infty} D_m} = \overline{D}$. We precede the statement and the proof of our theorem with the following lemma.

Lemma 4.3. *Let D be a bounded and convex domain in $\mathbb{C}^n$, $\tilde{x} \in D$, and let $\{D_m\}_{m=1}^{\infty}$ be a sequence of bounded and convex domains in $\mathbb{C}^n$ such that $D_m \supset D_{m+1}$ for all $m \in \mathbb{N}$ and $\overline{\bigcap_{m=1}^{\infty} D_m} = \overline{D}$. Then*

$$\lim_m \inf_{x \in D_m} \sup\{0 \le t \le 1 : \tilde{x} + t(x - \tilde{x}) \in D\} = 1.$$

Proof. Without any loss of generality we may assume that $\tilde{x} = 0 \in D$ and that $\overline{B_{\|\cdot\|}(0, r)} \subset D \subset D_1 \subset B_{\|\cdot\|}(0, R)$ for some $0 < r < R$. Assume that

$$\lim_m \inf_{x \in D_m} \sup\{0 \le t \le 1 : tx \in D\} = s < 1.$$

We obviously have $s \ge \frac{r}{R} > 0$. Also, there exist sequences $\{t_m\}_{m=1}^{\infty}$ and $\{x_m\}_{m=1}^{\infty}$ such that $x_m \in D_m$, $t_m x_m \in D \setminus \overline{B_{\|\cdot\|}(0, r)}$ and

$$\inf_{x \in D_m} \sup\{0 \le t \le 1 : tx \in D\} - \frac{1}{m} \le t_m$$

$$\le \sup\{0 \le t \le 1 : tx_m \in D\} \le \inf_{x \in D_m} \sup\{0 \le t \le 1 : tx \in D\} + \frac{1}{m}$$

for each $m = 1, 2, \ldots$. Hence $\lim_m t_m = s$. In addition, the compactness of $\overline{D_m}$, the inclusions $D_m \supset D_{m+1}$ for all $m \in \mathbb{N}$ and the convexity of each D_m imply the equality $\overline{\bigcap_{m=1}^{\infty} D_m} = \bigcap_{m=1}^{\infty} \overline{D_m} = \overline{D}$. By passing, if need be, to a subsequence, we may assume that there exists $x \in \overline{D}$ such that $\lim_m \|x_m - x\| = 0$. Next, since $\|x\| \ge r > 0$ and D is a convex open domain containing the closed ball $B_{\|\cdot\|}(0, r)$, we have $B_{\|\cdot\|}(sx, (1-s)r) \subset D$. Therefore $sx + (1-s)\frac{r}{2\|x\|} x$ is an element of D and this implies that

$$t_m x_m + (1-s)\frac{r}{2\|x\|} x_m = [t_m + (1-s)\frac{r}{2\|x\|}] x_m \in D$$

for all sufficiently large m. Hence we get

$$\inf_{x\in D_m}\sup\{0\le t\le 1: tx\in D\}-\frac{1}{m}\le t_m<t_m+(1-s)\frac{r}{2\|x\|}$$
$$\le \sup\{0\le t\le 1: tx_m\in D\}\le \inf_{x\in D_m}\sup\{0\le t\le 1: tx\in D\}+\frac{1}{m}.$$

Taking m to infinity, we get the following contradiction:

$$s=\lim_m t_m<s+(1-s)\frac{r}{2\|x\|}=\lim_m[t_m+(1-s)\frac{r}{2\|x\|}]$$
$$\le \lim_m[\inf_{x\in D_m}\sup\{0\le t\le 1: tx\in D\}+\frac{1}{m}]=s.$$

This completes our proof. □

Theorem 4.4. *Let D be a bounded and convex domain in $\mathbb{C}^n$, and let $\{D_m\}_{m=1}^{\infty}$ be a sequence of bounded and convex domains in $\mathbb{C}^n$ such that $D_m\supset D_{m+1}$ for all $m\in\mathbb{N}$ and $\overline{\bigcap_{m=1}^{\infty}D_m}=\overline{D}$. Then the sequence $\{k_{D_m}\}_{m=1}^{\infty}$ of the corresponding Kobayashi distances converges as m tends to infinity to k_D, uniformly on compact sets.*

Proof. Without any loss of generality we may assume that $0\in D$. Observe that $k_{D_m}(x,y)\le k_D(x,y)$ for each $x,y\in D$ and $m=1,2,...$ because $D\subset D_m$ for each $m=1,2,...$. This implies that

$$\limsup_{m\to\infty} k_{D_m}(x,y)\le k_D(x,y).$$

Next, since by Lemma 4.3,

$$\lim_m \inf_{x\in D_m}\sup\{0\le t\le 1:\tilde{x}+t(x-\tilde{x})\in D\}=1,$$

we get an increasing sequence $\{t_m\}_{m=1}^{\infty}$ such that $\lim_{m\to\infty}t_m=1$, $0<t_m<1$ and $t_mD_m\subset D$ for each m. Hence for $x,y\in D$, we obtain

$$k_{D_m}(x,y)=k_{t_mD_m}(t_mx,t_my)\ge k_D(t_mx,t_my)$$

and therefore

$$\liminf_{m\to\infty} k_{D_m}(x,y)\ge \lim_{m\to\infty}k_D(t_mx,t_my)=k_D(x,y).$$

So we have $\lim_{m\to\infty}k_{D_m}(x,y)=k_D(x,y)$. Applying Theorem 2.2, we observe that this convergence to k_D is uniform on compact subsets of D. □

Using the above results and compactness, it is not difficult to construct a decreasing or increasing sequence of equi-bounded and locally equi-uniformly linearly convex domains in $\mathbb{C}^n$.

Next, consider the Banach spaces $\ell^2\times\ell^2$ with the norms $\|(x,y)\|_{2,p}:=(\|x\|_2^p+\|y\|_2^p)^{\frac{1}{p}}$ for $(x,y)\in\ell^2\times\ell^2$, where $\|\cdot\|_2$ is the standard norm in the Hilbert space ℓ^2 and $1<p<\infty$. Let $B_{2,p}$ be the open unit ball in $(\ell^2\times\ell^2,\|(\cdot,\cdot)\|_{2,p})$. Observe that each three points in $B_{2,p}$ may be considered points in the strictly convex open unit ball $B_{2,p}^6$ in $(\mathbb{C}^3\times\mathbb{C}^3,\|(\cdot,\cdot)\|_{2,p})$ and since $(B_{2,p}^6,k_{B_{2,p}^6})$ is locally uniformly linearly convex (see Remark 3.1), the same is true for the metric space $(B_{2,p},k_{B_{2,p}})$ (see [4] and [6]). Let $1<p_1<p_2<\infty$ be fixed. By Theorems 2.4, 4.1 and 4.4, the family

$\{B^6_{2,p}\}_{p_1\le p\le p_2}$, treated as a family of domains in $(\mathbb{C}^6, \|\cdot\|)$, is locally equi-uniformly linearly convex. Hence the family $\{B_{2,p}\}_{p_1\le p\le p_2}$ is also locally equi-uniformly linearly convex.

5. A few facts from metric fixed point theory

Throughout this section we use the notations of [23].

Let (M, d) be a metric space. The symbol $\overline{B}(x, r)$ will stand for the closed ball of radius $r > 0$ centered at $x \in M$. For any nonempty and bounded subset $A \subset M$, we set

$$
\begin{aligned}
r_x(A) &= \sup\{d(x, a) : a \in A\}, \quad x \in M;\\
r(A) &= \inf\{r_a(A) : a \in A\};\\
\delta(A) &= \operatorname{diam}(A) = \sup\{r_a(A) : a \in A\} = \sup\{d(x, y) : x, y \in A\}.
\end{aligned}
$$

Recall that $r(A)$ is called the Chebyshev radius of A.

For a nonempty and bounded set A of M, set

$$\operatorname{cov}(A) = \bigcap\{\overline{B}(x, r) : x \in M,\ A \subset \overline{B}(x, r)\}.$$

We will say that a nonempty and bounded set A is an admissible set if and only if $A = \operatorname{cov}(A)$, that is, A is an intersection of closed balls. The family of all admissible subsets of M will be denoted by $\mathfrak{A}(M)$.

A family $\mathcal{S} \subset 2^M$ is called a convexity structure if

(i) $\emptyset, M \in \mathcal{S}$,

(ii) $\{x\} \in \mathcal{S}$ for each $x \in M$,

(iii) $\mathcal{S}$ contains the closed balls of M,

(iv) $\mathcal{S}$ is closed under arbitrary intersections.

Observe that the smallest convexity structure is the family $\mathfrak{A}(M)$ of all admissible subsets of M.

We say that a convexity structure $\mathcal{S}$ of M is compact if each descending chain of nonempty sets in $\mathcal{S}$ has a nonempty intersection.

A convexity structure $\mathcal{S}$ is said to be normal if for each $A \in \mathcal{S}$, we have either $\delta(A) = 0$ or $r(A) < \delta(A)$.

The following theorem due to M. A. Khamsi plays a crucial role in our subsequent considerations.

Theorem 5.1. [23] *Let (M, d) be a bounded metric space with a convexity structure $\mathfrak{A}(M)$ (that is, the family of all admissible subsets of M). If $\mathfrak{A}(M)$ is compact and normal, then any commuting family $\mathcal{F}$ of nonexpansive self-mappings of M has a common fixed point.*

6. The complex Banach space $(\ell^\infty(X_j, J), \|\cdot\|_\infty)$

In this section we introduce and study the complex Banach space $(\ell^\infty(X_j, J), \|\cdot\|_\infty)$.

Definition 6.1. Let J be an infinite set of indices and for each $j \in J$, let $(X_j, \|\cdot\|_j)$ be a complex reflexive Banach space. Then

$$\ell^\infty(X_j, J) := \{x = \{x_j\}_{j\in J} \in \prod_{j\in J} X_j : \sup_{j\in J} \|x_j\|_j < \infty\}.$$

We endow $\ell^\infty(X_j, J)$ with the supremum norm $\|\cdot\|_\infty$.

In $(\ell^\infty(X_j, J), \|\cdot\|_\infty)$ we also use another topology, namely, the topology $\sigma(X, \mathcal{N})$ generated by the following norming set:

$$\mathcal{N} = \{x^* = \{x_j^*\} \in (\ell^\infty(X_j, J))^* : x_j^* \in X_j^* \text{ for each } j \in J \text{ and there exists } j' \in J$$
$$\text{such that } x_{j"}^* = 0 \text{ for each } j" \neq j'\}.$$

Now we recall a well-known fact regarding holomorphic mappings [12].

Theorem 6.1. *Let J be an infinite set of indices and for each $j \in J$, let $(X_j, \|\cdot\|_j)$ be a complex reflexive Banach space. Let X be a complex Banach space and D a nonempty open subset of X. If $f : D \longrightarrow \ell^\infty(X_j, J)$ is locally bounded, then the following two statements are equivalent:*
(i) $f = \{f_j\}$ is holomorphic;
(ii) each $f_j : D \longrightarrow X_j$ is holomorphic.

In $(\ell^\infty(X_j, J), \|\cdot\|_\infty)$ we only consider certain special domains, namely, those domains generated by the Cartesian product of equi-bounded and convex domains $D_j \subset (X_j, \|\cdot\|)$, $j \in J$.

Definition 6.2. Consider the Banach space $(\ell^\infty(X_j, J), \|\cdot\|_\infty)$. For each $j \in J$, we denote by $B_j(x_j, r)$ the open ball in $(X_j, \|\cdot\|_j)$ centered at $x_j \in X_j$ and of radius $r > 0$. Consider a family of bounded and convex domains $\{D_j\}_{j\in J}$, where $\emptyset \neq D_j \subset X_j$ for each $j \in J$. Assume that there exist numbers $0 < r_1 < r_2$ and a point $\tilde{x} = \{\tilde{x}_j\}_{j\in J} \in \ell^\infty(X_j, J)$ such that

$$B_j(\tilde{x}_j, r_1) \subset D_j \subset B_j(\tilde{x}_j, r_2)$$

for each $j \in J$. Then the domain

$$D = \operatorname{int}(\prod_{j\in J} D_j) \in \ell^\infty(X_j, J),$$

where "int" denotes the interior of a set, is called the bounded and convex domain in $\ell^\infty(X_j, J)$ generated by the Cartesian product of the equi-bounded and convex domains $D_j \subset (X_j, \|\cdot\|)$, $j \in J$.

Using this definition and the above-mentioned norming set in $\ell^\infty(X_j, J)$, we can get a formula for the Kobayashi distance in a bounded and convex domain in $\ell^\infty(X_j, J)$, which is generated by the Cartesian product of equi-bounded and convex domains $D_j \subset (X_j, \|\cdot\|)$, $j \in J$ (see also [8] and [29]).

Theorem 6.2. *Let J be an infinite set of indices and for each $j \in J$, let $(X_j, \|\cdot\|_j)$ be a complex and reflexive Banach space. Let D be a bounded and convex domain in*

$\ell^{\infty}(X_j, J)$ generated by the Cartesian product of equi-bounded and convex domains $D_j \subset (X_j, \|\cdot\|)$, $j \in J$. Then the Kobayashi distance in D is given by

$$k_D\,(x, y) = \sup_{j\in J} k_{D_j}\,(x_j, y_j)$$

for each $x = \{x_j\}$ and $y = \{y_j\}$ in D.

Proof. Using the standard projection of $\ell^{\infty}\,(X_j, J)$ onto X_j, we get

$$k_{D_j}(x_j, y_j) \leq k_D(x, y)$$

for each $j \in J$ and consequently,

$$\sup_{j\in J} k_{D_j}(x_j, y_j) \leq k_D(x, y).$$

On the other hand, if we take two distinct points $x = \{x_j\}$ and $y = \{y_j\}$ in D, then by Theorem 2.7, for each $j' \in J$ with $x_{j'} \neq y_{j'}$, one can find a normalized complex $k_{D_{j'}}$-geodesic $f_{j'} : \Delta \to D_{j'}$, so that $f(0) = x_{j'}$, $f(z_{j'}) = y_{j'}$, $z_{j'} > 0$ and $k_{D_{j'}}(x_{j'}, y_{j'}) = k_{\Delta}(0, z_{j'})$. For other $j" \in J$, let $f_{j"} : \Delta \to D_{j"}$ be the constant function $f(z) = x_{j"} = y_{j"}$ and let $z_{j"} = 0$. Now, we put

$$0 < \tilde{z} = \sup_{j\in J} z_j < 1.$$

Then setting

$$s_j = \frac{z_j}{\tilde{z}}$$

and

$$g_j(z) = f_j(s_j z)$$

for $z \in \Delta$ and $j \in J$, we get a mapping $g : \Delta \to D$ such that $g(0) = x$ and $g(\tilde{z}) = y$. By Theorem 6.1, this mapping g is holomorphic. Hence we have

$$k_D\,(x, y) \leq k_{\Delta}(0, \tilde{z}) = \sup_{j\in J} k_{D_j}\,(x_j, y_j)\,.$$

So

$$k_D\,(x, y) = \sup_{j\in J} k_{D_j}\,(x_j, y_j)$$

and the proof is complete. □

7. Local admissibility of a domain D in $\ell^{\infty}(X_j, J)$ generated by the Cartesian product of equi-bounded, convex and locally equi-uniformly linearly convex domains D_j in $(X_j, \|\cdot\|)$, $j \in J$

Before we consider the common fixed point set of a commuting family of holomorphic mappings in a domain D in $\ell^{\infty}(X_j, J)$ generated by the Cartesian product of equi-bounded, convex and locally equi-uniformly linearly convex domains D_j in $(X_j, \|\cdot\|)$, $j \in J$, we present the following theorem.

Theorem 7.1. *Let J be an infinite set of indices and for each $j \in J$, let $(X_j, \|\cdot\|_j)$ be a complex and reflexive Banach space. Let a domain D in $\ell^\infty(X_j, J)$ be generated by the Cartesian product of equi-bounded and convex domains D_j in $(X_j, \|\cdot\|)$, $j \in J$. Let $G = \prod_{j\in J} G_j$ be a k_D-bounded product of nonempty, closed and convex subsets G_j of D_j, $j \in J$. If all (D_j, k_{D_j}) are locally equi-uniformly linearly convex, then the family $\mathfrak{A}(G)$ of all admissible sets in the metric space (G, k_D) is compact and normal.*

Proof. It is sufficient to observe that each nonempty admissible set E in (G, k_D) is a product of nonempty, closed and convex subsets of D_j, which are weakly compact, and that by assumption, each metric space $\left(D_j, k_{D_j}\right)$ is locally equi-uniformly linearly convex. $\square$

Let J be an infinite set of indices and for each $j \in J$, let $(X_j, \|\cdot\|_j)$ be a complex and reflexive Banach space. Let a domain D in $\ell^\infty(X_j, J)$ be generated by the Cartesian product of equi-bounded and convex domains D_j in $(X_j, \|\cdot\|)$, $j \in J$. We denote by $\mathrm{Fix}\,(f)$ the fixed point set of a self-mapping f of D and by $\mathrm{Fix}\,(\mathcal{F})$ the common fixed point set of a family $\mathcal{F}$ of self-mappings of D.

Corollary 7.2. *Let J be an infinite set of indices and for each $j \in J$, let $(X_j, \|\cdot\|_j)$ be a complex and reflexive Banach space. Let a domain D in $\ell^\infty(X_j, J)$ be generated by the Cartesian product of equi-bounded and convex domains D_j in $(X_j, \|\cdot\|)$, $j \in J$. Let $G = \prod_{j\in J} G_j$ be a k_D-bounded product of nonempty, closed and convex subsets of D_j. Assume that all (D_j, k_{D_j}) are locally equi-uniformly linearly convex. If $\mathcal{F}$ is a commuting family of k_D-nonexpansive self-mappings of G, then $\mathcal{F}$ has a common fixed point in G.*

Proof. It is sufficient to apply Theorems 5.1 and 7.1. $\square$

Corollary 7.3. *Let J be an infinite set of indices and for each $j \in J$, let $(X_j, \|\cdot\|_j)$ be a complex and reflexive Banach. Let a domain D in $\ell^\infty(X_j, J)$ be generated by the Cartesian product of equi-bounded and convex domains D_j in $(X_j, \|\cdot\|)$, $j \in J$. Assume that all (D_j, k_{D_j}) are locally equi-uniformly linearly convex. Let $\mathcal{F}$ be a commuting family of k_D-nonexpansive self-mappings of D and let $G = \prod_{j\in J} G_j$ be a k_D-bounded product of nonempty, closed and convex subsets of D_j, which is $\mathcal{F}$-invariant. If $\mathcal{F}$ has a common fixed point in D, then $\mathcal{F}$ has a common fixed point in G.*

Proof. Let x be a common fixed point of $\mathcal{F}$ in D and let $\overline{B_{k_D}(x, r)}$ be a closed ball in (D, k_D). For sufficiently large $r > 0$, the set $\tilde{G} = G \cap \overline{B_{k_D}(x, r)} \subset G$ is a nonempty, k_D-bounded and $\mathcal{F}$-invariant product of closed and convex subsets of D_j. By Corollary 7.2, $\mathcal{F}$ has a common fixed point in $\tilde{G}$. $\square$

Remark 7.1. *We remark in passing that the papers* [35], [32], [8] *and* [10] *are closely related to this section.*

8. The common fixed point set of commuting holomorphic self-mappings of a domain D in $\ell^\infty(X_j, J)$ generated by the Cartesian product of equi-bounded, convex and locally equi-uniformly linearly convex domains D_j in $(X_j, \|\cdot\|)$, $j \in J$

Now we are able to formulate and establish the main theorem of this paper.

Theorem 8.1. *Let J be an infinite set of indices and for each $j \in J$, let $(X_j, \|\cdot\|_j)$ be a complex and reflexive Banach space. Let D be a domain in $\ell^\infty(X_j, J)$ generated by the Cartesian product of equi-bounded and convex domains D_j in $(X_j, \|\cdot\|)$, $j \in J$. Assume that all (D_j, k_{D_j}) are locally equi-uniformly linearly convex. For any family $\mathcal{F}$ of commuting holomorphic (k_D-nonexpansive) self-mappings of D with a nonempty common fixed point set* $\mathrm{Fix}(\mathcal{F})$, *the set* $\mathrm{Fix}(\mathcal{F})$ *is a holomorphic (k_D-nonexpansive) retract of D.*

Proof. We base our proof on Bruck's method [2], [3] and confine our attention to the holomorphic case. Let

$$\mathcal{M}_\infty = \{g : g \text{ is a holomorphic self-mapping of } D,\ \ \mathrm{Fix}\,(\mathcal{F}) \subset \mathrm{Fix}\,(g)\}$$

and let $x_0 = \{x_{0j}\} \in \mathrm{Fix}\,(\mathcal{F})$ be fixed. The set $\mathcal{M}_\infty$ is a subset of the following Cartesian product:

$$\prod_{x \in D} \prod_{j \in J} \left\{ y_j \in D_j : k_{D_j}\,(y_j, x_{0j}) \le k_D\,(x, x_0) \right\} = \prod_{x \in D} \prod_{j \in J} D'_{x,j}.$$

Endowing each $D'_{x,j}$ with the weak topology, we obtain that each $D'_{x,j}$ is weakly compact and therefore, by Tychonoff's Theorem, the product $\prod_{x \in D} \prod_{j \in J} D'_{x,j}$ is compact in the product topology or equivalently, the product $\prod_{x \in D} (\prod_{j \in J} D'_{x,j})$ is compact in the product topology of $\sigma(X, \mathcal{N})$, where $\sigma(X, \mathcal{N})$ is generated by the following norming set:

$$\mathcal{N} = \{x^* = \{x_j^*\} \in (\ell^\infty\,(X_j, J))^* : x_j^* \in X_j^* \text{ for each } j \in J \text{ and there exists } j' \in J$$
$$\text{such that } x_{j''}^* = 0 \text{ for each } j'' \ne j'\}$$

(see Section 6). The set $\mathcal{M}_\infty$ is closed in the above product topology or equivalently, in the topology of coordinate pointwise $\sigma(X, \mathcal{N})$-convergence. This holds by Theorem 2.9 for holomorphic mappings and by Theorem 2.10 in the case of k_D-nonexpansive mappings. Observe that $\overline{D}$ is compact in the topology $\sigma(X, \mathcal{N})$. Now, we define a preorder $\le$ in $\mathcal{M}_\infty$ as follows: $g \le h$ if and only if

$$k_D\,(g\,(x)\,, w) \le k_D\,(h\,(x)\,, w)$$

for all $w \in \mathrm{Fix}\,(\mathcal{F})$ and $x \in D$. It follows from the compactness of $\prod_{x \in D} (\prod_{j \in J} D'_{x,j})$ in the product topology of $\sigma(X, \mathcal{N})$, the lower semicontinuity of k_D with respect to the topology $\sigma(X, \mathcal{N})$ (see Theorem 2.8) and the Kuratowski-Zorn Lemma that the set

$\mathcal{M}_\infty$ contains a minimal element r. We claim this r is a holomorphic retraction of D onto $\mathrm{Fix}\,(\mathcal{F})$. Indeed, we only need to show that

$$r(D) \subset \mathrm{Fix}\,(\mathcal{F})\,.$$

Assume, contrary to our claim, that there exists $y \in D$ such that $y_0 = r\,(y) \notin \mathrm{Fix}\,(\mathcal{F})$. Then by the minimality of r in $\mathcal{M}_\infty$ and the inequality $r \circ r \le r$, we obtain

$$k_D\left(r\left(y_0\right), w\right) = k_D\left(r\left(r\left(y\right)\right), w\right)$$

$$= k_D\left(r\left(y\right), w\right) = k_D\left(y_0, w\right) > 0$$

for all $w \in \mathrm{Fix}\,(\mathcal{F})$. We can even glean more information regarding the structure of the set $\mathcal{M}_\infty$. Since for each $j \in J$, after interchanging the j-coordinate functions between two arbitrarily chosen mappings from $\mathcal{M}_\infty$, we also end up with a mapping in $\mathcal{M}_\infty$, and since for each $g, h \in \mathcal{M}_\infty$ and $0 \le \beta \le 1$, the mapping $\beta g + (1-\beta)\,h$ also belongs to $\mathcal{M}_\infty$, for each $x \in D$, the set

$$\mathcal{M}_\infty(x) = \{g(x) : g \in \mathcal{M}_\infty\}$$

is equal to $\prod_{j\in J} D_j$", where each D_j" is convex and weakly compact. Let

$$C = \{(g \circ r)(y_0) : g \in \mathcal{M}_\infty\}\,.$$

Using the information on the shape of $\mathcal{M}_\infty(x)$, we see that C is k_D-bounded and $C = \prod_{j\in J} C_j$, where each C_j is convex and weakly compact. Directly from the definitions of $\mathcal{M}_\infty$, C and r, we obtain that the set C is $\mathcal{F}$-invariant and hence by Corollary 7.3, $C \cap \mathrm{Fix}\,(\mathcal{F}) \neq \emptyset$. Now, we choose an arbitrary point $w_0 = (g \circ r)\,(y_0) \in C \cap \mathrm{Fix}\,(\mathcal{F})\,.$ Then by the minimality of r in $\mathcal{M}_\infty$ and the inequality $g \circ r \le r$, we get the following contradiction:

$$0 < k_D\left(r\left(y_0\right), w_0\right) = k_D\left(\left(g \circ r\right)\left(y_0\right), w_0\right)$$

$$= k_D\left(\left(g \circ r\right)\left(y_0\right), \left(g \circ r\right)\left(y_0\right)\right) = 0.$$

This completes the proof of Theorem 8.1. □

Remark 8.1. *Recall that the example given in* [30] *shows that the assumption in the above theorem that the common fixed point set Fix*($\mathcal{F}$) *is nonempty is essential.*

Remark 8.2. *Theorem 8.1 is an extension of the theorems established in* [7] *and* [10].

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