Fixed Point Theory, 16(2015), No. 1, 3-14
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

SOME STABILITY CONCEPTS FOR DARBOUX PROBLEM FOR PARTIAL FRACTIONAL DIFFERENTIAL EQUATIONS ON UNBOUNDED DOMAIN

SAÏD ABBAS* AND MOUFFAK BENCHOHRA**

* Laboratory of Mathematics
University of Saida
PO Box 138, 20000 Saida, Algeria
E-mail: abbasmsaid@yahoo.fr

**Laboratoire de Mathématiques, Université de Sidi Bel-Abbès
B.P. 89, 22000, Sidi Bel-Abbès, Algérie
Department of Mathematics,
and
King Abdulaziz University
P.O. Box 80203, Jeddah 21589, Saudi Arabia
E-mail: benchohra@univ-sba.dz

Abstract. In the present paper, we investigate some Ulam's type stability concepts for the Darboux problem of partial differential equations of fractional order on unbounded domain. We use a nonlinear alternative of Leray-Schauder type for contraction maps in Fréchet spaces and a fractional version of Gronwall's inequality.
Key Words and Phrases: Fractional differential equations, left-sided mixed Riemann-Liouville integral, Caputo fractional order derivative, Darboux problem, Fréchet space, Ulam stability, fixed point.
2010 Mathematics Subject Classification: 26A33, 35L70, 34A40, 45N05, 47H10.

1. Introduction

The fractional calculus deals with extensions of derivatives and integrals to noninteger orders. It represents a powerful tool in applied mathematics to study a myriad of problems from different fields of science and engineering, with many break-through results found in mathematical physics, finance, hydrology, biophysics, thermodynamics, control theory, statistical mechanics, astrophysics, cosmology and bioengineering [12, 22, 26]. There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs of Abbas *et al.* [6], Kilbas *et al.* [18], Miller and Ross [19], the papers of Abbas *et al.* [1, 2, 3, 5, 4, 7], Vityuk and Golushkov [28], and the references therein.

The stability of functional equations was originally raised by Ulam in 1940 in a talk given at Wisconsin University. The problem posed by Ulam was the following: Under what conditions does there exist an additive mapping near an approximately additive mapping? (for more details see [27]). The first answer to Ulam's question was given by Hyers in 1941 in the case of Banach spaces in [13]. Thereafter, this type of stability is called the Ulam-Hyers stability. In 1978, Rassias [23] provided a remarkable generalization of the Ulam-Hyers stability of mappings by considering variables. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus, the stability question of functional equations is how do the solutions of the inequality differ from those of the given functional equation? Considerable attention has been given to the study of the Ulam-Hyers and Ulam-Hyers-Rassias stability of all kinds of functional equations; one can see the monographs of [14, 15]. Bota-Boriceanu and Petrusel [8], Petru *et al.* [20, 21], and Rus [24, 25] discussed the Ulam-Hyers stability for operatorial equations and inclusions. Castro and Ramos [9] considered the Hyers-Ulam-Rassias stability for a class of Volterra integral equations. Ulam stability for fractional differential equations with Caputo derivative are proposed by Wang *et al.* [29, 30]. Some stability results for fractional integral equation are obtanied by Wei *et al.* [31]. More details from historical point of view, and recent developments of such stabilities are reported in the monographs [14, 15, 17] and the papers [16, 24, 29, 30, 31].

Motivated by those papers, in this paper, we discuss the Ulam stabilities for the following fractional partial differential equation

$$^{c}D^{r}_{\theta}u(x,y) = f(x,y,u(x,y)); \quad \text{if } (x,y) \in J := [0,\infty) \times [0,\infty), \tag{1.1}$$

where $\theta = (0,0)$, $^{c}D^{r}_{\theta}$ is the fractional Caputo derivative of order $r = (r_1, r_2) \in (0,1] \times (0,1]$, $f : J \times E \to E$ is a given function, E is a (real or complex) Banach space. We use a nonlinear alternative of Leray-Schauder type for contraction maps in Fréchet spaces and a fractional version of Gronwall's inequality. We prove that the equation (1.1) is generalized Ulam-Hyers-Rassias stable.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. For each $p \in \mathbb{N}^*$, denote $L^1(J_p)$ the space of Bochner-integrable functions $u : J_p \to E$ with the norm

$$\|u\|_{L^1} = \int_0^p \int_0^p \|u(x,y)\|_E dydx,$$

where $J_p := [0,p] \times [0,p]$, and $\|.\|_E$ denotes a complete norm on E.
As usual, by $AC(J_p)$ we denote the space of absolutely continuous functions from J_p into E, and $\mathcal{C}_p := C(J_p)$ is the Banach space of all continuous functions from J_p into E with the norm $\|.\|_p$ defined by

$$\|u\|_p = \sup_{(x,y)\in J_p} \|u(x,y)\|_E. \tag{2.1}$$

Let $\theta = (0,0)$, $r_1,\ r_2 > 0$ and $r = (r_1, r_2)$. For $f \in L^1(J_p)$, the expression

$$(I^{r}_{\theta}f)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t)dtds,$$

is called the left-sided mixed Riemann-Liouville integral of order r, where $\Gamma(.)$ is the (Euler's) Gamma function defined by $\Gamma(\xi)=\int_0^{\infty}t^{\xi-1}e^{-t}dt;\ \xi>0$.

In particular,

$$(I_\theta^\theta u)(x,y)=u(x,y),\ (I_\theta^\sigma u)(x,y)=\int_0^x\int_0^y u(s,t)dtds;\ \text{for almost all } (x,y)\in J_p,$$

where $\sigma=(1,1)$.
For instance, $I_\theta^r u$ exists for all $r_1,r_2\in(0,\infty)$, when $u\in L^1(J_p)$. Note also that when $u\in C(J_p)$, then $(I_\theta^r u)\in C(J_p)$, moreover

$$(I_\theta^r u)(x,0)=(I_\theta^r u)(0,y)=0;\ x,y\in[0,p].$$

Example 2.1. Let $\lambda,\omega\in(0,\infty)$ and $r=(r_1,r_2)\in(0,\infty)\times(0,\infty)$, then

$$I_\theta^r x^\lambda y^\omega=\frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)}x^{\lambda+r_1}y^{\omega+r_2},\ \text{for almost all } (x,y)\in J_p.$$

By $1-r$ we mean $(1-r_1,1-r_2)\in[0,1)\times[0,1)$. Denote by $D^2_{xy}:=\frac{\partial^2}{\partial x\partial y}$, the mixed second order partial derivative.

Definition 2.2. [28] Let $r\in(0,1]\times(0,1]$ and $u\in L^1(J_p)$. The Caputo fractional-order derivative of order r of u is defined by the expression

$${}^cD_\theta^r u(x,y)=(I_\theta^{1-r}D^2_{xy}u)(x,y).$$

The case $\sigma=(1,1)$ is included and we have

$$({}^cD_\theta^\sigma u)(x,y)=(D^2_{xy}u)(x,y);\ \text{for almost all } (x,y)\in J_p.$$

Example 2.3. Let $\lambda,\omega\in(0,\infty)$ and $r=(r_1,r_2)\in(0,1]\times(0,1]$, then

$${}^cD_\theta^r x^\lambda y^\omega=\frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda-r_1)\Gamma(1+\omega-r_2)}x^{\lambda-r_1}y^{\omega-r_2};\ \text{for almost all } (x,y)\in J_p.$$

Let X be a Fréchet space with a family of semi-norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}^*}$. We assume that the family of semi-norms $\{\|\cdot\|_n\}$ verifies:

$$\|x\|_1\le\|x\|_2\le\ \|x\|_3\le...\quad for\ every\ x\in X.$$

Let $Y\subset X$, we say that Y is bounded if for every $n\in\mathbb{N}$, there exists $\overline{M}_n>0$ such that

$$\|y\|_n\le\overline{M}_n\quad for\ all\ y\in Y.$$

To X we associate a sequence of Banach spaces $\{(X^n,\|\cdot\|_n)\}$ as follows: For every $n\in\mathbb{N}$, we consider the equivalence relation $\sim_n$ defined by: $x\sim_n y$ if and only if $\|x-y\|_n=0$ for $x,y\in X$. We denote $X^n=(X|_{\sim_n},\|\cdot\|_n)$ the quotient space, the completion of X^n with respect to $\|\cdot\|_n$. To every $Y\subset X$, we associate a sequence $\{Y^n\}$ of subsets $Y^n\subset X^n$ as follows : For every $x\in X$, we denote $[x]_n$ the equivalence class of x of subset X^n and we defined $Y^n=\{[x]_n: x\in Y\}$. We denote $\overline{Y^n}$, $int_n(Y^n)$ and $\partial_n Y^n$, respectively, the closure, the interior and the boundary of Y^n with respect to $\|\cdot\|_n$ in X^n. For more information about this subject see [10].

Let $\mathcal{C}:=C(J)$ be the space of all continuous functions from J into E. For each $p\in\mathbb{N}^*$ define in the set $\mathcal{C}$ the semi-norms $\|x\|_p$ defined by (2.1). The set $\mathcal{C}$ is Fréchet space with the family of semi-norms $\{\|u\|_p\}_{p\in\mathbb{N}^*}$.

Now, we consider the Ulam stability of fractional differential equation (1.1). Let ϵ be a positive real number and $\Phi : J \to [0, \infty)$ be a continuous function. We consider the following inequalities

$$\|{}^cD^r_\theta u(x, y) - f(x, y, u(x, y))\|_E \le \epsilon; \quad \text{if } (x, y) \in J. \tag{2.2}$$

$$\|{}^cD^r_\theta u(x, y) - f(x, y, u(x, y))\|_E \le \Phi(x, y); \quad \text{if } (x, y) \in J. \tag{2.3}$$

$$\|{}^cD^r_\theta u(x, y) - f(x, y, u(x, y))\|_E \le \epsilon\Phi(x, y); \quad \text{if } (x, y) \in J. \tag{2.4}$$

Definition 2.4. [24] Problem (1.1) is Ulam-Hyers stable if there exists a real number $c_f > 0$ such that for each $\epsilon > 0$ and for each solution $u \in \mathcal{C}$ of the inequality (2.2), there exists a solution $v \in \mathcal{C}$ of problem (1.1) with

$$\|u(x, y) - v(x, y)\|_E \le \epsilon c_f; \ (x, y) \in J.$$

Definition 2.5. [24] Problem (1.1) is generalized Ulam-Hyers stable if there exists $\theta_f \in C([0, \infty), [0, \infty))$, $\theta_f(0) = 0$ such that, for each $\epsilon > 0$ and for each solution $u \in \mathcal{C}$ of the inequality (2.2), there exists a solution $v \in \mathcal{C}$ of problem (1.1) with

$$\|u(x, y) - v(x, y)\|_E \le \theta_f(\epsilon); \ (x, y) \in J.$$

Definition 2.6. [24] Problem (1.1) is Ulam-Hyers-Rassias stable with respect to Φ if there exists a real number $c_{f,\Phi} > 0$ such that for each $\epsilon > 0$ and for each solution $u \in \mathcal{C}$ of the inequality (2.4), there exists a solution $v \in \mathcal{C}$ of problem (1.1) with

$$\|u(x, y) - v(x, y)\|_E \le \epsilon c_{f,\Phi}\Phi(x, y); \ (x, y) \in J.$$

Definition 2.7. [24] Problem (1.1) is generalized Ulam-Hyers-Rassias stable with respect to Φ if there exists a real number $c_{f,\Phi} > 0$ such that for each solution $u \in \mathcal{C}$ of the inequality (2.3), there exists a solution $v \in \mathcal{C}$ of problem (1.1) with

$$\|u(x, y) - v(x, y)\|_E \le c_{f,\Phi}\Phi(x, y); \ (x, y) \in J.$$

Remark 2.8. It is clear that

(i) Definition 2 $\Rightarrow$ Definition 2,
(ii) Definition 2 $\Rightarrow$ Definition 2,
(iii) Definition 2 for $\Phi(x, y) = 1$ $\Rightarrow$ Definition 2.

Remark 2.9. A function $u \in \mathcal{C}$ is a solution of the inequality (2.2) if and only if there exist a function $g \in \mathcal{C}$ (which depend on u) such that

(i) $\|g(x, y)\|_E \le \epsilon$,
(ii) ${}^cD^r_\theta u(x, y) = f(x, y, u(x, y)) + g(x, y); \quad \text{if } (x, y) \in J.$

One can have similar remarks for the inequalities (2.3) and (2.4). So, the Ulam stabilities of the fractional differential equations are some special types of data dependence of the solutions of fractional differential equations.

Definition 2.10. Let X be a Fréchet space. A function $N : X \longrightarrow X$ is said to be a contraction if for each $n \in \mathbb{N}^*$ there exists $k_n \in [0, 1)$ such that

$$\|N(u) - N(v)\|_n \le k_n\|u - v\|_n \ \text{ for all } u, v \in X.$$

In the sequel, we need the following fixed point theorem in Fréchet spaces, given by Frigon and Granas [10].

Theorem 2.11. [10] *Let X be a Fréchet space and $Y \subset X$ a closed subset in X. Let $N : Y \longrightarrow X$ be a contraction such that $N(Y)$ is bounded. Then one of the following statements holds:*

(i) *The operator N has a unique fixed point;*
(ii) *There exists $\lambda \in [0,1),\ n \in \mathbb{N}^*$ and $u \in \partial_n Y^n$ such that $\|u - \lambda N(u)\|_n = 0$.*

In the sequel we will make use of the following generalization of Gronwall's lemma for two independent variables and singular kernel.

Lemma 2.12. (Gronwall lemma) [11] *Let $\upsilon : J_p \to [0,\infty)$ be a real function and $\omega(.,.)$ be a nonnegative, locally integrable function on J_p. If there are constants $c > 0$ and $0 < r_1, r_2 < 1$ such that*

$$\upsilon(x,y) \leq \omega(x,y) + c\int_0^x \int_0^y \frac{\upsilon(s,t)}{(x-s)^{r_1}(y-t)^{r_2}}dtds,$$

then there exists a constant $\delta = \delta(r_1, r_2)$ such that

$$\upsilon(x,y) \leq \omega(x,y) + \delta c\int_0^x \int_0^y \frac{\omega(s,t)}{(x-s)^{r_1}(y-t)^{r_2}}dtds,$$

for every $(x,y) \in J_p$.

3. Main results

In this section, we present conditions for the Ulam stability of problem (1.1).

For each $p \in \mathbb{N}^*$, consider the following Darboux problem of partial differential equations

$$\begin{cases} {}^cD_\theta^r u(x,y) = f(x,y,u(x,y)); & \text{if } (x,y) \in J_p, \\ u(x,0) = \varphi(x); \ x \in [0,p], \\ u(0,y) = \psi(y); \ y \in [0,p], \\ \varphi(0) = \psi(0), \end{cases} \tag{3.1}$$

where $\varphi, \psi : [0,p] \to E$ are given absolutely continuous functions.

In the sequel, we need the following Lemma given by the authors [2].

Lemma 3.1. [2] *Let $0 < r_1, r_2 \leq 1,\ \mu(x,y) = \varphi(x) + \psi(y) - \varphi(0)$. A function $u \in \mathcal{C}_p$ is a solution of the fractional integral equation*

$$u(x,y) = \mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,u(s,t))dtds, \tag{3.2}$$

if and only if u is a solution of the problem (3.1).

Lemma 3.2. *If $u \in \mathcal{C}$ is a solution of the inequality (2.2) then u is a solution of the following integral inequality*

$$\begin{aligned} \|u(x,y) - \mu(x,y) - \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,u(s,t))dtds\|_E \\ \leq \frac{\epsilon\ p^{r_1+r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)};\ \textit{if } (x,y) \in J_p. \end{aligned} \tag{3.3}$$

Proof. By Remark 2.9, for $(x,y) \in J$, we have

$${}^cD_\theta^r u(x,y) = f(x,y,u(x,y)) + g(x,y).$$

Then, from Lemma 3.1, for $(x,y)\in J$, we get

$$u(x,y)=\mu(x,y)+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}(g(s,t)+f(s,t,u(s,t)))dtds.$$

Thus, for $(x,y)\in J_p$ we obtain

$$\begin{aligned}&\left\|u(x,y)-\mu(x,y)-\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}f(s,t,u(s,t))dtds\right\|_E\\&=\left\|\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}g(s,t)dtds\right\|_E\\&\leq\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}\|g(s,t)\|_Edtds\\&\leq\frac{\epsilon\ p^{r_1+r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}.\end{aligned}$$

Hence, we obtain (3.3).

Remark 3.3. We have similar results for the solutions of the inequalities (2.3) and (2.4).

Consider now the following Darboux problem of partial differential equations

$$\begin{cases}{}^cD_\theta^ru(x,y)=f(x,y,u(x,y)); & \text{if } (x,y)\in J,\\ u(x,0)=\varphi(x);\ x\in[0,\infty),\\ u(0,y)=\psi(y);\ y\in[0,\infty),\\ \varphi(0)=\psi(0),\end{cases}\tag{3.4}$$

where $\varphi,\psi:[0,\infty)\to E$ are given absolutely continuous functions.

Now, we make use of the nonlinear alternative of Leray-Schauder type for contraction maps on Fréchet spaces (Theorem 2.11), for proving the uniqueness of the solution of the system (3.4).

Theorem 3.4. *Assume that the following hypotheses hold*

(H_1) *The function $f:J\times E\times E\to E$ is continuous,*
(H_2) *For each $p\in\mathbb{N}^*$, there exists $l_{p,f}\in\mathcal{C}_p$ such that for each $(x,y)\in J_p$*

$$\|f(x,y,u)-f(x,y,v)\|_E\leq l_{p,f}(x,y)\|u-v\|_E,\ \textit{for each } u,\ v\in E.$$

If

$$\frac{l^*_{p,f}p^{r_1+r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}<1,\tag{3.5}$$

where

$$l^*_{p,f}=\sup_{(x,y)\in J_p}l_{p,f}(x,y),$$

then the system (3.4) has a unique solution on J.

Proof. Transform the problem (3.4) into a fixed point problem. Consider the operator $N : \mathcal{C}_p \to \mathcal{C}_p$ defined by,

$$(Nu)(x,y) = \begin{cases} \phi(x,y), & (x,y) \in \tilde{J}, \\ \mu(x,y) & \\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} & \\ \times f(s,t,u(s,t))dtds, & (x,y) \in J. \end{cases}$$

Let u be a possible solution of the problem $u = \lambda N(u)$ for some $0 < \lambda < 1$. This implies that for each $(x,y) \in J_p$, we have

$$u(x,y) = \lambda\mu(x,y) + \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,u(s,t))dtds.$$

This implies by (H_2) that for each $(x,y) \in J_p$, we have

$$\begin{aligned} \|u(x,y)\|_E &\leq \|\mu(x,y)\|_E + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\|f(s,t,0)\|_E dtds \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\|f(s,t,u(s,t)) - f(s,t,0)\|_E dtds \\ &\leq \|\mu(x,y)\|_E + \frac{f^* p^{r_1+r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} l_{p,f}(s,t)\|u(s,t)\|_E dtds \\ &\leq \|\mu\|_p + \frac{f^* p^{r_1+r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \\ &+ \frac{l^*_{p,f}}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}(s,t)\|u(s,t)\|_E dtds, \end{aligned}$$

where

$$f^* = \sup_{(x,y)\in J_p} \|f(x,y,0)\|_E.$$

Set

$$w = \|\mu\|_p + \frac{f^* p^{r_1+r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}.$$

From Lemma 2.12, there exists a constant $\delta = \delta(r_1, r_2)$ such that we have for $(x,y) \in J_p$,

$$\begin{aligned} \|u(x,y)\|_E &\leq w\left(1 + \frac{\delta l^*_{p,f}}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} dtds\right) \\ &\leq w\left(1 + \frac{\delta l^*_{p,f} p^{r_1+r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}\right) := M_p. \end{aligned}$$

Thus

$$\|u(x,y)\|_E \leq M_p.$$

Set

$$U = \{u \in \mathcal{C}_p : \|u\|_p \leq 1 + M_p \text{ for all } p \in \mathbb{N}\}.$$

We shall show that $N : U \longrightarrow \mathcal{C}_p$ is a contraction map. Indeed, consider $v, w \in U$. Then for each $(x,y) \in J_p$, we obtain

$$\|N(v) - N(w)\|_p \leq \frac{l^*_{p,f} p^{r_1+r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \|v - w\|_p.$$

Hence by (3.5), $N : U \longrightarrow \mathcal{C}_p$ is a contraction.
By our choice of U, there is no $u \in \partial U^n$ such that $u = \lambda N(u)$, for $\lambda \in (0,1)$.
As a consequence of Theorem 2.11, we deduce that N has a unique fixed point u in U which is a solution to problem (3.4).
Theorem 3.5 *Assume that the assumptions* (H_1), (H_2) *and the following hypothesis hold*

(H_3) *For each* $p \in \mathbb{N}^*$, $\Phi \in L^1(J_p, [0,\infty))$ *and there exists* $\lambda_\Phi > 0$ *such that, for each* $(x,y) \in J_p$ *we have*

$$(I^r_\theta \Phi)(x,y) \leq \lambda_\Phi \Phi(x,y).$$

If the condition (3.5) holds, then equation (1.1) is generalized Ulam-Hyers-Rassias stable.
Proof. Let $u \in \mathcal{C}$ be a solution of the inequality (2.3). By Theorem 3.4, there v is a unique solution of the problem (3.4). Then for $(x,y) \in J$, we have

$$v(x,y) = \mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,v(s,t))dtds.$$

By differential inequality (2.3), for each $(x,y) \in J$, we have

$$\begin{aligned}\|u(x,y) - \mu(x,y) - &\frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,u(s,t))dtds\|_E \\ &\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \Phi(s,t)dtds.\end{aligned}$$

Thus, by (H_3) for each $(x,y) \in J_p$, we get

$$\begin{aligned}\|u(x,y) - \mu(x,y) - &\frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,u(s,t))dtds\|_E \\ &\leq \lambda_\Phi \Phi(x,y).\end{aligned}$$

Hence for each $(x,y)\in J_p$, it follows that

$$\begin{aligned}\|u(x,y)-v(x,y)\|_E &\le \|u(x,y)-\mu(x,y)\\ &\quad -\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\\ &\quad \times f(s,t,u(s,t))dtds\|_E\\ &\quad +\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\\ &\quad \times \|f(s,t,u(s,t))-f(s,t,v(s,t))\|_E dtds\\ &\le \lambda_\Phi\Phi(x,y)\\ &\quad +\frac{l^*_{p,f}}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\\ &\quad \times \|u(s,t)-v(s,t)\|_E dtds.\end{aligned}$$

From Lemma 2.12, there exists a constant $\gamma=\gamma(r_1,r_2)$ such that

$$\begin{aligned}\|u(x,y)-v(x,y)\|_E &\le \lambda_\Phi\Phi(x,y)\\ &\quad +\frac{\gamma l^*_{p,f}\lambda_\Phi}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\Phi(s,t)dtds\\ &\le (1+\gamma l^*_{p,f}\lambda_\Phi)\lambda_\Phi\Phi(x,y)\\ &:= c_{f,\Phi}\Phi(x,y).\end{aligned}$$

Consequently, equation (1.1) is generalized Ulam-Hyers-Rassias stable.

4. An example

Let $E=l^1=\left\{w=(w_1,w_2,...,w_n,...):\sum_{n=1}^{\infty}|w_n|<\infty\right\}$, be the Banach space with norm

$$\|w\|_E=\sum_{n=1}^{\infty}|w_n|.$$

Consider the following infinite system of partial hyperbolic fractional implicit differential equations of the form

$$^cD^r_\theta u_n(x,y)=f_n(x,y,u_n(x,y));\quad \text{if } (x,y)\in J:=[0,\infty)\times[0,\infty), \tag{4.1}$$

with the initial conditions

$$\begin{cases}u_n(x,0)=(x,0,...,0,...);\ x\in[0,\infty),\\ u_n(0,y)=(y^2,0,...,0,...);\ y\in[0,\infty),\end{cases} \tag{4.2}$$

where $(r_1,r_2)\in(0,1]\times(0,1]$,

$u=(u_1,u_2,...,u_n,...)$, $\ {}^cD^r_\theta u=({}^cD^r_\theta u_1,{}^cD^r_\theta u_2,...,{}^cD^r_\theta u_n,...)$, $\ f=(f_1,f_2,...,f_n,...)$,

and for each $n\in\mathbb{N}^*$,

$$f_n(x,y,u(x,y))=\frac{c_p}{(1+5|u_n(x,y)|)e^{x+y+3}};\ (x,y)\in J,\ \textit{and}\ \ p\in\mathbb{N}^*.$$

For each $u,\ \overline{u} \in E,\ n \in \mathbb{N}^*$ and $(x,y) \in J_p$ we have

$$\|f_n(x,y,u_n) - f_n(x,y,\overline{u}_n)\|_E \leq \frac{5c_p}{e^3}\|u_n - \overline{u}_n|.$$

Thus, for each $u, \overline{u} \in E$ and $(x,y) \in J_p$, we get

$$\begin{aligned}\|f(x,y,u(x,y)) - f(x,y,\overline{u}(x,y))\|_E &= \sum_{n=1}^{\infty} |f_n(x,y,u(x,y)) - f_n(x,y,\overline{u}(x,y))| \\ &\leq \frac{1}{10e^3}\sum_{n=1}^{\infty}|u_n - \overline{u}_n| \\ &= \frac{5c_p}{e^3}\|u - \overline{u}\|_E.\end{aligned}$$

Hence conditions (H_1) and (H_2) are satisfied with $l^*_{p,f} = \frac{5c_p}{e^3}$. We shall show that condition (3.5) holds. Indeed

$$\frac{l^*_{p,f}p^{r_1+r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} = \frac{5}{e^3} < 1,$$

which is satisfied for each $(r_1, r_2) \in (0,1] \times (0,1]$. The hypothesis (H_3) is satisfied with $\Phi(x,y) = xy^2$ and $\lambda_\Phi = 8p^{r_1+r_2}$. Indeed, for each $(x,y) \in J_p$ we get

$$(I^r_\theta \Phi)(x,y) = \frac{\Gamma(2)\Gamma(3)}{\Gamma(2+r_1)\Gamma(3+r_2)}x^{1+r_1}y^{2+r_2} \leq 8p^{r_1+r_2}xy^2 = \lambda_\Phi \Phi(x,y).$$

Consequently Theorem 3.5 implies that equation (4.1) is generalized Ulam-Hyers-Rassias stable.

References

[1] S. Abbas, D. Baleanu, M. Benchohra, *Global attractivity for fractional order delay partial integro-differential equations*, Adv. Difference Eq., **2012**, 19 pages, doi:10.1186/1687-1847-2012-62.

[2] S. Abbas, M. Benchohra, *Darboux problem for perturbed partial differential equations of fractional order with finite delay*, Nonlinear Anal. Hybrid Syst., **3**(2009), 597-604.

[3] S. Abbas, M. Benchohra, *Fractional order partial hyperbolic differential equations involving Caputo's derivative*, Stud. Univ. Babeş-Bolyai Math., **57**(4)(2012), 469-479.

[4] S. Abbas, M. Benchohra, J.J. Nieto, *Global uniqueness results for fractional order partial hyperbolic functional differential equations*, Adv. in Difference Eq., **2011**, Art. ID 379876, 25 pp.

[5] S. Abbas, M. Benchohra, A. Cabada, *Partial neutral functional integro-differential equations of fractional order with delay*, Bound. Value Prob., **2012**(2012), 128, 15 pp.

[6] S. Abbas, M. Benchohra, G.M. N'Guérékata, Topics in Fractional Differential Equations, Springer, New York, 2012.

[7] S. Abbas, M. Benchohra, A.N. Vityuk, *On fractional order derivatives and Darboux problem for implicit differential equations*, Frac. Calc. Appl. Anal., **15**(2012), 168-182.

[8] M.F. Bota-Boriceanu, A. Petruşel, *Ulam-Hyers stability for operatorial equations and inclusions*, Analele Univ. Al.I. Cuza Iaşi, **57**(2011), 65-74.

[9] L.P. Castro, A. Ramos, *Hyers-Ulam-Rassias stability for a class of Volterra integral equations*, Banach J. Math. Anal., **3**(2009), 36-43.

[10] M. Frigon, A. Granas, *Théorèmes d'existence pour des inclusions différentielles sans convexité*, C.R. Acad. Sci. Paris, Ser. I, **310**(1990), 819-822.

[11] D. Henry, *Geometric Theory of Semilinear Parabolic Partial Differential Equations*, Springer-Verlag Berlin, 1989.

[12] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[13] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci., **27**(1941), 222-224.

[14] D.H. Hyers, G. Isac, Th.M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, 1998.

[15] S.-M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, 2001.

[16] S.-M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis*, Springer, New York, 2011.

[17] S.-M. Jung, *A fixed point approach to the stability of a Volterra integral equation*, Fixed Point Theory Appl., **2007**(2007), Article ID 57064, 9 pages.

[18] A.A. Kilbas, Hari M. Srivastava, Juan J. Trujillo, *Theory and Applications of Fractional Differential Equations.* North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[19] K.S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York, 1993.

[20] T.P. Petru, M.-F. Bota, *Ulam-Hyers stabillity of operational inclusions in complete gauge spaces*, Fixed Point Theory, **13**(2012), 641-650.

[21] T.P. Petru, A. Petruşel, J.-C. Yao, *Ulam-Hyers stability for operatorial equations and inclusions via nonself operators*, Taiwanese J. Math., **15**(2011), 2169-2193.

[22] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[23] Th.M. Rassias, *On the stability of linear mappings in Banach spaces*, Proc. Amer. Math. Soc., **72**(1978), 297-300.

[24] I.A. Rus, *Ulam stability of ordinary differential equations*, Studia Univ. Babes-Bolyai, Math., **54**(4)(2009), 125-133.

[25] I.A. Rus, *Remarks on Ulam stability of the operatorial equations,* Fixed Point Theory, **10**(2009), 305-320.

[26] V.E. Tarasov, *Fractional Dynamics. Applications of Fractional Calculus to Dynamics of Particles, Fields and Media*, Springer, Heidelberg, 2010.

[27] S.M. Ulam, *A Collection of Mathematical Problems*, Interscience Publishers, New York, 1968.

[28] A.N. Vityuk, A.V. Golushkov, *Existence of solutions of systems of partial differential equations of fractional order,* Nonlinear Oscil., **7**(3)(2004), 318-325.

[29] J. Wang, L. Lv, Y. Zhou, *Ulam stability and data dependence for fractional differential equations with Caputo derivative*, E.J. Qual. Theory Diff. Eq., (63)(2011), 1-10.

[30] J. Wang, L. Lv, Y. Zhou, *New concepts and results in stability of fractional differential equations*, Commun. Nonlinear Sci. Numer. Simul., **17**(2012), 2530-2538.

[31] W. Wei, X. Li, X. Li, *New stability results for fractional integral equation*, Comput. Math. Appl., **64**(2012), 3468-3476.

Received: February 13, 2013; Accepted: March 11, 2013.