Fixed Point Theory, 15(2014), No. 2, 623-634
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

SOLVABILITY OF THE STATIONARY MATHEMATICAL MODEL OF A NON-NEWTONIAN FLUID MOTION WITH OBJECTIVE DERIVATIVE

ANDREY ZVYAGIN

Research Institute of Mathematics, Voronezh State University
394006, Voronezh, Russian Federation
E-mail: zvyagin.a@mail.ru

Abstract. Leray–Schauder topological degree theory and approximation-topological approach are used to the boundary value problem for a system of equations that describes the stationary mathematical model of weak aqueous polymer solutions motion with the smoothed Jaumann's derivative. Solvability of this problem in a weak sense is studied.
Key Words and Phrases: Non-Newtonian fluid, solvability in a weak sense, approximation problem, existence theorem, Leray-Schauder degree theory for completely continuous vector fields.
2010 Mathematics Subject Classification: 76A05, 47H10.

1. Introduction

The motion of an incompressible fluid with the constant density filling a bounded domain $\Omega \subset \mathbb{R}^n, n = 2, 3$, on a time interval $[0, T]$, $T > 0$, is described by the system of equations in the Cauchy form (see, for example, [1]):

$$\frac{\partial v}{\partial t} + \sum_{i=1}^{n} v_i \frac{\partial v}{\partial x_i} + \text{grad } p = \text{Div } \sigma + f, \tag{1.1}$$

$$\text{div } v = 0, \quad (t, x) \in (0, T) \times \Omega, \tag{1.2}$$

where $v(x, t) = (v_1, ..., v_n)$ is the velocity vector of a particle at the point x at time t, $(v_1, ..., v_n)$ are the components of v, $p = p(x, t)$ is the fluid pressure at the point x at time t, and $f = f(x, t)$ is the density of external forces (also called volumetric) acting on the fluid. The symbol Div σ stands for the vector

$$\Big(\sum_{j=1}^{n} \frac{\partial \sigma_{1j}}{\partial x_j}, \sum_{j=1}^{n} \frac{\partial \sigma_{2j}}{\partial x_j}, ..., \sum_{j=1}^{n} \frac{\partial \sigma_{nj}}{\partial x_j}\Big),$$

whose coordinates are the divergence of rows of the matrix $\sigma = (\sigma_{ij}(x))$, where σ is the deviator of the stress tensor.

The work was partially supported by grants ? 130100041, ? 140131228 of Russian Foundation of Basic Research and by the Ministry of Education and Science of Russia in frameworks of state task for higher education organizations in science for 20142016 (project number 772).

System (1.1)–(1.2) describes flows of all kinds of fluids, but it contains the deviator of the stress tensor which is not expressed explicitly via the unknowns of the system. As a rule, to express the deviator of the stress tensor via the unknowns of system (1.1)–(1.2) one uses relations between the deviator of the stress tensor and the tensor of deformation velocities $\mathcal{E}(v) = (\mathcal{E}_{ij}(v))_{j=1,\dots,n}^{i=1,\dots,n}$, $\mathcal{E}_{ij}(v) = \frac{1}{2}(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i})$, and their time derivatives. By establishing the connection between the deviator of the stress tensor and the tensor of deformation velocities and their derivatives, we determine the type of fluid. This relation is called the constitutive or rheological relation and is usually obtained by the mechanistic model method (see, for example, [2]). In rheology, a real structure is often replaced by some model by supposing that the behavior of this model is analogous to the behavior of the structure. This model consists of elements that do not exist in the real body: springs, pistons, hoists, etc. Note that these relations are hypotheses to be checked out for concrete fluids by experimental data.

The rheological relation that describes the motion of viscoelastic medium is the following

$$\sigma = 2\nu\mathcal{E} + 2\varkappa\dot{\mathcal{E}}, \tag{1.3}$$

where $\nu > 0$ is the viscosity of fluid and $\varkappa > 0$ is the time of retardation (delay). This model of fluid motion describes the motion of a viscous non-Newtonian fluid that needs time to start moving under the action of force instantly applied.

In rheological relation (1.3) we have a time derivative $\dot{\mathcal{E}}$. Unfortunately, method of mechanistic models does not indicate which derivation we have to use (total, partial or any special derivative). Mathematical investigations have begun with the consideration of the partial derivative in (1.3). The corresponding model is called the Voigt model. Then A.P. Oskolkov considered the case of the total derivative [3]. But later in his work errors were found [4]. In the work [5] a complete proof of the existence of weak solutions in the model (1.1)–(1.2) with a total derivative was given. Note that the stationary case of this problem was considered in [6].

In the recent years rational mechanics [7] has influenced scientists in such a way, that they have started to investigate the rheological relations that are independent of the observer, i.e. that they do not vary under the Galilean change of variables:

$$t^* = t + a, \tag{1.4}$$

$$x^* = x_0^*(t) + Q(t)(t - t_0), \tag{1.5}$$

where a is a time value, x_0 is a point in a space, x_0^* is a time function, Q is a time function with values in the set of orthogonal tensors.

In other words, if the original tensor function changes according to law (1.4)–(1.5), will the rheological relation be the same in the different reference frames? In the case of partial and total derivatives the answer is negative. The notion of an objective derivative enables an affirmative answer to this question.

Definition 1.1. *Let G be a symmetric tensor-valued function of two tensor arguments and $T(t,x)$ be a symmetric tensor-valued function. An operator of the form*

$$\frac{DT(t,x)}{Dt} = \frac{dT(t,x)}{dt} + G(\nabla v(t,x), T(t,x))$$

is called an objective derivative, if for any change of frame (1.4)–(1.5) *the equality*

$$\frac{D^*T^*(t,x)}{Dt^*} = Q(t)\frac{DT(t,x)}{Dt}Q(t)^T$$

holds for all frame-indifferent symmetric tensor-valued functions $T(t,x)$.

An example of an objective derivative of a tensor is the smoothed Jaumann's derivative (see [8]):

$$\frac{DT(t,x)}{Dt} = \frac{dT(t,x)}{dt} + T(t,x)W_\rho(t,x) - W_\rho(t,x)T(t,x),$$

$$W_\rho(v)(t,x) = \int\limits_{\mathbb{R}^n} \rho(x-y)W(t,y)\,dy,$$

where $\rho : \mathbb{R}^n \to \mathbb{R}$ is a smooth function with compact support such that $\int\limits_{\mathbb{R}^n} \rho(y)\,dy = 1$ and $\rho(x) = \rho(y)$ for x and y with the same Euclidean norm;

$$W(v) = (W_{ij}(v))_{j=1,\dots,n}^{i=1,\dots,n},\ W_{ij}(v) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)$$

is the vorticity tensor.

Substituting the right-hand side of (1.3) with the smoothed Jaumann's derivative for σ in equations (1.1)–(1.2) and considering stationary case, we obtain

$$\sum_{i=1}^{n} v_i \frac{\partial v}{\partial x_i} - \nu\Delta v - 2\varkappa \mathrm{Div}\left(v_k \frac{\partial \mathcal{E}(v)}{\partial x_k}\right) - \\ - 2\varkappa \mathrm{Div}\Big(\mathcal{E}(v)W_\rho(v) - W_\rho(v)\mathcal{E}(v)\Big) + \mathrm{grad}\, p = f, \quad x \in \Omega, \quad (1.6)$$

$$\mathrm{div}\, v = 0, \quad x \in \Omega. \quad (1.7)$$

For system (1.6)–(1.7) we consider the boundary value problem with the boundary condition

$$v|_{\partial\Omega} = 0. \quad (1.8)$$

In the present paper we study the existence of weak solution of boundary value problem (1.6)-(1.8) which describes the motion of weak aqueous polymer solutions filling a bounded domain $\Omega \subset \mathbb{R}^n, n = 2, 3$, governed by the rheological relation with smoothed Jaumann's derivative.

For this investigation the approximation and topological methods are used (see, for example, [9], [10]). The boundary value problem is considered as an operator equation. The involved operators often do not possess good properties, therefore certain approximating of this equation is considered. Then the solvability of this approximating equation is studied in a more smoothed space. For this purpose, we apply the technique of the Leray-Schauder topological degree. The last step is the passage to the limit in the approximating equation as the approximating parameters tend to zero, and the solutions of the approximating equation converge to a solution of the original equation.

2. Weak setting

Denote by $C_0^\infty(\Omega)^n$ the space of functions of class C^∞ mapping Ω to $\mathbb{R}^n$ with compact support in Ω. Also we will need the definitions of the following function spaces: $\mathcal{V} = \{v(x) = (v_1, \dots, v_n) \in C_0^\infty(\Omega)^n : \operatorname{div} v = 0\}$; V is the closure $\mathcal{V}$ with respect to the norm of space $W_2^1(\Omega)^n$ with the scalar product

$$((v, w)) = \int\limits_\Omega \nabla v : \nabla w \, dx.$$

Here the symbol $\nabla v : \nabla w$, $v = (v_1, \dots, v_n)$, $w = (w_1, \dots, w_n)$, denotes the component-wise matrix multiplication:

$$\nabla v : \nabla w = \sum_{i,j=1}^n \frac{\partial v_i}{\partial x_j} \frac{\partial w_i}{\partial x_j}.$$

Let X be the closure of $\mathcal{V}$ with respect to the norm of space $W_2^3(\Omega)^n$. Consider the space X with the norm:

$$\|v\|_X = \Big(\int\limits_\Omega \nabla(\Delta v) : \nabla(\Delta v) \, dx \Big)^{1/2}.$$

Definition 2.1. *Let f belong to V^*. A weak solution of boundary value problem* (1.6)–(1.8) *is a function $v \in V$ such that for any $\varphi \in X$ it satisfies the equality*

$$\nu \int\limits_\Omega \nabla v : \nabla \varphi \, dx - \int\limits_\Omega \sum_{i,j=1}^n v_i v_j \frac{\partial \varphi_j}{\partial x_i} \, dx - \varkappa \int\limits_\Omega \sum_{i,j,k=1}^n v_k \frac{\partial v_i}{\partial x_j} \frac{\partial^2 \varphi_j}{\partial x_i \partial x_k} \, dx -$$
$$\varkappa \int\limits_\Omega \sum_{i,j,k=1}^n v_k \frac{\partial v_j}{\partial x_i} \frac{\partial^2 \varphi_j}{\partial x_i \partial x_k} \, dx + 2\varkappa \int\limits_\Omega (\mathcal{E}(v) W_\rho(v) - W_\rho(v) \mathcal{E}(v)) : \nabla \varphi \, dx = \langle f, \varphi \rangle. \tag{2.1}$$

The main result of this paper is the following theorem:

Theorem 2.1. *For any $f \in V^*$ boundary value problem* (1.6)–(1.8) *has at least one weak solution $v_* \in V$.*

3. Approximating problem

While studying problem (1.6)–(1.8), we use the approximation-topological approach to problems of hydrodynamics [10]. In fact, we investigate an approximating problem with a small parameter $\varepsilon > 0$:

Approximating problem. *To find a function $v \in X$, which for any $\varphi \in X$ satisfies the following equality*

$$\varepsilon \int\limits_\Omega \nabla (\Delta v) : \nabla (\Delta \varphi) \, dx - \int\limits_\Omega \sum_{i,j=1}^n v_i v_j \frac{\partial \varphi_j}{\partial x_i} \, dx + \nu \int\limits_\Omega \nabla v : \nabla \varphi \, dx$$

$$-\varkappa\int\limits_\Omega \sum_{i,j,k=1}^n v_k \frac{\partial v_i}{\partial x_j}\frac{\partial^2\varphi_j}{\partial x_i\partial x_k}\,dx - \varkappa\int\limits_\Omega \sum_{i,j,k=1}^n v_k \frac{\partial v_j}{\partial x_i}\frac{\partial^2\varphi_j}{\partial x_i\partial x_k}\,dx$$
$$+2\varkappa\int\limits_\Omega \Big(\mathcal{E}(v)W_\rho(v) - W_\rho(v)\mathcal{E}(v)\Big) : \nabla\varphi\,dx = \langle f,\varphi\rangle. \qquad (3.1)$$

Note that (3.1) differs from (2.1) by the presence of the term

$$\varepsilon\int\limits_\Omega \nabla(\Delta v) : \nabla(\Delta\varphi)\;dx.$$

At the first step we obtain a priori estimate of equality (3.1) in space X and show by means of the topological degree methods that there exists a solution of the approximating problem in X. Also we obtain in space V an estimate of solutions to the approximating problem which does not depend on parameter ε. Then we construct a sequence of such solutions and show that it admits a subsequence that converges to a weak solution of boundary value problem (1.6)–(1.8) as the parameter of approximation ε tends to zero.

Consider the following operators:

$$A : V \to V^*, \quad \langle Av,\varphi\rangle = \int\limits_\Omega \nabla v : \nabla\varphi\,dx, \quad v,\varphi\in V;$$

$$N : X \to X^*, \quad \langle Nv,\varphi\rangle = \int\limits_\Omega \nabla(\Delta v) : \nabla(\Delta\varphi)\,dx, \quad v,\varphi\in X;$$

$$B_1 : L_4(\Omega)^n \to V^*, \quad \langle B_1(v),\varphi\rangle = \int\limits_\Omega \sum_{i,j=1}^n v_i v_j \frac{\partial\varphi_j}{\partial x_i}\,dx,\; v\in L_4(\Omega)^n,\; \varphi\in V;$$

$$B_2 : V \to X^*, \quad \langle B_2(v),\varphi\rangle = \int\limits_\Omega \sum_{i,j,k=1}^n v_k\frac{\partial v_i}{\partial x_j}\frac{\partial^2\varphi_j}{\partial x_i\partial x_k}\,dx, \quad v\in V,\;\varphi\in X;$$

$$B_3 : V \to X^*, \quad \langle B_3(v),\varphi\rangle = \int\limits_\Omega \sum_{i,j,k=1}^n v_k\frac{\partial v_j}{\partial x_i}\frac{\partial^2\varphi_j}{\partial x_i\partial x_k}\,dx, \quad v\in V,\;\varphi\in X;$$

$$D : V \to X^*, \quad \langle D(v),\varphi\rangle = \int\limits_\Omega \Big(\mathcal{E}(v)W_\rho(v) - W_\rho(v)\mathcal{E}(v)\Big) : \nabla\varphi\,dx, \quad v\in V,\varphi\in X.$$

Since in equality (3.1) the function $\varphi\in X$ is arbitrary, this relation is equivalent to the following operator equation:

$$\varepsilon Nv + \nu Av - B_1(v) - \varkappa B_2(v) - \varkappa B_3(v) + 2\varkappa D(v) = f \qquad (3.2)$$

Thus a weak solution of the approximating problem is a solution $v\in X$ of operator equation (3.2).

We also define the following operators:

$$L : X \to X^*, \quad L(v) = \varepsilon Nv;$$
$$K : X \to X^*, \quad K(v) = \nu Av - B_1(v) - \varkappa B_2(v) - \varkappa B_3(v) + 2\varkappa D(v).$$

The problem of finding a solution of equation (3.2) is equivalent to the problem of finding a solution for the following operator equation:

$$L(v) + K(v) = f. \tag{3.3}$$

We will use the next assertions (the proofs of Lemma 3.1 – Lemma 3.4 can be found, for example, in [5]).

Lemma 3.1. *The operator $A : V \to V^*$ is continuous and it obeys the estimate:*

$$\|Av\|_{V^*} \leqslant C_1\|v\|_V.$$

Moreover, the operator $A : X \to X^*$ is completely continuous.

Lemma 3.2. *The operator $L : X \to X^*$ is continuous, invertible and it obeys the estimate:*

$$\|Lv\|_{X^*} \leqslant \varepsilon\|v\|_X.$$

Moreover, the inverse operator $L^{-1} : X^* \to X$ is continuous.

Lemma 3.3. *The operator $B_1 : L_4(\Omega)^n \to V^*$ is continuous and it obeys the estimate:*

$$\|B_1 v\|_{V^*} \leqslant C_2\|v\|^2_{L_4(\Omega)^n}.$$

Moreover, the operator $B_1 : X \to X^*$ is completely continuous.

Lemma 3.4. *The mapping $B_i : V \to X^*$, $i = 2, 3$, is continuous and it obeys the estimate:*

$$\|B_i v\|_{X^*} \leqslant C_3\|v\|^2_V.$$

Moreover, the operator $B_i : X \to X^*$ is completely continuous.

Lemma 3.5. *The operator $D : V \to X^*$ is continuous and it obeys the estimate:*

$$\|D(v)\|_{X^*} \leqslant C_4\|v\|^2_V. \tag{3.4}$$

Proof. We start by estimating $\mathcal{E}$ and W_ρ.

$$\|\mathcal{E}(v)\|^2_{L_2(\Omega)} = \sum_{i,j=1}^n \|\mathcal{E}_{ij}(v)\|^2_{L_2(\Omega)} \leqslant C_5 \sum_{i,j=1}^n \int_\Omega \Big(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\Big)^2 dx$$

$$= C_5 \sum_{i,j=1}^n \int_\Omega \Big(\frac{\partial v_i}{\partial x_j}\frac{\partial v_i}{\partial x_j} + 2\frac{\partial v_i}{\partial x_j}\frac{\partial v_j}{\partial x_i} + \frac{\partial v_j}{\partial x_i}\frac{\partial v_j}{\partial x_i}\Big) dx$$

$$= C_5 \sum_{i,j=1}^n \Big[\int_\Omega \frac{\partial v_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\, dx - 2\int_\Omega v_i \frac{\partial^2 v_j}{\partial x_i \partial x_j}\, dx + \int_\Omega \frac{\partial v_j}{\partial x_i}\frac{\partial v_j}{\partial x_i}\, dx\Big]$$

$$= C_5\Big[\int_\Omega \nabla v : \nabla v\, dx + \int_\Omega \nabla v : \nabla v\, dx\Big] \leqslant 2C_5\|v\|^2_V.$$

Therefore, $\|\mathcal{E}(v)\|_{L_2(\Omega)} \leqslant C_6\|v\|_V$.

$$\|(W_\rho)_{ij}(v)\|_{L_2(\Omega)} \leqslant \|(W_\rho)_{ij}(v)\|_{L_\infty(\Omega)}$$

$$\leqslant \frac{1}{2}\sup_{x\in\Omega}\Big|\int_\Omega \rho(x-y)\Big(\frac{\partial v_i(y)}{\partial y_j} - \frac{\partial v_j(y)}{\partial y_i}\Big)\, dy\Big|$$

$$\leqslant \frac{1}{2}\sup_{x\in\Omega}\Big|\int\limits_{\Omega} -\frac{\partial\rho(x-y)}{\partial y_j}v_i(y) + \frac{\partial\rho(x-y)}{\partial y_i}v_j(y)\,dy\Big|$$
$$\leqslant \|grad\ \rho\|_{L_2(\Omega)}\|v(t)\|_{L_2(\Omega)}.$$

By definition, for any $v \in V, \varphi \in X$ we have

$$|\langle D(v),\varphi\rangle| = \left|\int\limits_{\Omega}\Big(\mathcal{E}(v)W_\rho(v) - W_\rho(v)\mathcal{E}(v)\Big) : \nabla\varphi\,dx\right| \leqslant$$
$$\leqslant C_7\Big[\|\mathcal{E}(v)\|_{L_2(\Omega)}\|W_\rho(v)\|_{L_2(\Omega)} + \|W_\rho(v)\|_{L_2(\Omega)}\|\mathcal{E}(v)\|_{L_2(\Omega)}\Big]\|\nabla\varphi\|_{C(\Omega)^n} \leqslant$$
$$\leqslant C_8\|v\|_V^2\|\varphi\|_X.$$

This yields estimate (3.4).
Now prove that the operator D continuous. For any $v^m, v^0 \in V$ we have:

$$\Big|\langle D(v^m),\varphi\rangle - \langle D(v^0),\varphi\rangle\Big| = \Big|\int\limits_{\Omega}\Big(\mathcal{E}(v^m)W_\rho(v^m) - W_\rho(v^m)\mathcal{E}(v^m)\Big) : \nabla\varphi\,dx$$
$$-\int\limits_{\Omega}\Big(\mathcal{E}(v^0)W_\rho(v^0) - W_\rho(v^0)\mathcal{E}(v^0)\Big) : \nabla\varphi\,dx\Big| \leqslant C_9\Big|\int\limits_{\Omega}\mathcal{E}(v^m)W_\rho(v^m)$$
$$-W_\rho(v^m)\mathcal{E}(v^m) - \mathcal{E}(v^0)W_\rho(v^0) + W_\rho(v^0)\mathcal{E}(v^0)\,dx\Big|\|\varphi\|_X$$
$$\leqslant C_9\Big|\int\limits_{\Omega}\mathcal{E}(v^m)\Big(W_\rho(v^m) - W_\rho(v^0)\Big) + \Big(\mathcal{E}(v^m) - \mathcal{E}(v^0)\Big)W_\rho(v^0)$$
$$-W_\rho(v^m)\Big(\mathcal{E}(v^m) - \mathcal{E}(v^0)\Big) - \Big(W_\rho(v^m) - W_\rho(v^0)\Big)\mathcal{E}(v^0)\,dx\Big|\|\varphi\|_X$$
$$\leqslant C_{10}\Big[\|\mathcal{E}(v^m)\|_{L_2(\Omega)}\|W_\rho(v^m - v^0)\|_{L_2(\Omega)} + \|\mathcal{E}(v^m - v^0)\|_{L_2(\Omega)}$$
$$\times\|W_\rho(v^0)\|_{L_2(\Omega)} + \|W_\rho(v^m)\|_{L_2(\Omega)}\|\mathcal{E}(v^m - v^0)\|_{L_2(\Omega)} + \|W_\rho(v^m - v^0)\|_{L_2(\Omega)}$$
$$\times\|\mathcal{E}(v^0)\|_{L_2(\Omega)}\Big]\|\varphi\|_X \leqslant C_{11}\Big[\|v^m\|_V\|v^m - v^0\|_V + \|v^m - v^0\|_V\|v^0\|_V$$
$$+\|v^m\|_V\|v^m - v^0\|_V + \|v^m - v^0\|_V\|v^0\|_V\Big]\|\varphi\|_X$$
$$\leqslant C_{12}\Big(\|v^m\|_V + \|v^0\|_V\Big)\|v^m - v^0\|_V\|\varphi\|_X.$$

Thus we get $\left\|D(v^m) - D(v^0)\right\|_X \leqslant C_{13}\left(\|v^m\|_V + \left\|v^0\right\|_V\right)\left\|v^m - v^0\right\|_V$.
Let the sequence $\{v^m\} \subset V$ converge to some function $v^0 \in V$. Then the continuity of the mapping $D : V \to X$ follows from the previous inequality. □

Lemma 3.6. *The operator $K : X \to X^*$ is completely continuous.*
Proof. The complete continuity of the operator $K : X \to X^*$ follows from the complete continuity of the operators

$A : X \to X^*$ Lemma 3.1;

$B_1 : X \to X^*$ Lemma 3.3;

$B_2 : X \to X^*$ Lemma 3.4;

$$B_3 : X \to X^* \quad \text{Lemma 3.4;}$$
$$D : X \to X^* \quad \text{Lemma 3.5.}$$
□

Along with equation (3.3) consider the following family of operator equations:

$$L(v) + \lambda K(v) = \lambda f, \quad \lambda \in [0,1], \tag{3.5}$$

which coincides with equation (3.3) for $\lambda = 1$.

Theorem 3.1. *If $v \in X$ is a solution of operator equation (3.5) for some $\lambda \in [0,1]$, then the following estimate holds:*

$$\varepsilon \|v\|_X^2 \leqslant C_{14}, \text{ where } C_{14} = \frac{\|f\|_{V^*}^2}{2\nu}. \tag{3.6}$$

Moreover, if $\lambda = 1$, then the following estimate holds:

$$\nu \|v\|_V^2 \leqslant C_{15}, \text{ where } C_{15} = \frac{\|f\|_{V^*}^2}{\nu}. \tag{3.7}$$

Proof. Let $v \in X$ be a solution of (3.5). Then for any $\varphi \in X$ the following equation holds:

$$\begin{aligned} &\varepsilon \int_\Omega \nabla(\Delta v) : \nabla(\Delta \varphi)\, dx - \lambda \int_\Omega \sum_{i,j=1}^n v_i v_j \frac{\partial \varphi_j}{\partial x_i}\, dx + \lambda\nu \int_\Omega \nabla v : \nabla \varphi\, dx \\ &-\lambda\varkappa \int_\Omega \sum_{i,j,k=1}^n v_k \frac{\partial v_i}{\partial x_j} \frac{\partial^2 \varphi_j}{\partial x_i \partial x_k}\, dx - \lambda\varkappa \int_\Omega \sum_{i,j,k=1}^n v_k \frac{\partial v_j}{\partial x_i} \frac{\partial^2 \varphi_j}{\partial x_i \partial x_k}\, dx \\ &+2\lambda\varkappa \int_\Omega \Big(\mathcal{E}(v) W_\rho(v) - W_\rho(v)\mathcal{E}(v)\Big) : \nabla \varphi\, dx = \lambda \langle f, \varphi \rangle \end{aligned} \tag{3.8}$$

Note that

$$\begin{aligned} &\int_\Omega \sum_{i,j,k=1}^n v_k \frac{\partial v_i}{\partial x_j} \frac{\partial^2 \varphi_j}{\partial x_i \partial x_k}\, dx + \int_\Omega \sum_{i,j,k=1}^n v_k \frac{\partial v_j}{\partial x_i} \frac{\partial^2 \varphi_j}{\partial x_i \partial x_k}\, dx \\ &= 2 \int_\Omega \sum_{i,j,k=1}^n v_k(t) \mathcal{E}_{ij}(v) \frac{\partial^2 \varphi_j}{\partial x_i \partial x_k}\, dx = -2 \int_\Omega \sum_{i,j,k=1}^n v_k \frac{\partial \mathcal{E}_{ij}(v)}{\partial x_k} \frac{\partial \varphi_j}{\partial x_i}\, dx \\ &-2 \int_\Omega \sum_{i,j,k=1}^n \frac{\partial v_k}{\partial x_k} \mathcal{E}_{ij}(v) \frac{\partial \varphi_j}{\partial x_i}\, dx = -2 \int_\Omega \sum_{i,j,k=1}^n v_k \frac{\partial \mathcal{E}_{ij}(v)}{\partial x_k} \frac{\partial \varphi_j}{\partial x_i}\, dx. \end{aligned}$$

Then (3.8) can be rewritten in the form

$$\begin{aligned} &\varepsilon \int_\Omega \nabla(\Delta v) : \nabla(\Delta \varphi)\, dx - \lambda \int_\Omega \sum_{i,j=1}^n v_i v_j \frac{\partial \varphi_j}{\partial x_i}\, dx \\ &+\lambda\nu \int_\Omega \nabla v : \nabla \varphi\, dx + 2\lambda\varkappa \int_\Omega \sum_{i,j,k=1}^n v_k \frac{\partial \mathcal{E}_{ij}(v)}{\partial x_k} \frac{\partial \varphi_j}{\partial x_i}\, dx \\ &+2\lambda\varkappa \int_\Omega \Big(\mathcal{E}(v) W_\rho(v) - W_\rho(v)\mathcal{E}(v)\Big) : \nabla \varphi\, dx = \lambda \langle f, \varphi \rangle. \end{aligned}$$

Since the latter equation holds for all $\varphi \in X$, it is true for $\varphi = v$ as well:

$$\varepsilon \int_\Omega \nabla(\Delta v) : \nabla(\Delta v)\, dx - \lambda \int_\Omega \sum_{i,j=1}^n v_i v_j \frac{\partial v_j}{\partial x_i}\, dx$$

$$+\lambda\nu \int_\Omega \nabla v : \nabla v dx + 2\lambda\varkappa \int_\Omega \sum_{i,j,k=1}^n v_k \frac{\partial \mathcal{E}_{ij}(v)}{\partial x_k} \frac{\partial v_j}{\partial x_i}\, dx$$

$$+2\lambda\varkappa \int_\Omega \Big(\mathcal{E}(v)W_\rho(v) - W_\rho(v)\mathcal{E}(v)\Big) : \nabla v\, dx = \lambda\langle f, v\rangle. \tag{3.9}$$

We reduce the terms on the left-hand side of the equation (3.9) in the following way:

$$\varepsilon \int_\Omega \nabla(\Delta v) : \nabla(\Delta v)\, dx = \varepsilon \|v\|_X^2;$$

$$\int_\Omega \sum_{i,j=1}^n v_i v_j \frac{\partial v_j}{\partial x_i}\, dx = \int_\Omega \sum_{i,j=1}^n v_i \frac{\partial(v_j v_j)}{\partial x_i}\, dx = -\int_\Omega \sum_{i,j=1}^n \frac{\partial v_i}{\partial x_i} v_j v_j\, dx = 0;$$

$$\int_\Omega \Big(\mathcal{E}(v)W_\rho(v) - W_\rho(v)\mathcal{E}(v)\Big) : \nabla v\, dx = \frac{1}{2}\int_\Omega \Big(\mathcal{E}(v)W_\rho(v) - W_\rho(v)\mathcal{E}(v)\Big) :$$

$$: \Big(\mathcal{E}(v) + W(v)\Big)\, dx = \frac{1}{2}\int_\Omega \Big(\mathcal{E}(v)W_\rho(v) - W_\rho(v)\mathcal{E}(v)\Big) : \mathcal{E}(v)\, dx$$

$$+\frac{1}{2}\int_\Omega \Big(\mathcal{E}(v)W_\rho(v) - W_\rho(v)\mathcal{E}(v)\Big) : W(v)\, dx = \frac{1}{2}\sum_{i,j,k=1}^n \int_\Omega \Big(\mathcal{E}_{ij}(W_\rho)_{jk}\mathcal{E}_{ik}$$

$$-(W_\rho)_{jk}\mathcal{E}_{ki}\mathcal{E}_{ji}\Big)\, dx + \frac{1}{2}\sum_{i,j,k=1}^n \int_\Omega \Big(\mathcal{E}_{ij}(W_\rho)_{jk}W_{ik} - (W_\rho)_{kj}\mathcal{E}_{ji}W_{ki}\Big)\, dx$$

$$= \frac{1}{2}\sum_{i,j,k=1}^n \int_\Omega \mathcal{E}_{ij}(W_\rho)_{jk}\mathcal{E}_{ik} - \mathcal{E}_{ij}(W_\rho)_{jk}\mathcal{E}_{ik}\, dx + \frac{1}{2}\sum_{i,j,k=1}^n \int_\Omega \mathcal{E}_{ij}(W_\rho)_{jk}W_{ik}$$

$$-\mathcal{E}_{ij}(W_\rho)_{jk}W_{ik}\, dx = 0;$$

$$\int_\Omega \sum_{i,j,k=1}^n v_k \frac{\partial \mathcal{E}_{ij}(v)}{\partial x_k}\frac{\partial v_j}{\partial x_i}\, dx = \frac{1}{2}\Big(\int_\Omega \sum_{i,j,k=1}^n v_k \frac{\partial \mathcal{E}_{ij}(v)}{\partial x_k}\frac{\partial v_j}{\partial x_i}\, dx$$

$$+\int_\Omega \sum_{i,j,k=1}^n v_k \frac{\partial \mathcal{E}_{ij}(v)}{\partial x_k}\frac{\partial v_i}{\partial x_j}\, dx\Big) = \int_\Omega \sum_{i,j,k=1}^n v_k \frac{\partial\left(\mathcal{E}_{ij}(v)\mathcal{E}_{ij}(v)\right)}{\partial x_k}\, dx$$

$$= -\int_\Omega \sum_{k=1}^n \frac{\partial v_k}{\partial x_k} \sum_{i,j=1}^n \mathcal{E}_{ij}(v)\mathcal{E}_{ij}(v) dx = 0.$$

Here we take into account that the strain-rate tensor $\mathcal{E}(v)$ is symmetric and tensors $W_\rho(v)$ and $W(v)$ are skew-symmetric. Hence, equation (3.9) can be rewritten in the following form:

$$\varepsilon\|v\|_X^2 + \lambda\nu\|v\|_V^2 = \lambda\langle f, v\rangle.$$

Using the upper estimate of the right-hand side of the latter equation

$$\lambda\langle f, v\rangle \leqslant \lambda\,|\langle f, v\rangle| \leqslant \lambda\|f\|_{V^*}\|v\|_V \leqslant \|f\|_{V^*}\|v\|_V \leqslant \lambda\frac{\|f\|_{V^*}^2}{2\delta} + \lambda\frac{\delta\|v\|_V^2}{2}$$

for $\delta = \nu$ we get

$$\varepsilon\|v\|_X^2 + \lambda\nu\|v\|_V^2 \leqslant \lambda\frac{\|f\|_{V^*}^2}{2\nu} + \lambda\frac{\nu\|v\|_V^2}{2},$$

$$\varepsilon\|v\|_X^2 + \lambda\frac{\nu\|v\|_V^2}{2} \leqslant \lambda\frac{\|f\|_{V^*}^2}{2\nu},\ \varepsilon\|v\|_X^2 \leqslant \lambda\frac{\|f\|_{V^*}^2}{2\nu} \leqslant \frac{\|f\|_{V^*}^2}{2\nu}.$$

Similarly for $\lambda = 1$ we get $\nu\|v\|_V^2 \leqslant \lambda\dfrac{\|f\|_{V^*}^2}{\nu}$. This proves (3.6) and (3.7). □

Theorem 3.2. *Operator equation* (3.3) *has at least one weak solution v 2 X:*

Proof. To prove this theorem we use the Leray-Schauder degree theory for completely continuous vector fields. By virtue of a priori estimate (3.6), all solutions of family of equations (3.5) are contained in the ball $B_R \subset X$ of radius $R = C_{14} + 1$. By Lemma 3.6 the mapping $[-K(\cdot)+f] : X \to X^*$ is completely continuous. By virtue of Lemma 3.2 the operator $L^{-1} : X^* \to X$ is continuous.

Thus, the mapping $L^{-1}\,[-K(\cdot) + f] : X \to X$ is completely continuous. Then the mapping $G : [0, 1] \times X \to X$, $G(\lambda, v) = \lambda L^{-1}\,[-K(v) + f]$ is completely continuous with respect to the two-dimensioned argument (λ, v). From the above, we get that the completely continuous vector field $\Phi(\lambda, v) = v - G(\lambda, v)$ does not vanish on the boundary of B_R. By the homotopy invariance of the degree we get

$$\deg_{LS}(\Phi(0, \cdot), B_R, 0) = \deg_{LS}(\Phi(1, \cdot), B_R, 0).$$

Recall that $\Phi(0, \cdot) = I$ and by the degree normalization property $\deg_{LS}(I, B_R, 0) = 1$. Hence, $\deg_{LS}(\Phi(1, \cdot), B_R, 0) = 1$.

Thus, we see that there exists at least a solution $v \in X$ of the equation

$$v - L^{-1}\,[-K(v) + f] = 0$$

and, therefore, of equation (3.3).

Since there is a solution $v \in X$ of equation (3.3), from the above it follows that the approximating problem has at least one weak solution $v \in X$. □

4. Proof of Theorem 2.1

Proof. In (2.1), let us take $\varepsilon_m = \dfrac{1}{m}$. The sequence $\{\varepsilon_m\}$ converges to zero as $m \to +\infty$. By Theorem 3.2 for any ε_m there exists a weak solution $v_m \in X \subset V$ of the approximation problem. Thus, each v_m satisfies to the equation

$$\varepsilon_m \int\limits_\Omega \nabla\left(\Delta v_m\right) : \nabla\left(\Delta\varphi\right) dx - \int\limits_\Omega \sum_{i,j=1}^n (v_m)_i (v_m)_j \frac{\partial\varphi_j}{\partial x_i}\, dx + \nu \int\limits_\Omega \nabla v_m : \nabla\varphi\, dx$$

$$-\varkappa\int\limits_{\Omega}\sum_{i,j,k=1}^{n}(v_m)_k\frac{\partial(v_m)_i}{\partial x_j}\frac{\partial^2\varphi_j}{\partial x_i\partial x_k}\,dx-\varkappa\int\limits_{\Omega}\sum_{i,j,k=1}^{n}(v_m)_k\frac{\partial(v_m)_j}{\partial x_i}\frac{\partial^2\varphi_j}{\partial x_i\partial x_k}\,dx$$

$$+2\varkappa\int\limits_{\Omega}\Big(\mathcal{E}(v_m)W_\rho(v_m)-W_\rho(v_m)\mathcal{E}(v_m)\Big):\nabla\varphi\,dx=\langle f,\varphi\rangle. \tag{4.1}$$

Then by the definition of weak convergence

$$\nu\int\limits_{\Omega}\nabla v_m:\nabla\varphi\,dx\to\nu\int\limits_{\Omega}\nabla v_*:\nabla\varphi\,dx\quad\text{as } m\to+\infty,\ \varphi\in X.$$

Then, without loss of generality (passing to a subsequence if needed), from (3.6) we see that

$$\lim_{m\to\infty}\left|\varepsilon_m\int\limits_{\Omega}\nabla\left(\Delta v_m\right):\nabla\left(\Delta\varphi\right)\,dx\right|=\lim_{m\to\infty}\sqrt{\varepsilon_m}\lim_{m\to\infty}\left|\sqrt{\varepsilon_m}\int\limits_{\Omega}\nabla\left(\Delta v_m\right):\nabla\left(\Delta\varphi\right)\,dx\right|$$

so we get $\varepsilon_m\int\limits_{\Omega}\nabla\left(\Delta v_m\right):\nabla\left(\Delta\varphi\right)\,dx\to0, m\to+\infty$.

For the remaining integrals we have

$$\varkappa\int\limits_{\Omega}\sum_{i,j,k=1}^{n}(v_m)_k\frac{\partial(v_m)_i}{\partial x_j}\frac{\partial^2\varphi_j}{\partial x_i\partial x_k}\,dx\to\varkappa\int\limits_{\Omega}\sum_{i,j,k=1}^{n}(v_*)_k\frac{\partial(v_*)_i}{\partial x_j}\frac{\partial^2\varphi_j}{\partial x_i\partial x_k}\,dx\qquad m\to+\infty;$$

$$\varkappa\int\limits_{\Omega}\sum_{i,j,k=1}^{n}(v_m)_k\frac{\partial(v_m)_j}{\partial x_i}\frac{\partial^2\varphi_j}{\partial x_i\partial x_k}\,dx\to\varkappa\int\limits_{\Omega}\sum_{i,j,k=1}^{n}(v_*)_k\frac{\partial(v_*)_j}{\partial x_i}\frac{\partial^2\varphi_j}{\partial x_i\partial x_k}\,dx\qquad m\to+\infty.$$

Indeed, here the sequence v_m converges to v_* strongly in $L_4(\Omega)^n$ and $\nabla(v_m)$ converges to ∇v_* weakly in $L_2(\Omega)^{n^2}$. Thus, their product converges to the product of their limits.

In the last term we have

$$\int\limits_{\Omega}\left(\mathcal{E}(v_m)W_\rho(v_m)-\mathcal{E}(v_*)W_\rho(v_*)\right):\nabla\varphi\,dx$$

$$=\int\limits_{\Omega}\left(\mathcal{E}(v_m)(W_\rho(v_m)-W_\rho(v_*))+(\mathcal{E}(v_m)-\mathcal{E}(v_*))W_\rho(v_*)\right):\nabla\varphi\,dx$$

$$\leq\|\mathcal{E}(v_m)\|_{L_2(\Omega)}\|\nabla\varphi\|_{L_2(\Omega)}\|W_\rho(v_m-v_*)\|_{L_\infty(\Omega)}+\|W_\rho(v_*)\|_{L_\infty(\Omega)}\int\limits_{\Omega}\mathcal{E}(v_m-v_*):\nabla\varphi\,dx$$

$$\leq\|\mathcal{E}(v_m)\|_{L_2(\Omega)}\|\nabla\varphi\|_{L_2(\Omega)}\|(v_m-v_*)\|_{L_2(\Omega)}+\|W_\rho(v_*)\|_{L_\infty(\Omega)}\int\limits_{\Omega}\mathcal{E}(v_m-v_*):\nabla\varphi\,dx$$

$$\leq\|\mathcal{E}(v_m)\|_{L_2(\Omega)}\|\nabla\varphi\|_{L_2(\Omega)}\|(v_m-v_*)\|_{L_4(\Omega)}++\|W_\rho(v_*)\|_{L_\infty(\Omega)}\int\limits_{\Omega}\mathcal{E}(v_m-v_*):\nabla\varphi\,dx.$$

Recall that the sequence v_m converges to v_* strongly in $L_4(\Omega)^n$ and $\nabla(v_m)$ converges to ∇v_* weakly in $L_2(\Omega)^{n^2}$. Therefore, we have

$$\int\limits_\Omega \mathcal{E}(v_m)W_\rho(v_m) : \nabla\varphi\,dx \to \int\limits_\Omega \mathcal{E}(v_*)W_\rho(v_*) : \nabla\varphi\,dx \qquad m \to +\infty.$$

Similarly we obtain

$$\int\limits_\Omega W_\rho(v_m)\mathcal{E}(v_m) : \nabla\varphi\,dx \to \int\limits_\Omega W_\rho(v_*)\mathcal{E}(v_*) : \nabla\varphi\,dx \qquad m \to +\infty.$$

Thus, passing to the limit in equation (4.1) as $m \to +\infty$, we see that the limit function v_* satisfies the following equation:

$$\nu\int\limits_\Omega \nabla v_* : \nabla\varphi\,dx - \int\limits_\Omega \sum_{i,j=1}^n (v_*)_i(v_*)_j \frac{\partial\varphi_j}{\partial x_i}\,dx - \varkappa\int\limits_\Omega \sum_{i,j,k=1}^n (v_*)_k \frac{\partial(v_*)_i}{\partial x_j}\frac{\partial^2\varphi_j}{\partial x_i\partial x_k}\,dx$$

$$-\varkappa\int\limits_\Omega \sum_{i,j,k=1}^n (v_*)_k \frac{\partial(v_*)_j}{\partial x_i}\frac{\partial^2\varphi_j}{\partial x_i\partial x_k}\,dx + 2\varkappa\int\limits_\Omega (\mathcal{E}(v_*)W_\rho(v_*) - W_\rho(v_*)\mathcal{E}(v_*)) : \nabla\varphi\,dx$$

$$= \langle f, \varphi\rangle.$$

This proves that $v_* \in V$. This completes the proof of Theorem 2.1. □

References

[1] R.V.Goldstein, V.A. Gorodtsov, *Mechanics of Continuous Media, Part 1: Fundamentals and Classical Models of Fluids*, Nauka, Fizmatlit, Moscow, 2000, (in Russian).

[2] M. Reiner, *Rheology*, Handbuch der Physik, S. Flugge (Ed.), Bd. VI, Springer, 1958.

[3] A.P. Oskolkov, *On some quasilinear systems occuring in the study of motion of viscous fluids*, Zap. Nauchn. Semin. Leningr. Otd. Mat. Inst. Steklova, **52**(1975), 128-157, (in Russian).

[4] O.A. Ladyzhenskaya, *On some gaps in two of my papers on the Navier-Stokes equations and the way of closing them*, Zap. Nauchn. Semin. POMI, **271**(2000), 151-155, (in Russian).

[5] V.G. Zvyagin, M.V. Turbin, *Mathematical problems of viscoelastic media hydrodynamics*, Kracand (URSS), Moscow, 2012, (in Russian).

[6] A.V. Zvyagin, *Solvability of a stationary model of motion of weak aqueous polymer solutions*, Russian Mathematics (Iz. VUZ), **55**(2011), no. 2, 90-92.

[7] C. Truesdell, *A First Course in Rational Continuum Mechanics*, The John Hopkins University, Baltimore, 1972.

[8] V.G. Zvyagin, D.A. Vorotnikov, *Approximating-topological methods in some problems of hydrodinamics*, J. Fixed Point Theory Appl., **3**(2008), no. 1, 23-49.

[9] M. Kamenskii, V. Obukhovskii, P. Zecca, *Condensing multivalued maps and semilinear differential inclusions in Banach spaces*, De Gruyter Series In Nonlinear Anal. Appl., 7, Walter de Gruyter, Berlin-New York, 2001.

[10] V.G. Zvyagin, D.A. Vorotnikov, *Topological approximayion methods for evolutionary problems of nonlinear hydrodinamics*, De Gruyter Series in Nonlinear Anal. Appl., 12, Walter de Gruyter, Berlin-New York, 2008.

Received: December 10, 2012; Accepted: November 2, 2013