Fixed Point Theory, 15(2014), No. 2, 609-622
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

GENERALIZED KY FAN MINIMAX INEQUALITIES FOR SET-VALUED MAPPINGS

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Abstract. In this paper, by virtue of the Ky Fan section theorem, the Kakutani-Fan- Glicksberg fixed point theorem and two nonlinear scalarization functions, we investigate some generalized Ky Fan minimax inequalities for set-valued mappings, in which the minimization and the maximization of set-valued mappings are taken in the sense of vector optimization.

Key Words and Phrases: Ky Fan minimax inequality, set-valued mapping, nonlinear scalarization function, fixed point theorem, vector optimization.
2010 Mathematics Subject Classification: 49J35, 49K35, 90C47, 47H10.

1. Introduction

It is well known that the Ky Fan minimax inequality plays a very important role in many fields, such as variational inequality, game theory, control theory, and fixed point theory. Yen [30] obtained two existence theorems of variational inequalities by virtue of a generalized Ky Fan minimax inequality. Ha [15] established a fixed point theorem by applying an extended Ky Fan minimax inequality. Park [26] obtained a generalization of Nash equilibrium theorem by using the Ky Fan minimax inequality. Because of its wide applications, Ky Fan minimax inequalities relative to scalar functions have been generalized in various ways; see [16, 5, 6, 7, 8, 33].

In recent years, based on the development of vector optimization, a great deal of papers have devoted to the study of Ky Fan minimax inequalities for vector-valued functions. Chen [3] obtained a Ky Fan minimax inequality for a vector-valued function on H-spaces by using a generalized Fan's section theorem. Chang et al. [2] proved a Ky Fan minimax inequality for a vector-valued function on W-spaces. Li and Wang [22] established some Ky Fan minimax inequalities for vector-valued functions. Luo [24] obtained some generalized Ky Fan minimax inequalities for vector-valued functions by applying the Browder fixed point theorem and the Kakutani-Fan-Glicksberg fixed point theorem. Yang et al. [29] established minimax theorems for vector-valued

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mappings in abstract convex spaces. There are also many papers studying other types of minimax inequalities for vector-valued functions. Nieuwenhuis [25] obtained a minimax inequality when the vector-valued function is of the form $f(x, y) = x + y$. Ferro [10, 11] proved some minimax inequalities for general vector-valued functions. Tanaka [28] obtained some minimax inequalities for vector-valued functions by using the existence results of cone saddle points for vector-valued functions. Gong [14] obtained a strong minimax inequality and established an equivalent relationship between the strong minimax inequality and a strong cone saddle point theorem for a vector-valued function. Li et al. [19] investigated a minimax inequality and a saddle point theorem for a vector-valued function in the sense of lexicographic order, respectively.

There are also articles that have investigated minimax theorems for set-valued mappings. Li et al. [20] obtained some minimax inequalities for set-valued mappings by using a section theorem and a separation theorem of convex sets. Li et al. [21] studied some generalized minimax inequalities for set-valued mappings by using a nonlinear scalarization function. Zhang et al. [31, 32] established some minimax inequalities for set-valued mappings. Motivated by these earlier work, we establish some generalized Ky Fan minimax inequalities for set-valued mappings by using some fixed point theorems and two nonlinear scalarization functions.

The rest of the paper is organized as follows. In Section 2, we introduce notations and preliminary results. In Section 3, we establish a Ky Fan minimax inequality for a scalar set-valued mapping. In Section 4, we obtain some generalized Ky Fan minimax inequalities for set-valued mappings.

2. Preliminaries

Throughout this paper, unless otherwise specified, let X and V be real Hausdorff topological vector spaces. Assume that S is a pointed closed convex cone in V with its interior $\text{int}S \neq \emptyset$. We define the binary relation:

$$x \leq_S y \Leftrightarrow x \in y - S, \qquad \forall \ x, y \in V.$$

Some fundamental terminologies are presented as follows.

Definition 2.1. [17] Let $A \subset V$ be a nonempty subset. (i) A point $z \in A$ is said to be a minimal point of A iff $A \bigcap (z - S) = \{z\}$, and $\text{Min}A$ denotes the set of all minimal points of A.

(ii) A point $z \in A$ is said to be a weakly minimal point of A iff $A \bigcap (z - \text{int}S) = \emptyset$, and $\text{Min}_{\text{w}}A$ denotes the set of all weakly minimal points of A.

(iii) A point $z \in A$ is said to be a maximal point of A iff $A \bigcap (z + S) = \{z\}$, and $\text{Max}A$ denotes the set of all maximal points of A.

(iv) A point $z \in A$ is said to be a weakly maximal point of A iff $A \bigcap (z + \text{int}S) = \emptyset$, and $\text{Max}_{\text{w}}A$ denotes the set of all weakly maximal points of A.

It is easy to verity that $\text{Min}A \subset \text{Min}_{\text{w}}A \quad \text{and} \quad \text{Max}A \subset \text{Max}_{\text{w}}A$.

Definition 2.2. [1] Let $F : X \to 2^V$ be a set-valued mapping.

(i) F is said to be upper semicontinuous(u.s.c.) at $x_0 \in X$ iff, for any neighborhood $N(F(x_0))$ of $F(x_0)$, there exists a neighborhood $N(x_0)$ of x_0 such that

$$F(x) \subset N(F(x_0)), \quad \forall x \in N(x_0).$$

(ii) F is said to be lower semicontinuous(l.s.c.) at $x_0 \in X$ iff, for any net $\{x_\alpha\} \subset X$ such that $x_\alpha \to x_0$ and any $y_0 \in F(x_0)$, there exists a net $y_\alpha \in F(x_\alpha)$ such that $y_\alpha \to y_0$.

(iii) F is said to be continuous at $x_0 \in X$ iff F is both u.s.c. and l.s.c. at x_0.

Lemma 2.3. Let X_0 be a compact subset of X. Suppose that $F : X_0 \times X_0 \to 2^V$ is a continuous set-valued mapping and for each $(x, y) \in X_0 \times X_0$, $F(x, y)$ is a nonempty compact set. Then $\Phi(x) = \mathrm{Max_w} \bigcup_{y \in X_0} F(x, y)$ is u.s.c. and compact-valued on X_0.

Proof. It follows from Lemma 2.2 in [20] that Φ is u.s.c.. By the compactness of X_0 and the closeness of the weakly maximal point set, Φ is also compact-valued. □

Lemma 2.4. [1] Let X_0 be a compact subset of X, and let $F : X_0 \to 2^V$ be a set-valued mapping. If X_0 is compact and F is u.s.c. and compact-valued, then $F(X_0) = \bigcup_{x \in X_0} F(x)$ is compact.

Lemma 2.5. [10] Let A be a nonempty compact subset of V. Then

(i)Min$A \neq \emptyset$;

$(ii)A \subset \mathrm{Min}A + S$; (iii)Max$A \neq \emptyset$; and $(iv)A \subset \mathrm{Max}A - S$.

Definition 2.6. [12] Given $e \in \mathrm{int}S$ and $a \in V$, the nonlinear scalarization functions ξ_{ea} and $h_{ea} : V \to R$ are, respectively, defined by

$$\xi_{ea}(z) = \min\{t \in R : z \in a + te - S\},$$

and

$$h_{ea}(z) = \max\{t \in R : z \in a + te + S\}.$$

Next, we give some useful properties of the above scalarization functions.

Lemma 2.7. [13, 4] Let $e \in \mathrm{int}S$ and $a \in V$. The following properties hold:

(i) $\xi_{ea}(z) < r \Leftrightarrow z \in a + re - \mathrm{int}S$; $h_{ea}(z) > r \Leftrightarrow z \in a + re + \mathrm{int}S$;

(ii) $\xi_{ea}(z) \leq r \Leftrightarrow z \in a + re - S$; $h_{ea}(z) \geq r \Leftrightarrow z \in a + re + S$;

(iii) $\xi_{ea}(\cdot)$ and $h_{ea}(\cdot)$ are continuous functions;

(iv) ξ_{ea} and h_{ea} are strictly monotonically increasing (monotonically increasing), that is, if $z_1 - z_2 \in \mathrm{int}S \Rightarrow f(z_1) > f(z_2)(z_1 - z_2 \in S \Rightarrow f(z_1) \geq f(z_2))$, where f denotes ξ_{ea} or h_{ea}.

Definition 2.8. Let X_0 be a nonempty convex subset of X, and let $F : X_0 \to 2^V$ be a set-valued mapping.

(i) F is said to be properly S-quasiconcave [21] on X_0 iff, for each $x_1,\ x_2 \in X_0$ and $l \in [0, 1]$, either

$$F(x_1) \subset F(lx_1 + (1-l)x_2) - S \quad \text{or} \quad F(x_2) \subset F(lx_1 + (1-l)x_2) - S.$$

(ii) F is said to be S-quasiconcave [23] on X_0 iff, for any point $z \in V$, the level set

$$Lev_F(z) := \{x \in X_0 : \exists t \in F(x),\ \text{s.t. } t \in z + S\}$$

is convex.

Lemma 2.9. Let X_0 be a nonempty convex subset of X. Let $F : X_0 \to 2^V$ be a set-valued mapping, $e \in \mathrm{int}S$ and $a \in V$:

(i) If F is properly S-quasiconcave on X_0, then $\xi_{ea} \circ F$ is R_+-quasiconcave on X_0;

(ii) If F is S-quasiconcave on X_0, then $h_{ea} \circ F$ is R_+-quasiconcave on X_0;

Proof. (i) For any $w \in R$, we only need to prove that

$$Lev_{\xi_{ea}F}(w) = \{x \in X_0 : \exists t \in \xi_{ea}(F(x)),\ \text{s.t. } t \geq w\}$$

is a convex set. Let $x_1,\ x_2 \in Lev_{\xi_{ea}F}(w)$ and $l \in [0,1]$. Then, there exist $z_1 \in F(x_1)$ and $z_2 \in F(x_2)$ such that $\xi_{ea}(z_1) \geq w$ and $\xi_{ea}(z_2) \geq w$. By Lemma 2.7 (i),

$$z_1 \notin a + we - \text{int}S \quad \text{and} \quad z_2 \notin a + we - \text{int}S.$$

Thus, we have

$$(z_1 + S)\bigcap(a + we - \text{int}S) = \emptyset \quad \text{and} \quad (z_2 + S)\bigcap(a + we - \text{int}S) = \emptyset.$$

Since F is properly S-quasiconcave on X_0, there exists $z \in F(lx_1+(1-l)x_2)$ such that $z \notin a+we-\text{int}S$. By Lemma 2.7 (i), $\xi_{ea}(z) \geq w$. Namely, $lx_1+(1-l)x_2 \in Lev_{\xi_{ea}F}(w)$.

(ii) Take any $w \in R$. We have to show that

$$Lev_{h_{ea}F}(w) = \{x \in X_0 : \exists t \in h_{ea}(F(x)),\ \text{s.t.}\ t \geq w\}.$$

is a convex set. Let $x_1,\ x_2 \in Lev_{h_{ea}F}(w)$ and $l \in [0,1]$. Then, there exist $z_1 \in F(x_1)$ and $z_2 \in F(x_2)$ with $h_{ea}(z_i) \geq w,\ i = 1,2$. By Lemma 2.7 (ii), we have

$$z_i \in a + we + S, \quad \text{for } i = 1,2.$$

Since F is S-quasiconcave on X_0, there exists $z \in F(lx_1 + (1-l)x_2)$ such that

$$z \in a + we + S.$$

This shows that $h_{ea}(z) \geq w$. Thus $lx_1 + (1-l)x_2 \in Lev_{h_{ea}F}(w)$. □

Lemma 2.10. ([27], Lemma 3.2) Let $F : X \to 2^R$ be a continuous set-valued mapping with compact values. Then, the function $h : X \to R$, defined by $h(x) = \max F(x)$, is continuous.

Theorem 2.11. [16, 7] Let X_0 be a nonempty compact convex subset of X and let A be a subset of $X_0 \times X_0$ such that

(a) for each $y \in X_0$, the set $\{x \in X_0 : (x,y) \in A\}$ is closed in X_0

(b) for each $x \in X_0$, the set $\{y \in X_0 : (x,y) \notin A\}$ is convex or empty.

If $(x,x) \in A$ for each $x \in X_0$, then there exists a point $x_0 \in X_0$ such that $\{x_0\} \times X_0 \subset A$.

Theorem 2.12. [9, 18] Let X be a real locally convex Hausdorff topological vector space. Let X_0 be a nonempty compact convex subset of X. If $T : X_0 \to 2^{X_0}$ is u.s.c., and for any $x \in X_0$, $T(x)$ is a nonempty, closed and convex set, then T has a fixed point.

3. Ky Fan minimax inequalities of scalar set-valued mappings

Lemma 3.1. Let X_0 be a nonempty convex subset of X, and let $F : X_0 \to 2^R$ be a set-valued mapping. Then the following two statements are equivalent:

(i) for any $r \in R$, $\{x \in X_0 : \exists w \in F(x),\ \text{s.t.}\ w \geq r\}$ is convex;

(ii) for any $t \in R$, $\{x \in X_0 : \exists w \in F(x),\ \text{s.t.}\ w > t\}$ is convex.

Proof. $(i) \Rightarrow (ii)$ For any $t \in R, \lambda \in [0,1]$, and $x_1, x_2 \in \{x \in X_0 : \exists w \in F(x),\ \text{s.t.}\ w > t\}$. Then, $\exists w_1 \in F(x_1)$ such that $w_1 > t$ and $\exists w_2 \in F(x_2)$ such that $w_2 > t$. So, we have

$$r = \min\{w_1, w_2\} > t \text{ and } x_1, x_2 \in \{x \in X_0 : \exists w \in F(x),\ \text{s.t.}\ w \geq r\}.$$

Since $\{x \in X_0 : \exists w \in F(x),\ \text{s.t.}\ w \geq r\}$ is convex, we have

$$\lambda x_1 + (1-\lambda)x_2 \in \{x \in X_0 : \exists w \in F(x),\ \text{s.t.}\ w \geq r > t\}.$$

Thus, $\lambda x_1 + (1-\lambda)x_2 \in \{x \in X_0 : \exists w \in F(x),\ \text{s.t. } w > t\}$.

$(ii) \Rightarrow (i)$ For any $r \in R, \lambda \in [0,1]$, and $x_1,\ x_2 \in \{x \in X_0 : \exists w \in F(x),\ \text{s.t. } w \geq r\}$, naturally we have that $x_1,\ x_2 \in \{x \in X_0 : \exists w \in F(x),\ \text{s.t. } w > r - \varepsilon\}$, for all $\varepsilon > 0$. By (ii), we have $\{x \in X_0 : \exists w \in F(x),\ \text{s.t. } w > r - \varepsilon\}$ is convex, that is,

$$\lambda x_1 + (1-\lambda)x_2 \in \{x \in X_0 : \exists w \in F(x),\ \text{s.t. } w > r - \varepsilon\}.$$

Since ε is arbitrary, then $\lambda x_1 + (1-\lambda)x_2 \in \{x \in X_0 : \exists w \in F(x),\ \text{s.t. } w \geq r\}$. □

Now, we present a Ky Fan minimax inequality for a scalar set-valued mapping.

Theorem 3.2. Let X_0 be a nonempty compact convex subset of X. Suppose that the following conditions are satisfied:

(i) $F : X_0 \times X_0 \to 2^R$ is a continuous set-valued mapping with compact values;

(ii) for any $x \in X_0$, $F(x, \cdot)$ is R_+-quasiconcave on X_0.

Then,

$$\min \bigcup_{x \in X_0} \max F(x, X_0) \leq \max \bigcup_{x \in X_0} F(x, x). \tag{1}$$

Proof. By assumptions and Lemmas 2.3-2.5,

$$\min \bigcup_{x \in X_0} \max F(x, X_0) \neq \emptyset \quad \text{and} \quad \max \bigcup_{x \in X_0} F(x, x) \neq \emptyset.$$

Choose any real number t such that

$$t > \max \bigcup_{x \in X_0} F(x, x)$$

and let

$$A = \{(x, y) \in X_0 \times X_0 : \forall z \in F(x, y), z \leq t\}.$$

We will prove that A satisfies all conditions of Theorem 2.11.

First, we show that for each $y \in X_0$, the set $\{x \in X_0 : (x, y) \in A\}$ is a closed set. Indeed, for each $y \in X_0$, let $x_\alpha \in \{x \in X_0 : (x, y) \in A\}$ and $x_\alpha \to x_0$. By the l.s.c. of $F(\cdot, y)$, for any $z_0 \in F(x_0, y)$, there exists $z_\alpha \in F(x_\alpha, y)$ such that $z_\alpha \to z_0$. Since $(x_\alpha, y) \in A$ for any α, we have that $z_\alpha \leq t$. Thus,

$$x_0 \in \{x \in X_0 : \forall z \in F(x, y), z \leq t\}$$

and hence $\{x \in X_0 : (x, y) \in A\}$ is closed.

Second, we show that for each $x \in X_0$, the set $\{y \in X_0 : (x, y) \notin A\}$ is a convex set. Indeed, by the assumption of A, we have that for each $x \in X_0$,

$$\{y \in X_0 : (x, y) \notin A\} = \{y \in X_0 : \exists z \in F(x, y), z > t\}.$$

By the condition (ii) and Lemma 3.1, we see that $\{y \in X_0 : (x, y) \notin A\}$ is a convex set.

Moreover, since $\max F(x, x) \leq \max \bigcup_{x \in X_0} F(x, x)$, it is clear that $(x, x) \in A$, for each $x \in X_0$. Then, by Theorem 2.11, there exists $x_0 \in X_0$ such that $\{x_0\} \times X_0 \subset A$, i.e.,

$$\max \bigcup_{x \in X_0} F(x_0, x) \leq t.$$

By the assumption on t, we have

$$\min \bigcup_{x\in X_0} \max F(x, X_0) \leq \max \bigcup_{x\in X_0} F(x, x)$$

and hence (1) holds. This completes the proof. $\square$

4. Ky Fan minimax inequalities for set-valued mappings

In this section, we present some Ky Fan minimax inequalities for set-valued mappings. Let $F : X_0 \times X_0 \to 2^V$ and $e \in \mathrm{int}S$, arbitrarily fixed.

Theorem 4.1. Let X_0 be a nonempty compact convex subset of X. Suppose that the following conditions are satisfied:

(i) F is continuous with compact values;

(ii) for any $x \in X_0$, $F(x, \cdot)$ is S-quasiconcave on X_0.

Then,

$$\mathrm{Max_w} \bigcup_{x\in X_0} F(x, x) \subset \mathrm{Min} \bigcup_{x\in X_0} \mathrm{Max_w} F(x, X_0) + V \setminus (-\mathrm{int}S). \tag{2}$$

Proof. Since F is continuous and X_0 is compact, it follows from Lemmas 2.4 and 2.5 that

$$\mathrm{Max_w} \bigcup_{x\in X_0} F(x, x) \neq \emptyset.$$

Let $z \in \mathrm{Max_w} \bigcup_{x\in X_0} F(x, x)$. Then, we have

$$(z + \mathrm{int}S) \bigcap (\bigcup_{x\in X_0} F(x, x)) = \emptyset.$$

By Lemma 2.7 (i), we have

$$h_{ez}(u) \leq 0, \qquad \forall u \in \bigcup_{x\in X_0} F(x, x). \tag{3}$$

By the continuity of h_{ez} and F, and the compactness of X_0, there exist $x_1 \in X_0$ and $z_1 \in F(x_1, x_1)$ such that $\max \bigcup_{x\in X_0} h_{ez}(F(x, x)) = h_{ez}(z_1)$. Then, by Lemma 2.7 (iv), we have $z_1 \in \mathrm{Max_w} \bigcup_{x\in X_0} F(x, x) \subset \bigcup_{x\in X_0} F(x, x)$. By (3), we have

$$\max \bigcup_{x\in X_0} h_{ez}(F(x, x)) \leq 0. \tag{4}$$

Consider the set-valued mapping $G = h_{ez}(F) : X_0 \times X_0 \to 2^R$. By Lemma 2.9, it is clear that all conditions of Theorem 3.2 are satisfied for G, and hence we have

$$\min \bigcup_{x\in X_0} \max G(x, X_0) \leq \max \bigcup_{x\in X_0} G(x, x). \tag{5}$$

By (4) and (5), $\min \bigcup_{x\in X_0} \max h_{ez}(F(x, X_0)) \leq 0$. Thus, there exists $x' \in X_0$ such that $\max \bigcup_{y\in X_0} h_{ez}(F(x', y)) \leq 0$. Then, there exist $y' \in X_0$ and $z' \in F(x', y')$ such that $h_{ez}(z') = \max\bigcup_{y\in X_0} h_{ez}(F(x',y)) \le 0$. Therefore, by Lemma 2.7 (i) and (iv), we have $z' \notin z + \mathrm{int}S$ and $z' \in \mathrm{Max_w}\bigcup_{y\in X_0} F(x',y)$. We get

$$\begin{aligned} z &\in z' + V \setminus (-\mathrm{int}S) \subset \mathrm{Max_w} \bigcup_{y\in X_0} F(x',y) + V \setminus (-\mathrm{int}S) \\ &\subset \mathrm{Min} \bigcup_{x\in X_0} \mathrm{Max_w} F(x,X_0) + S + V \setminus (-\mathrm{int}S) \\ &= \mathrm{Min} \bigcup_{x\in X_0} \mathrm{Max_w} F(x,X_0) + V \setminus (-\mathrm{int}S). \end{aligned}$$

Hence, (2) holds. This completes the proof. □

Remark 4.2. If $F(x,\cdot)$ is S-concave for every $x \in X_0$, then it is clear that $F(x,\cdot)$ is S-quasiconcave for every $x \in X_0$. However, the converse is not valid. Thus, Theorem 4.1 of [32] is a special case of Theorem 4.1.

Example 4.3. Let $X = R$, $V = R^2$, $X_0 = [0,1]$, $S = \{(x,y)|\ x \ge 0, y \ge 0\}$ and $M = \{(u,0) \in R^2|\ -1 \le u \le 1\}$. Let $f : X_0 \times X_0 \to R^2$ and $F : X_0 \times X_0 \to 2^{R^2}$,

$$f(x,y) = (y, xy^2)$$

and

$$F(x,y) = f(x,y) + M.$$

Obviously, $F(x,\cdot)$ is S-quasiconcave on X_0 for any $x \in X_0$. Nevertheless, for any $x \in X_0$, $F(x,\cdot)$ is not S-concave on X_0. Therefore, Theorem 4.1 of [32] is not applicable. However, all conditions of Theorem 4.1 hold. So, inclusion (2) holds. Indeed, by simple computation,

$$\mathrm{Max_w} \bigcup_{x\in X_0} F(x,x) = \{(u,1)|\ 0 \le u \le 2\}$$

and

$$\mathrm{Min} \bigcup_{x\in X_0} \mathrm{Max_w} F(x,X_0) = \{(-1,0)\}.$$

Obviously,

$$\mathrm{Max_w} \bigcup_{x\in X_0} F(x,x) \subset \mathrm{Min} \bigcup_{x\in X_0} \mathrm{Max_w} F(x,X_0) + V \setminus (-\mathrm{int}S).$$

Theorem 4.4. Let X_0 be a nonempty compact convex subset of X. Suppose that the following conditions are satisfied:

(i) F is continuous with compact values;

(ii) for any $x \in X_0$, $F(x,\cdot)$ is properly S-quasiconcave on X_0.

Then,

$$\mathrm{Min_w} \bigcup_{x\in X_0} \mathrm{Max_w} F(x,X_0) \subset \mathrm{Max} \bigcup_{x\in X_0} F(x,x) + V \setminus \mathrm{int}S. \tag{6}$$

Proof. Since F is continuous and X_0 is compact, it follows from Lemmas 2.3-2.5 that

$$\mathrm{Min_w} \bigcup_{x\in X_0} \mathrm{Max_w} F(x,X_0) \ne \emptyset.$$

Let $z \in \mathrm{Min_w} \bigcup_{x\in X_0} \mathrm{Max_w} F(x, X_0)$. Then, we have

$$(z - \mathrm{int}S) \bigcap (\bigcup_{x\in X_0} \mathrm{Max_w} F(x, X_0)) = \emptyset.$$

By Lemma 2.7 (i), we have

$$\xi_{ez}(u) \geq 0, \qquad \forall u \in \bigcup_{x\in X_0} \mathrm{Max_w} F(x, X_0). \tag{7}$$

By the continuity of ξ_{ez} and F, and the compactness of X_0, for each $x \in X_0$, there exist $y_1 \in X_0$ and $z_1 \in F(x, y_1)$ such that $\max \xi_{ez}(F(x, X_0)) = \xi_{ez}(z_1)$. Then, by Lemma 2.7 (iv), we have $z_1 \in \mathrm{Max_w} F(x, X_0) \subset \bigcup_{x\in X_0} \mathrm{Max_w} F(x, X_0)$. By (7), $\max \xi_{ez}(F(x, X_0)) \geq 0$. It follows from the arbitrariness of $x \in X_0$ that

$$\min \bigcup_{x\in X_0} \max \xi_{ez}(F(x, X_0)) \geq 0. \tag{8}$$

Consider the set-valued mapping $W = \xi_{ez}(F) : X_0 \times X_0 \to 2^R$. By Lemma 2.9, it is clear that all conditions of Theorem 3.2 are satisfied for W, and hence we have

$$\min \bigcup_{x\in X_0} \max W(x, X_0) \leq \max \bigcup_{x\in X_0} W(x, x). \tag{9}$$

By (8) and (9), $\max \bigcup_{x\in X_0} \xi_{ez}(F(x, x)) \geq 0$. Thus, there exist $x' \in X_0$ and $z' \in F(x', x')$ such that $\xi_{ez}(z') \geq 0$. By Lemma 2.7 (i), $z' \notin z - \mathrm{int}S$. Thus, we have

$$\begin{aligned} z &\in z' + V \setminus \mathrm{int}S \subset \bigcup_{x\in X_0} F(x', x') + V \setminus \mathrm{int}S \\ &\subset \mathrm{Max} \bigcup_{x\in X_0} F(x, x) - S + V \setminus \mathrm{int}S \\ &= \mathrm{Max} \bigcup_{x\in X_0} F(x, x) + V \setminus \mathrm{int}S. \end{aligned}$$

Hence, (6) holds. This completes the proof. □

Theorem 4.5. Let X_0 be a nonempty compact convex subset of X. Suppose that the following conditions are satisfied:

(i) F is continuous with compact values;
(ii) for any $x \in X_0$, $F(x, \cdot)$ is properly S-quasiconcave on X_0;
(iii) for each $x \in X_0$,

$$\mathrm{Max_w} \bigcup_{x\in X_0} F(x, x) - F(x, x) \subset S.$$

Then,

$$\mathrm{Max_w} \bigcup_{x\in X_0} F(x, x) \subset \mathrm{Min} \bigcup_{x\in X_0} \mathrm{Max_w} F(x, X_0) + S. \tag{10}$$

Proof. Since F is continuous and X_0 is compact, it follows from Lemmas 2.3-2.5 that,

$$\mathrm{Min} \bigcup_{x\in X_0} \mathrm{Max_w} F(x, X_0) \neq \emptyset.$$

Suppose that $z \in V$ and $z \notin \bigcup_{x \in X_0} \mathrm{Max_w} F(x, X_0) + S$, i.e.,

$$(z - S) \bigcap (\bigcup_{x \in X_0} \mathrm{Max_w} F(x, X_0)) = \emptyset.$$

By Lemma 2.7 (ii), we have

$$\xi_{ez}(u) > 0, \qquad \forall u \in \bigcup_{x \in X_0} \mathrm{Max_w} F(x, X_0). \tag{11}$$

By the continuity of ξ_{ez} and F and the compactness of X_0, for each $x \in X_0$, there exist $y_1 \in X_0$ and $z_1 \in F(x, y_1)$ such that $\max \bigcup_{y \in X_0} \xi_{ez}(F(x, y)) = \xi_{ez}(z_1)$. By Lemma 2.7 (iv), $z_1 \in \mathrm{Max_w} \bigcup_{y \in X_0} F(x, y) \subset \bigcup_{x \in X_0} \mathrm{Max_w} F(x, X_0)$. Therefore, by (11), we have $\max \bigcup_{y \in X_0} \xi_{ez}(F(x, y)) > 0$. Since $x \in X_0$ is arbitrary, we have

$$\min \bigcup_{x \in X_0} \max \xi_{ez}(F(x, X_0)) > 0. \tag{12}$$

Consider the set-valued mapping $W = \xi_{ez}(F) : X_0 \times X_0 \to 2^R$. By Lemma 2.9, it is clear that all conditions of Theorem 3.2 are satisfied for W, and hence we have

$$\min \bigcup_{x \in X_0} \max W(x, X_0) \leq \max \bigcup_{x \in X_0} W(x, x). \tag{13}$$

By (12) and (13), $\max \bigcup_{x \in X_0} \xi_{ez}(F(x, x)) > 0$. Then, there exist $x' \in X_0$ and $z' \in F(x', x')$ such that $\xi_{ez}(z') > 0$. By Lemma 2.7 (ii), we have

$$z' \notin z - S. \tag{14}$$

If $z \in \mathrm{Max_w} \bigcup_{x \in X_0} F(x, x)$, then by the condition (iii) we have

$$F(x, x) \subset z - S, \qquad \forall x \in X_0,$$

which contradicts (14). Thus, we have

$$z \notin \mathrm{Max_w} \bigcup_{x \in X_0} F(x, x).$$

Remembering that

$$\mathrm{Min} \bigcup_{x \in X_0} \mathrm{Max_w} F(x, X_0) + S \supset \bigcup_{x \in X_0} \mathrm{Max_w} F(x, X_0) + S.$$

Hence, (10) holds. This completes the proof. □

Remark 4.6. (i) Clearly, if $V = R$ and $S = R_+$, then condition (iii) of Theorem 4.5 always holds.

(ii) The following example is given to illustrate that condition (iii) in Theorem 4.5 is essential.

Example 4.7. Let $X = R$, $V = R^2$, $X_0 = [0, 1] \subset X$, $S = \{(x, y) | y \geq |x|\}$ and $M = \{(u, 0) | -1 \leq u \leq 1\}$. Let $f : [0, 1] \times [0, 1] \to R^2$ and $F : [0, 1] \times [0, 1] \to 2^{R^2}$,

$$f(x, y) = \begin{cases} (x, 0), & y \leq x \\ (x, 2(y - x)), & y \geq x \end{cases}$$

and

$$F(x, y) = f(x, y) + M.$$

Obviously, F is continuous with compact values and $F(x,\cdot)$ is properly S-quasiconcave for every $x \in X_0$. For any $x \in X_0$, we have

$$F(x,x) = f(x,x) + M = (x,0) + M.$$

Then,

$$\mathrm{Max_w} \bigcup_{x\in X_0} F(x,x) = \{(u,0)| -1 \le u \le 2\}.$$

Take $x_0 = 0$. Thus,

$$\mathrm{Max_w} \bigcup_{x\in X_0} F(x,x) - F(x_0,x_0) = \{(u,0)| -2 \le u \le 3\} \not\subset S.$$

Obviously, all assumptions of Theorem 4.5 except for (iii) are satisfied. Moreover,

$$\mathrm{Min} \bigcup_{x\in X_0} \mathrm{Max_w} F(x,X_0) = \{(u,0)|0 \le u \le 2\}.$$

Clearly,

$$\mathrm{Max} \bigcup_{x\in X_0} F(x,x) \not\subset \mathrm{Min} \bigcup_{x\in X_0} \mathrm{Max_w} F(x,X_0) + S.$$

Hence, the condition (iii) in Theorem 4.5 is essential.

Theorem 4.8. Let X_0 be a nonempty compact convex subset of X. Suppose that the following conditions are satisfied:

(i) F is continuous with compact values;

(ii) for any $x \in X_0$, $F(x,\cdot)$ is S-quasiconcave on X_0;

(iii) for each $x \in X_0$,

$$\mathrm{Min_w} \bigcup_{x\in X_0} \mathrm{Max_w} F(x,X_0) \subset \mathrm{Max} F(x,X_0) - S.$$

Then,

$$\mathrm{Min_w} \bigcup_{x\in X_0} \mathrm{Max_w} F(x,X_0) \subset \mathrm{Max_w} \bigcup_{x\in X_0} F(x,x) - S. \tag{15}$$

Proof. Since F is continuous and X_0 is compact, it follows from Lemmas 2.4 and 2.5 that,

$$\mathrm{Max_w} \bigcup_{x\in X_0} F(x,x) \neq \emptyset.$$

Suppose that $z \in V$ and $z \notin \mathrm{Max_w} \bigcup_{x\in X_0} F(x,x) - S$, i.e.,

$$(z+S) \bigcap (\mathrm{Max_w} \bigcup_{x\in X_0} F(x,x)) = \emptyset.$$

By Lemma 2.7 (ii), we have

$$h_{ez}(u) < 0, \qquad \forall u \in \mathrm{Max_w} \bigcup_{x\in X_0} F(x,x). \tag{16}$$

By the continuity of h_{ez} and F and the compactness of X_0, there exist $x_1 \in X_0$ and $z_1 \in F(x_1, x_1)$ such that $\max \bigcup_{x\in X_0} h_{ez}(F(x,x)) = h_{ez}(z_1)$. By Lemma 2.7 (iv), $z_1 \in \mathrm{Max_w} \bigcup_{x\in X_0} F(x,x)$. Therefore, by (16), we have

$$\max \bigcup_{x\in X_0} h_{ez}(F(x,x)) < 0. \tag{17}$$

Consider the set-valued mapping $G = h_{ez}(F) : X_0 \times X_0 \to 2^R$. By Lemma 2.9, it is clear that all conditions of Theorem 3.2 are satisfied for G, and hence we have

$$\min \bigcup_{x\in X_0} \max G(x, X_0) \le \max \bigcup_{x\in X_0} G(x,x). \tag{18}$$

By (17) and (18), $\min \bigcup_{x\in X_0} \max h_{ez}(F(x, X_0)) < 0$. There exists $x_1 \in X_0$ such that $\max h_{ez}(F(x_1, X_0)) < 0$. By Lemma 2.7 (ii),

$$z \notin z_1 - S, \qquad \forall z_1 \in F(x_1, X_0).$$

Hence, we have

$$z \notin \mathrm{Max}F(x_1, X_0) - S. \tag{19}$$

If $z \in \mathrm{Min_w} \bigcup_{x\in X_0} \mathrm{Max_w}F(x, X_0)$, then by the condition (iii) we have

$$z \in \mathrm{Max}F(x, X_0) - S, \qquad \forall x \in X_0,$$

which contradicts (19). Thus, we have

$$z \notin \mathrm{Min_w} \bigcup_{x\in X_0} \mathrm{Max_w}F(x, X_0).$$

Hence, (15) holds. This completes the proof. □

Remark 4.9. Clearly, if $V = R$ and $S = R_+$, then condition (iii) of Theorem 4.8 always holds.

Remark 4.10. In [31, 32], some similar Ky Fan minimax inequalities for set-valued mappings are also obtained. However, our proof methods are different from their ones.

Theorem 4.11. *Let X be a real locally convex Hausdorff topological vector space. Let X_0 be a nonempty compact convex subset of X. Suppose that the following conditions are satisfied:*

(i) F is continuous with compact values;

(ii) for any $x \in X_0$, $F(x,\cdot)$ is S-quasiconcave on X_0.

Then,

$$\exists\ z_1 \in \mathrm{Max} \bigcup_{x\in X_0} F(x,x) \quad \textit{and} \quad \exists\ z_2 \in \mathrm{Min} \bigcup_{x\in X_0} \mathrm{Max}_w F(x, X_0)$$

such that

$$z_1 \in z_2 + S. \tag{20}$$

Proof. Let $a \in V$. We define a multifunction $T : X_0 \to 2^{X_0}$ by the formula

$$T(x) = \{y \in X_0 : \max h_{ea}(F(x, X_0)) \in h_{ea}(F(x,y))\}, \qquad \text{for } x \in X_0.$$

First, by the continuity of h_{ea} and F and the compactness of X_0, it is clear that $T(x) \ne \emptyset$, for each $x \in X_0$.

Second, we show that $T(x)$ is a closed set, for each $x \in X_0$. Indeed, for each $x \in X_0$, let a net $\{y_\alpha : \alpha \in I\} \subset T(x)$ and $y_\alpha \to y_0$. Since $h_{ea}(F(x, y_\alpha)) \subset h_{ea}(F(x, X_0))$ $\forall\alpha$, $\max h_{ea}(F(x, X_0)) = \max h_{ea}(F(x, y_\alpha))$. By Lemma 2.10, $\max h_{ea}(F(x, \cdot))$ is a continuous real-valued function. Then, $\max h_{ea}(F(x, X_0)) = \max h_{ea}(F(x, y_0))$. Thus, $\max h_{ea}(F(x, X_0)) \in h_{ea}(F(x, y_0))$. We have

$$y_0 \in T(x) = \{y \in X_0 : \max h_{ea}(F(x, X_0)) \in h_{ea}(F(x, y))\}$$

and hence for each $x \in X_0$, $T(x)$ is a closed set.

Next, we show that $T(x)$ is a convex set, for each $x \in X_0$. Indeed, for each $x \in X_0$, let $y_1, y_2 \in T(x)$ and $\lambda \in [0, 1]$. By the condition (ii) and Lemma 2.9, there exists $z_0 \in h_{ea}(F(x, \lambda y_1 + (1-\lambda)y_2))$ such that $z_0 \geq \max h_{ea}(F(x, X_0))$. Since $h_{ea}(F(x, \lambda y_1 + (1-\lambda)y_2)) \subset h_{ea}(F(x, X_0))$, $z_0 \leq \max h_{ea}(F(x, X_0))$. Thus, $\max h_{ea}(F(x, X_0)) = z_0 \in h_{ea}(F(x, \lambda y_1 + (1-\lambda)y_2))$, i.e.,

$$\lambda y_1 + (1-\lambda)y_2 \in T(x) = \{y \in X_0 : \max h_{ea}(F(x, X_0)) \in h_{ea}(F(x, y))\}$$

and hence for each $x \in X_0$, $T(x)$ is a convex set.

Now, we prove that T is u.s.c. on X_0. Since X_0 is compact, we only need to show that T is a closed map (see [1]). Let a net

$$\{(x_\alpha, y_\alpha)\} \subset \mathrm{Graph}T := \{(x, y) \in X_0 \times X_0 : \max h_{ea}(F(x, X_0)) \in h_{ea}(F(x, y))\}$$

and $(x_\alpha, y_\alpha) \to (x_0.y_0)$.

Since $h_{ea}(F(x_\alpha, y_\alpha)) \subset h_{ea}(F(x_\alpha, X_0))$ $\forall\alpha$,

$$\max h_{ea}(F(x_\alpha, X_0)) = \max h_{ea}(F(x_\alpha, y_\alpha)).$$

By Lemmas 2.3 and 2.10, $\max h_{ea}(F(\cdot, X_0))$ and $\max h_{ea}(F(\cdot, \cdot))$ are two continuous real-valued functions. Then, $\max h_{ea}(F(x_0, X_0)) = \max h_{ea}(F(x_0, y_0))$. Thus, $\max h_{ea}(F(x_0, X_0)) \in h_{ea}(F(x_0, y_0))$, i.e., $(x_0, y_0) \in \mathrm{Graph}T$. Hence, T is u.s.c..

Then, by Theorem 2.12, there exists $x_0 \in X_0$ such that $x_0 \in T(x_0)$, i.e.,

$$\max h_{ea}(F(x_0, X_0)) \in h_{ea}(F(x_0, x_0)).$$

Let $z \in F(x_0, x_0)$ be such that

$$h_{ea}(z) = \max h_{ea}(F(x_0, X_0)).$$

By Lemma 2.7 (iv), $z \in \mathrm{Max_w}F(x_0, X_0)$, i.e.,

$$F(x_0, x_0) \bigcap \mathrm{Max_w}F(x_0, X_0) \neq \emptyset.$$

Then, by assumptions and Lemma 2.5, we have that

$$F(x_0, x_0) \subset \mathrm{Max} \bigcup_{x \in X_0} F(x, x) - S$$

and

$$\mathrm{Max_w}F(x_0, X_0) \subset \mathrm{Min} \bigcup_{x \in X_0} \mathrm{Max_w}F(x, X_0) + S.$$

Namely, for every $u \in F(\bar{x}, \bar{x})$ and $v \in \mathrm{Max}_w \bigcup_{y \in X_0} F(\bar{x}, y)$, there exist $z_1 \in \mathrm{Max} \bigcup_{x \in X_0} F(x, x)$ and $z_2 \in \mathrm{Min} \bigcup_{x \in X_0} \mathrm{Max}_w F(x, X_0)$ such that

$$u \in z_1 - S \qquad \text{and} \qquad v \in z_2 + S.$$

Particularly, taking $u = v$, we have $z_1 \in z_2 + S$. This completes the proof. □

Remark 4.12. If $F(x, \cdot)$ is properly S-quasiconcave for every $x \in X_0$, then it is clear that $F(x, \cdot)$ is S-quasiconcave for every $x \in X_0$. However, the converse is not valid. Thus, Theorem 3.1 of [31] is a special case of Theorem 4.11.

Example 4.13. Let $X = R$, $V = R^2$, $X_0 = [-1, 1]$, $S = R^2_+$ and $M = \{(u, 0)| -1 \leq u \leq 1\}$. Let $f : X_0 \times X_0 \to R^2$ and $F : X_0 \times X_0 \to 2^{R^2}$,

$$f(x, y) = \{(x(y, z))|z = \sqrt{1 - y^2}\}$$

and

$$F(x, y) = f(x, y) + M.$$

Obviously, $f(x, \cdot)$ is S-quasiconcave on X_0 for any $x \in X_0$. Nevertheless, for any $x \in X_0$, $f(x, \cdot)$ is not properly S-quasiconcave on X_0. Therefore, Theorem 3.1 of [31] is not applicable. However, by simple computation,

$$\mathrm{Min} \bigcup_{x \in X_0} \mathrm{Max_w} F(x, X_0) = \{(u, 0)| -1 \leq u \leq 2\}$$

and

$$\mathrm{Max} \bigcup_{x \in X_0} F(x, x) = \{(u, \frac{1}{2})| -\frac{1}{2} \leq u \leq \frac{3}{2}\}$$

Thus, taking $(-1, 0) \in \mathrm{Min} \bigcup_{x \in X_0} \mathrm{Max_w} F(x, X_0)$ and $(0, \frac{1}{2}) \in \mathrm{Max} \bigcup_{x \in X_0} F(x, x)$,

$$(0, \frac{1}{2}) \in (-1, 0) + S.$$

Remark 4.14. When F is a real-valued function and $S = R_+$, the minimax inequalities (1),(2),(6),(10) (15) and (20) reduce to the well-known Ky Fan minimax inequality, respectively.

Acknowledgement. The authors are grateful to the anonymous referees for their valuable comments and suggestions, which improved the paper.

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Received: October 26, 2012; Accepted: February 18, 2013