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STRONG CONVERGENCE FOR THE MANN ITERATION OF λ-STRICT PSEUDO-CONTRACTION

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Abstract. In this paper, we prove strong convergence of the Mann iteration of an λ-strict pseudo-contraction T in a real $q-$uniformly smooth Banach space. The results presented in this paper are interesting extensions and improvements upon those known ones of Marino and Xu [J. Math. Anal. Appl. 324(2007) 336-349], and are development and complementariness of the corresponding ones of Chai and Song [Fixed Point Theory and Applications, 2011(2011) 95], Cai and Hu [Computers & Mathematics with Applications 59(1)(2010), 149-160], Zhou [Nonlinear Anal. 69(2008) 3160-3173, Acta Mathematica Sinica, English Series, 26(2010), 743-758] and Zhang and Su [Convergence theorems for strict pseudo-contractions in q-uniformly smooth Banach spaces, Nonlinear Analysis, 70(9)(2009), 3236-3242; 71(2009) 4572-4580].
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1. Introduction

Let C be a nonempty subset of a Banach space E with the norm $\|\cdot\|$ and let T be a mapping from C to E. Throughout this paper, let $F(T) = \{x \in E : Tx = x\}$, the set of all fixed point of a mapping T and let $\mathbb{N}$ and $\mathbb{R}$ be the sets of positive integers and real numbers, respectively. The normalized duality mapping J from E into 2^{E^*} is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|\|x^*\|, \|x\| = \|x^*\|\}.$$

A mapping T is called *Lipschitzian* if there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\| \text{ for all } x, y \in C.$$

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T is said to be *nonexpansive* if $L = 1$ in the above inequality. T is called λ-*strictly pseudocontractive* if there exists $\lambda \in (0, 1)$ and $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \le ||x - y||^2 - \lambda||x - y - (Tx - Ty)||^2 \text{ for all } x, y \in D(T). \quad (1.1)$$

T is called *pseudocontractive* if $\lambda \equiv 0$ in (1.1). Obviously, each λ-strictly pseudocontractive mapping is a Lipschitzian and pseudocontractive mapping with $L = \frac{\lambda+1}{\lambda}$. In particular, a nonexpansive mapping is λ-strictly pseudocontractive mapping in a Hilbert space, but the conversion may be false.

A Banach space E is said to satisfy *Opial's condition* ([5]) if, for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \quad \forall y \in E \text{ with } x \neq y.$$

In particular, Opial's condition is independent of uniformly convex (smooth) since the l^p spaces satisfy this condition for $1 < p < \infty$ while it fails for the L^p $(p \neq 2)$ spaces. In fact, spaces satisfying Opial's condition need not even by isomorphic to uniformly convex spaces ([3]).

Marino and Xu [4] studied weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces. Recently, Zhou [8] proved a weak convergence theorem of λ-strictly pseudo-contractive mapping T in a 2-uniformly smooth Banach space.

Theorem Z. *(Zhou* [8, Theorem 2.1]*) Let E be a real 2-uniformly smooth Banach space with the smooth constant K, and let C be a closed convex subset of E, and let $T : C \to C$ be a λ-strict pseudo-contraction with $F(T) \neq \emptyset$. Suppose that E is either uniformly convex or satisfies Opial's condition. Given $u, x_0 \in C$, a sequence $\{x_n\}$ is generated by*

$$x_{n+1} = \alpha_n Tx_n + (1 - \alpha_n)x_n, \quad (1.2)$$

where $\{\alpha_n\}$ in $(0, 1)$ satisfies:

(i) $\alpha_n \in [0, \mu]$, $\mu = \min\{1, \frac{\lambda}{K^2}\}$;

(ii) $\sum_{n=0}^{\infty} \alpha_n(\lambda - K^2\alpha_n) = \infty$.

Then the sequence $\{x_n\}$ converges weakly to a fixed point z of T.

Recently, Zhou [9] further discussed the convergent properties of iterates of (a finite family of) λ-strict pseudo-contraction in a real q-uniformly smooth Banach space, and obtained the strong convergence of the modified Mann's iteration. Zhang and Su [10, 11] also showed the strong convergence of λ-strict pseudo-contraction for finding some fixed point of such mappings. Still in a real q-uniformly smooth Banach space, Cai and Hu [2] studied strong convergence of an iteration for a finite family of λ-strict pseudo-contraction. Very recently, Chai and Song [1] studied the strong convergence of the modified Mann's iteration (1.3).

Theorem CS. *(Chai and Song* [1, Theorem 3.1]*) Let E be a real 2-uniformly smooth Banach space with the smooth constant K, and let C be a closed convex subset of E, and let $T : C \to C$ be a λ-strict pseudo-contraction with $F(T) \neq \emptyset$. Given $u, x_0 \in C$, a sequence $\{x_n\}$ is generated by*

$$x_{n+1} = \beta_n u + (1 - \beta_n)[\alpha_n Tx_n + (1 - \alpha_n)x_n], \quad (1.3)$$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ *in* $(0,1)$ *satisfy the following control conditions:*

(i) $\alpha_n \in [a, \mu]$, $\mu = \min\{1, \frac{\lambda}{K^2}\}$ *for some constant* $a \in (0, \mu)$*;*

(ii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$*;*

(iii) $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$ *and* $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Then, the sequence $\{x_n\}$ *converges strongly to a fixed point* z *of* T.

In this paper, we will deal with strong convergence of the Mann iteration

$$x_{n+1} = \beta_n T x_n + (1 - \beta_n) x_n \tag{1.4}$$

where the sequence $\{\beta_n\}$ in $[0, \mu]$, $\mu = \min\{1, \frac{\lambda}{K^2}\}$ such that

$$\liminf_{n\to\infty} \beta_n(\lambda - K^2\beta_n) > 0. \tag{1.5}$$

Our results obviously develop and complement the corresponding ones of Chai and Song [1], Cai and Hu [2], Marino and Xu [4], Zhou [8, 9], Zhang and Su [10, 11] and others.

2. Preliminaries and basic results

For achieving our purposes, the following facts and results are needed.

Let $\rho_E : [0, \infty) \to [0, \infty)$ be the modulus of smoothness of E defined by

$$\rho_E(t) = \sup\left\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : x \in S(E), \|y\| \le t\right\}.$$

Let $q > 1$. A Banach space E is said to be *q-uniformly smooth* if there exists a fixed constant $c > 0$ such that $\rho_E(t) \le ct^q$ and *uniformly smooth* if $\lim_{t\to 0} \frac{\rho_E(t)}{t} = 0$. Clearly, a q-uniformly smooth space must be uniformly smooth. Typical example of uniformly smooth Banach spaces is L_p $(p > 1)$. More precisely, L_p is $\min\{p, 2\}$-uniformly smooth for every $p > 1$.

Lemma 2.1. (Zhou [8, Lemma 1.2]) *Let C be a nonempty subset of a real 2-uniformly smooth Banach space E with the best smooth constant K, and let $T : C \to C$ be a λ-strict pseudo-contraction. For any $\alpha \in (0, 1)$, we define $T_\alpha = (1-\alpha)x+\alpha Tx$. Then,*

$$\|T_\alpha x - T_\alpha y\|^2 \le \|x - y\|^2 - 2\alpha(\lambda - K^2\alpha)\|Tx - Ty - (x - y)\|^2 \text{ for all } x, y \in C. \tag{2.1}$$

In particular, as $\alpha \in (0, \frac{\lambda}{K^2}]$, $T_\alpha : C \to C$ is nonexpansive such that $F(T_\alpha) = F(T)$.

Lemma 2.2. *Let C be a nonempty closed and convex subset of a real 2-uniformly smooth Banach space E with the best smooth constant K and let $T : C \to C$ be a λ-strict pseudo-contraction with $F(T) \ne \emptyset$. Suppose that the sequence $\{x_n\}$ is defined by the Mann iteration (1.4) and the sequence $\{\beta_n\}$ in $[0, \mu]$, $\mu = \min\{1, \frac{\lambda}{K^2}\}$. Then*

(i) the sequence $\{x_n\}$ is bounded;

(ii) $\|x_{n+1} - u\| \le \|x_n - u\|$ for each $u \in F(T)$;

(iii) the limit $\lim_{n\to\infty} \|x_n - u\|$ exists for each $u \in F(T)$;

if, in addition, $\{\beta_n\}$ satisfy the condition (1.5), then

(iv) $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

Proof. Let $u \in F(T)$ and $T_{\beta_n} = \beta_n T + (1-\beta_n)I$. It follows from Lemma 2.1 that T_{β_n} is nonexpansive and $F(T_{\beta_n}) = F(T)$, and so

$$\|x_{n+1} - u\| = \|T_{\beta_n} x_n - u\| \le \|x_n - u\| \le \cdots \le \|x_1 - u\|.$$

So the sequence $\{x_n\}$ is bounded and the sequence $\{\|x_n - u\|\}$ is monotone non-increasing, and hence the limit $\lim_{n\to\infty} \|x_n - u\|$ exists for each $u \in F(T)$.

Now we show (iv). From Lemma 2.1, it follows that $T_{\beta_n} u = u$ for all n and

$$\begin{aligned}\|x_{n+1} - u\|^2 &= \|(\beta_n T x_n + (1-\beta_n)x_n) - u\|^2 = \|T_{\beta_n} x_n - T_{\beta_n} u\|^2 \\ &\le \|x_n - u\|^2 - \beta_n(\lambda - K^2\beta_n)\|Tx_n - x_n\|^2.\end{aligned}$$

Then we have

$$\beta_n(\lambda - K^2\beta_n)\|Tx_n - x_n\|^2 \le \|x_n - u\|^2 - \|x_{n+1} - u\|^2,$$

and so,

$$\sum_{n=1}^{\infty} \beta_n(\lambda - K^2\beta_n)\|Tx_n - x_n\|^2 \le \|x_1 - u\|^2 < +\infty.$$

From the condition (1.5), it follows that

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0.$$

This completes the proof. □

3. Main results

Let C be a nonempty subset of a Banach space E. A mapping $T : C \to C$ is said to satisfy *Condition I* if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0, f(r) > 0$ for $r \in (0,\infty)$ such that

$$\|x - Tx\| \ge f(d(x, F(T))) \text{ for all } x \in C,$$

where $d(x, F(T)) = \inf\{\|x - y\|; y \in F(T)\}$. This concept was introduced by Senter and Dotson in [7]; and several examples of such mappings were given.

Next we will show our main results.

Theorem 3.1. *Let C be a nonempty closed and convex subset of a real 2-uniformly smooth Banach space E with the best smooth constant K and let $T : C \to C$ be a λ-strict pseudo-contraction with $F(T) \neq \emptyset$ and satisfying Condition I. Suppose that the sequence $\{x_n\}$ is defined by the Mann iteration (1.4),*

$$x_{n+1} = \beta_n T x_n + (1-\beta_n)x_n,$$

where the sequence $\{\beta_n\}$ in $[0,\mu]$, $\mu = \min\{1, \frac{\lambda}{K^2}\}$ such that

$$\liminf_{n\to\infty} \beta_n(\lambda - K^2\beta_n) > 0.$$

Then the sequence $\{x_n\}$ converges strongly to a fixed point z of T.

Proof. It follows from Lemma 2.2 that the sequence $\{x_n\}$ is bounded and

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{3.1}$$

Then Condition I implies $\lim\limits_{n\to\infty} f(d(x_n, F(T))) = 0$, and hence

$$\lim_{n\to\infty} d(x_n, F(T)) = 0. \tag{3.2}$$

Next we show that the sequence $\{x_n\}$ is a Cauchy sequence of E. In fact, for any $n, m \in \mathbb{N}$ with $m > n$, then $\|x_m - u\| \le \|x_n - u\|$ for each $u \in F(T)$ by Lemma 2.2 (ii), and so

$$\|x_n - x_m\| \le \|x_n - u\| + \|u - x_m\| \le 2\|x_n - u\|. \tag{3.3}$$

Since u is arbitrary, then we may take the infimum for u in (3.3),

$$\|x_n - x_m\| \le 2\inf\{\|x_n - u\|; u \in F(T)\} = 2d(x_n, F(T)).$$

From (3.2), it follows that as $\lim\limits_{n\to\infty} \|x_n - x_m\| = 0$, which means that $\{x_n\}$ is a Cauchy sequence. So there exists $z \in E$ such that

$$\lim_{n\to\infty} \|x_n - z\| = 0.$$

Since

$$\begin{aligned} \|z - Tz\| \le& \|z - x_n\| + \|x_n - Tx_n\| + \|Tx_n - Tz\| \\ \le& \|z - x_n\| + \|x_n - Tx_n\| + \frac{1+\lambda}{\lambda}\|x_n - z\|, \end{aligned}$$

then from (3.1), it follows that $\|Tz - z\| = 0$, i.e., $z \in F(T)$. □

A mapping $T : C \to E$ is said to be *demicompact* (Petryshyn [6]) provided whenever a sequence $\{x_n\} \subset K$ is bounded and the sequence $\{x_n - Tx_n\}$ strongly converges, then there is a subsequence $\{x_{n_k}\}$ which strongly converges.

Theorem 3.2. *Let C be a nonempty closed and convex subset of a real 2-uniformly smooth Banach space E with the best smooth constant K and let $T : C \to C$ be a λ-strict pseudo-contraction with $F(T) \ne \emptyset$. Suppose that T is demicompact and the sequence $\{x_n\}$ is defined by the Mann iteration (1.4),*

$$x_{n+1} = \beta_n Tx_n + (1 - \beta_n)x_n,$$

where the sequence $\{\beta_n\}$ in $[0, \mu]$, $\mu = \min\{1, \frac{\lambda}{K^2}\}$ such that

$$\liminf_{n\to\infty} \beta_n(\lambda - K^2\beta_n) > 0.$$

Then the sequence $\{x_n\}$ converges strongly to a fixed point z of T.

Proof. It follows from Lemma 2.2 that the sequence $\{x_n\}$ is bounded and

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{3.4}$$

Then the demicompactness of T implies there is a subsequence $\{x_{n_k}\} \subset \{x_n\}$ and $z \in E$ such that

$$\lim_{k\to\infty} \|x_{n_k} - z\| = 0. \tag{3.5}$$

By (3.4), we also have $\lim_{k\to\infty} \|Tx_{n_k} - z\| = 0$. Since $\|Tx_{n_k} - Tz\| \leq \frac{1+\lambda}{\lambda}\|x_{n_k} - z\|$, then $\lim_{k\to\infty} \|Tx_{n_k} - Tz\| = 0$. So $z = Tz$, i.e., $z \in F(T)$.

From Lemma 2.2 (iii), it follows that the limit $\lim_{n\to\infty} \|x_n - z\|$ exists, and then

$$\lim_{n\to\infty} \|x_n - z\| = \lim_{k\to\infty} \|x_{n_k} - z\| = 0.$$

The proof is completed. □

Using the same proof techniques as in Theorem 3.1 and 3.2, we easily obtain the following result. Since the only difference is that $\alpha_n(\lambda - K^2\alpha_n)$ is replaced by $\alpha_n(q\lambda - K_q\alpha_n^{q-1})$ in its proof, we decide to omit the theorem proof.

Theorem 3.3. *Let K be a nonempty closed and convex subset of a real q-uniformly smooth Banach space E with the best smooth constant K_q $(q > 1)$ and let $T : K \to K$ be a λ-strict pseudo-contraction with $F(T) \neq \emptyset$. Suppose that T either is demicompact or satisfies Condition I. Assume that the sequence $\{x_n\}$ is defined by the Mann iteration (1.4), $x_{n+1} = \beta_n Tx_n + (1-\beta_n)x_n$, where the sequence $\{\beta_n\}$ in $[0, \mu]$, $\mu = \min\{1, \{\frac{q\lambda}{K_q}\}^{\frac{1}{q-1}}\}$ such that $\liminf_{n\to\infty} \beta_n(q\lambda - K_q\beta_n^{q-1}) > 0$. Then the sequence $\{x_n\}$ converges strongly to a fixed point z of T.*

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