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TWO REMARKS ON THE MODIFIED HALPERN ITERATIONS IN CAT(0) SPACES

SATIT SAEJUNG*,**

*Department of Mathematics, Faculty of Science, Khon Kaen University
Khon Kaen 40002, Thailand
E-mail: saejung@kku.ac.th

**The Centre of Excellence in Mathematics, Commission on Higher Education (CHE)
Sri Ayudthaya Road, Bangkok 10400, Thailand

Abstract. The purpose of this paper is to give a short and simple proof of a generalization of Suzuki's lemma [9] in metric spaces of hyperbolic type as well as to include some remarks on the strong convergence theorem of modified Halpern iteration in CAT(0) spaces, which is proved by Cuntavepanit and Panyanak [4].
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1. Introduction

A metric space (X, d) is said to be of *hyperbolic type* if there exists a family $\mathcal{L}$ of metric segments, that is, isometric images of real line segments, such that the following conditions hold:

- for each $x, y \in X$ there exists exactly one member $S[x, y] \in \mathcal{L}$, we also let $S[x, y] := \{\alpha x \oplus (1-\alpha)y \in X : \alpha \in [0, 1]\}$ where $d(\alpha x \oplus (1-\alpha)y, y) = \alpha d(x, y)$;
- for $x, y, z \in X$ and $\alpha \in [0, 1]$

$$d(\alpha x \oplus (1-\alpha)z, \alpha y \oplus (1-\alpha)z) \le \alpha d(x, y).$$

Clearly, every normed space is a metric space of hyperbolic type. Moreover, CAT(0) spaces (the precise definition will be given below) is of hyperbolic type.

To prove some strong convergence theorems of the modified Halpern iterations, Panyanak and Cuntavepanit [9] proved the following generalization of Suzuki's lemma:

Lemma 1.1. *Let $\{z_n\}, \{w_n\}$ and $\{v_n\}$ be bounded sequences in a metric space (X, d) of hyperbolic type and let $\{\alpha_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$. If $\lim_{n\to\infty} d(z_n, v_n) = 0$, $z_{n+1} = \alpha_n w_n \oplus (1-\alpha_n)v_n$ for all $n \in \mathbb{N}$ and $\limsup_{\to\infty}(d(w_{n+1}, w_n) - d(z_{n+1}, z_n)) \le 0$, then $\lim_{n\to\infty} d(w_n, z_n) = 0$.*

It is obvious that Lemma 1.1 $\implies$ Suzuki's lemma (see [12]).

Lemma 1.2 (Suzuki's lemma). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a metric space (X,d) of hyperbolic type and let $\{\alpha_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$. If $x_{n+1} = \alpha_n y_n \oplus (1-\alpha_n)x_n$ for all $n \in \mathbb{N}$ and $\limsup_{n\to\infty}(d(y_{n+1},y_n) - d(x_{n+1},x_n)) \le 0$, then $\lim_{n\to\infty} d(y_n,x_n) = 0$.*

Panyanak and Cuntavepanit gave the proof of Lemma 1.1 by following the idea of Suzuki's lemma and their proof is very long (it occupies more than 4 pages). Here we present a simple and short proof. In fact, we have Suzuki's lemma $\Longrightarrow$ Lemma 1.1.

Our proof needs the following lemma, known as Xu's lemma for sequences of real numbers (see [13]):

Lemma 1.3 (Xu's lemma). *If $\{s_n\}$, $\{t_n\}$ and $\{\alpha_n\}$ are sequences of real numbers such that the following hold:*

- *$s_n \ge 0$ and $s_{n+1} = (1-\alpha_n)s_n + \alpha_n t_n$ for all $n \in \mathbb{N}$;*
- *$\alpha_n \in [0,1]$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;*
- *$\limsup_{n\to\infty} t_n \le 0$,*

then $\lim_{n\to\infty} s_n = 0$.

A short and simple proof of Lemma 1.1. Let $\{z_n\}, \{w_n\}$ and $\{v_n\}$ be bounded sequences satisfying the conditions in Lemma 1.1. Define $\{x_n\}$ by $x_1 = z_1$ and $x_{n+1} := \alpha_n w_n \oplus (1-\alpha_n)x_n$ for all $n \ge 1$. It follows then that

$$\begin{aligned} d(z_{n+1},x_{n+1}) &= d(\alpha_n w_n \oplus (1-\alpha_n)v_n, \alpha_n w_n \oplus (1-\alpha_n)x_n) \\ &\le (1-\alpha_n)d(v_n,x_n) \\ &\le (1-\alpha_n)d(z_n,x_n) + (1-\alpha_n)d(v_n,z_n) \\ &= (1-\alpha_n)d(z_n,x_n) + \alpha_n \frac{(1-\alpha_n)d(v_n,z_n)}{\alpha_n}. \end{aligned}$$

Set $s_n \equiv d(z_n,x_n)$ and $t_n \equiv \frac{(1-\alpha_n)d(v_n,z_n)}{\alpha_n}$ and apply Xu's lemma to get that $\lim_{n\to\infty} d(z_n,x_n) = 0$. On the other hand, we apply Suzuki's lemma for the sequences $\{x_n\}$ and $\{w_n\}$. Since $\limsup_{\to\infty}(d(w_{n+1},w_n) - d(z_{n+1},z_n)) \le 0$ and $\lim_{n\to\infty} d(z_n,x_n) = 0$, we have

$$\begin{aligned} &\limsup_{n\to\infty}(d(w_{n+1},w_n) - d(x_{n+1},x_n)) \\ &\le \limsup_{n\to\infty}(d(w_{n+1},w_n) - d(z_{n+1},z_n)) \\ &\qquad + \limsup_{n\to\infty}(d(z_{n+1},z_n) - d(x_{n+1},x_n)) \\ &\le \limsup_{n\to\infty}(d(z_{n+1},x_{n+1}) + d(z_n,x_n)) = 0. \end{aligned}$$

We then conclude that $\lim_{n\to\infty} d(w_n,x_n) = 0$. Consequently,

$$\lim_{n\to\infty} d(w_n,z_n) \le \lim_{n\to\infty}(d(w_n,x_n) + d(x_n,z_n)) = 0.$$

This completes the proof. □

CAT(0) spaces are an example of metric spaces of hyperbolic type. Let us recall the precise definition of CAT(0) spaces.

Let (X, d) be a metric space and $x, y \in X$ with $l = d(x, y)$. A *geodesic path* from x to y is an isometry $c : [0, l] \to X$ such that $c(0) = x$ and $c(l) = y$. The image of a geodesic path is called a *geodesic segment*. A metric space X is a *(uniquely) geodesic space* if every two points of X are joined by only one geodesic segment. A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ in a geodesic space X consists of three points x_1, x_2, x_3 of X and three geodesic segments joining each pair of vertices. A *comparison triangle* of a geodesic triangle $\triangle(x_1, x_2, x_3)$ is the triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in the Euclidean space $\mathbb{R}^2$ such that $d(x_i, x_j) = d_{\mathbb{R}^2}(\overline{x_i}, \overline{x_j})$ for all $i, j = 1, 2, 3$.

A geodesic space X is a *CAT(0) space* if for each geodesic triangle $\triangle := \triangle(x_1, x_2, x_3)$ in X and its comparison triangle $\overline{\triangle} := \triangle(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in $\mathbb{R}^2$, the *CAT(0) inequality*

$$d(x, y) \leq d_{\mathbb{R}^2}(\overline{x}, \overline{y})$$

is satisfied by all $x, y \in \triangle$ and $\overline{x}, \overline{y} \in \overline{\triangle}$. The meaning of the CAT(0) inequality is that a geodesic triangle in X is at least thin as its comparison triangle in the Euclidean plane. A thorough discussion of these spaces and their important role in various branches of mathematics are given in [2] and [3]. The complex Hilbert ball with the hyperbolic metric is an example of a CAT(0) space (see [5]).

Lemma 1.4. *For elements x, y and z in a CAT(0) space and $t \in [0, 1]$, the following inequality holds:*

$$d^2((1-t)x \oplus ty, z) \leq (1-t)d^2(x, z) + td^2(y, z) - t(1-t)d^2(x, y).$$

2. Main results

We next give two remarks on the strong convergence theorem of modified Halpern iterations in CAT(0) spaces. Recall that a subset C of a CAT(0) space is *convex* if it contains all geodesic segments joining any two points in C; and $T : C \to C$ is *nonexpansive* if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in C$. For more detail on fixed point theory for nonexpansive mappings in CAT(0) spaces, we refer the reader to two interesting surveys by Kirk [6, 7]. The following result was proved by Cuntavepanit and Panyanak (see [4, Theorem 3.1]).

Theorem 2.1. *Let C be a closed and convex subset of a complete CAT(0) space and let $T : C \to C$ be a nonexpansive mapping with $F(T) := \{x \in C : x = Tx\} \neq \varnothing$. Suppose that $\{\alpha_n\}$ and $\{\lambda_n\}$ are sequences on $[0, 1]$ such that*

(A1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*
(A2) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$*;*
(A3) $\lim_{n\to\infty} \lambda_n = 0$ *and* $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.

If $u \in C$ and $\{x_n\}$ is iteratively defined by $x_1 \in C$ and

$$x_{n+1} = \alpha_n u \oplus (1-\alpha_n)(\lambda_n x_n \oplus (1-\lambda_n)Tx_n),$$

then the sequence $\{x_n\}$ converges to $z \in F(T)$ which is the nearest point of $F(T)$ to u.

First, we show that Theorem 2.1 can regarded as a consequence of the recent result proved by the author (see [10, Theorem 3.2]). Let us recall the following concept introduced by Aoyama et al. [1]. For a subset C of a complete CAT(0) space, let

$\{T_n\}_{n=1}^{\infty}$ be a countable family of mappings from C into itself. We say that a family $\{T_n\}$ satisfies *AKTT-condition* if

$$\sum_{n=1}^{\infty} \sup\{d(T_{n+1}z, T_n z) : z \in B\} < \infty$$

for each bounded subset of B of C. In this case, for each $x \in C$, it follows that $\{T_n x\}$ is a Cauchy sequence in C. In particular, if C is a closed subset and $\{T_n\}$ satisfies AKTT-condition, then we can define $T : C \to C$ such that

$$Tx = \lim_{n\to\infty} T_n x \quad (x \in C).$$

In this case, we also say that $(\{T_n\}, T)$ satisfies AKTT-condition.

Theorem 2.2. *Let X be a complete CAT(0) space and C a closed convex subset of X. Let $\{T_n\} : C \to C$ be a countable family of nonexpansive mappings with $\cap_{n=1}^{\infty} F(T_n) \neq \varnothing$. Suppose that $u, x_1 \in C$ are arbitrarily chosen and $\{x_n\}$ is defined by*

$$x_{n+1} = \alpha_n u \oplus (1 - \alpha_n) T_n x_n \quad \textit{for all } n \geq 1,$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying

(A1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(A2') $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$ *or* $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$.

Suppose, in addition, that

(M1) $(\{T_n\}, T)$ *satisfies AKTT-condition;*

(M2) $F(T) = \cap_{n=1}^{\infty} F(T_n)$.

Then $\{x_n\}$ converges to $z \in \cap_{n=1}^{\infty} F(T_n)$ which is nearest u.

Remark 2.3. Theorem 2.1 with an even *weaker* assumption can be proved as a corollary of Theorem 2.2. In fact, Theorem 2.1 holds under the following assumption:

(A1) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(A2') $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$ or $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$;

(A3') $\lim_{n\to\infty} \lambda_n < 1$ and $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.

To see this, let us define each mapping $T_n : C \to C$ by

$$T_n x := \lambda_n x \oplus (1 - \lambda_n) T x \quad (x \in C).$$

In the presence of (A3'), let $\lambda = \lim_{n\to\infty} \lambda_n \in [0, 1)$ and define $\widetilde{T} : C \to C$ by

$$\widetilde{T} x := \lambda x \oplus (1 - \lambda) T x \quad (x \in C).$$

Note that

- $d(T_{n+1}x, T_n x) = |\lambda_{n+1} - \lambda_n| d(x, Tx)$ for all $x \in C$ and $n \in \mathbb{N}$;
- $F(T_n) = F(T) = F(\widetilde{T})$ for all $n \in \mathbb{N}$.

It follows then that

(M1) $(\{T_n\}, \widetilde{T})$ satisfies AKTT-condition;

(M2) $F(\widetilde{T}) = \cap_{n=1}^{\infty} F(T_n)$.

Note that (A3) $\implies$ (A3') and if $\lambda_n \equiv 1/2$, then this lies beyond the scope of Theorem 2.1.

Secondly, we prove the following result which supplements Theorem 2.1. Note that if $\lambda_{2n-1} \equiv 1/3$ and $\lambda_{2n} \equiv 2/3$, then the condition (A3') (and hence (A3)) is not satisfied, that is, our result cannot be obtained from Theorem 2.1.

Theorem 2.4. *The conclusion of Theorem 2.1 holds under the following assumptions:*

(A1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(A3") $0 < \liminf_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \lambda_n < 1$.

To prove Theorem 2.4, we need the following two lemmas. The first is proved by the author in [10] and the second by Maingé in [8]. It should be noted that the original proof of Lemma 2.5 makes use of Banach limits. However, following the same proof, we can use the superior limit as well.

Lemma 2.5. *Let C be a closed convex subset of a complete CAT(0) space X and $T : C \to C$ be a nonexpansive mapping. Let $u \in C$ be fixed. For each $t \in (0, 1)$, the mapping $S_t : C \to C$ defined by*

$$S_t x = tu \oplus (1-t)Tx \quad \text{for } x \in C$$

has a unique fixed point $x_t \in C$, that is,

$$x_t = S_t x_t = tu \oplus (1-t)Tx_t. \tag{2.1}$$

Then $F(T) \neq \varnothing$ if and only if $\{x_t\}$ given by the formula (2.1) remains bounded as $t \to 0$. In this case, the following statements hold:

(1) *$\{x_t\}$ converges to $z \in F(T)$ which is nearest u;*

(2) *$\limsup_{n\to\infty}(d^2(u, z) - d^2(u, x_n)) \le 0$ for all bounded sequences $\{x_n\}$ with $d(x_n, Tx_n) \to 0$.*

Lemma 2.6. *If $\{s_n\}$ is a sequence of real numbers and there exists a subsequence $\{s_{n_k}\}$ such that $s_{n_k} < s_{n_k+1}$ for all $k \in \mathbb{N}$, then there exists a nondecreasing sequence $\{m_k\}$ of natural numbers such that $\lim_{k\to\infty} m_k = \infty$ and the following two inequalities*

$$s_{m_k} \le s_{m_k+1} \quad \text{and} \quad s_k \le s_{m_k+1}$$

hold for all $k \in \mathbb{N}$.

Proof of Theorem 2.4. The proof given here is adapted from the the recent result of the author [11]. Let z be the nearest point of $F(T)$ to u. We consider the following inequalities:

$$\begin{aligned} d^2(x_{n+1}, z) \le{} & \alpha_n d^2(u, z) + (1-\alpha_n) d^2(\lambda_n x_n \oplus (1-\lambda_n)Tx_n, z) \\ & - \alpha_n(1-\alpha_n) d^2(\lambda_n x_n \oplus (1-\lambda_n)Tx_n, u) \\ \le{} & \alpha_n d^2(u, z) + (1-\alpha_n)[\lambda_n d^2(x_n, z) + (1-\lambda_n) d^2(Tx_n, z) \\ & - \lambda_n(1-\lambda_n) d^2(Tx_n, x_n)] \\ & - \alpha_n(1-\alpha_n) d^2(\lambda_n x_n \oplus (1-\lambda_n)Tx_n, u) \\ \le{} & \alpha_n d^2(u, z) + (1-\alpha_n) d^2(x_n, z) \\ & - \lambda_n(1-\lambda_n)(1-\alpha_n) d^2(Tx_n, x_n) \\ & - \alpha_n(1-\alpha_n) d^2(\lambda_n x_n \oplus (1-\lambda_n)Tx_n, u). \end{aligned}$$

First, we observe that

$$\begin{aligned} d^2(x_{n+1}, z) &\le \alpha_n d^2(u, z) + (1-\alpha_n) d^2(x_n, z) \\ &\le \max\{d^2(u, z), d^2(x_n, z)\}. \end{aligned}$$

By induction, we conclude that $\{x_n\}$ is a bounded sequence.

Next, we consider the following two cases:

Case 1: There exists an integer N such that $d^2(x_{n+1}, z) \le d^2(x_n, z)$ for all $n \ge N$. In particular, $\lim_{n\to\infty} d^2(x_n, z)$ exists and hence

$$\lim_{n\to\infty} (d^2(x_n, z) - d^2(x_{n+1}, z)) = 0.$$

Note that

$$\begin{aligned} &\lambda_n(1-\lambda_n)(1-\alpha_n) d^2(Tx_n, x_n) \\ &\le \alpha_n d^2(u, z) + (1-\alpha_n) d^2(x_n, z) - d^2(x_{n+1}, z). \end{aligned}$$

It follows from $\lim_{n\to\infty} \alpha_n = 0$ that

$$\lim_{n\to\infty} \lambda_n(1-\lambda_n) d^2(Tx_n, x_n) = 0.$$

In the light of (A3"), we have

$$\lim_{n\to\infty} d^2(Tx_n, x_n) = 0.$$

Consequently, by Lemma 2.5, we have

$$\limsup_{n\to\infty} (d^2(u, z) - d^2(u, x_n)) \le 0. \tag{2.2}$$

Note that

$$\begin{aligned} d^2(x_{n+1}, z) \le{}& (1-\alpha_n) d^2(x_n, z) \\ &+ \alpha_n (d^2(u, z) - (1-\alpha_n) d^2(\lambda_n x_n \oplus (1-\lambda_n) Tx_n, u)) \end{aligned}$$

and, by (2.2),

$$\begin{aligned} &\limsup_{n\to\infty} (d^2(u, z) - (1-\alpha_n) d^2(\lambda_n x_n \oplus (1-\lambda_n) Tx_n, u)) \\ &= \limsup_{n\to\infty} (d^2(u, z) - d^2(u, x_n)) \le 0. \end{aligned}$$

It follows then from Xu's lemma that $\lim_{n\to\infty} d^2(x_n, z) = 0$.

Case 2: There exists a subsequence $\{n_k\} \subset \{n\}$ such that $d^2(x_{n_k+1}, z) > d^2(x_{n_k}, z)$ for all $k \in \mathbb{N}$. Using Lemma 2.6, we can find a nondecreasing sequence $\{m_k\}$ of natural numbers such that $\lim_{k\to\infty} m_k = \infty$ and the following two inequalities hold:

$$d^2(x_{m_k}, z) \le d^2(x_{m_k+1}, z) \quad \text{for all } k \in \mathbb{N}, \tag{2.3}$$

$$d^2(x_k, z) \le d^2(x_{m_k+1}, z) \quad \text{for all } k \in \mathbb{N}. \tag{2.4}$$

It follows from (2.3) that

$$\begin{aligned} d^2(x_{m_k}, z) &\le d^2(x_{m_k+1}, z) \\ &\le \alpha_{m_k} d^2(u, z) + (1-\alpha_{m_k}) d^2(x_{m_k}, z) \\ &\quad - \lambda_{m_k}(1-\lambda_{m_k})(1-\alpha_{m_k}) d^2(Tx_{m_k}, x_{m_k}). \end{aligned}$$

As $\lim_{n\to\infty} \alpha_n = 0$ and (A3"), we can conclude that

$$\lim_{k\to\infty} d^2(Tx_{m_k}, x_{m_k}) = 0.$$

Consequently, by Lemma 2.5, we have

$$\limsup_{k\to\infty}(d^2(u,z) - d^2(u, x_{m_k})) \le 0. \tag{2.5}$$

Again, in the presence of (2.3), we also have

$$\begin{aligned} d^2(x_{m_k+1}, z) &\le \alpha_{m_k} d^2(u,z) + (1-\alpha_{m_k}) d^2(x_{m_k}, z) \\ &\quad - \alpha_{m_k}(1-\alpha_{m_k}) d^2(\lambda_{m_k} x_{m_k} \oplus (1-\lambda_{m_k}) Tx_{m_k}, u) \\ &\le \alpha_{m_k} d^2(u,z) + (1-\alpha_{m_k}) d^2(x_{m_k+1}, z) \\ &\quad - \alpha_{m_k}(1-\alpha_{m_k}) d^2(\lambda_{m_k} x_{m_k} \oplus (1-\lambda_{m_k}) Tx_{m_k}, u). \end{aligned}$$

In particular, since $\alpha_{m_k} > 0$,

$$d^2(x_{m_k+1}, z) \le d^2(u,z) - (1-\alpha_{m_k}) d^2(\lambda_{m_k} x_{m_k} \oplus (1-\lambda_{m_k}) Tx_{m_k}, u).$$

It follows then from (2.4) and (2.5) that

$$\begin{aligned} &\limsup_{k\to\infty} d^2(x_k, z) \\ &\le \limsup_{k\to\infty} d^2(x_{m_k+1}, z) \\ &\le \limsup_{k\to\infty}(d^2(u,z) - (1-\alpha_{m_k}) d^2(\lambda_{m_k} x_{m_k} \oplus (1-\lambda_{m_k}) Tx_{m_k}, u)) \\ &= \limsup_{k\to\infty}(d^2(u,z) - d^2(\lambda_{m_k} x_{m_k} \oplus (1-\lambda_{m_k}) Tx_{m_k}, u)) \\ &= \limsup_{k\to\infty}(d^2(u,z) - d^2(u, x_{m_k})) \le 0. \end{aligned}$$

This completes the proof. □

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