Fixed Point Theory, 15(2014), No. 2, 579-594
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

THE METHOD OF MONOTONE ITERATIONS FOR MIXED MONOTONE OPERATORS IN PARTIALLY ORDERED SETS AND ORDER-ATTRACTIVE FIXED POINTS

MIRCEA-DAN RUS

Department of Mathematics, Technical University of Cluj-Napoca
Str. Memorandumului nr. 28, 400114 Cluj-Napoca, Romania
E-mail: rus.mircea@math.utcluj.ro

Abstract. We use the method of monotone iterations to obtain fixed point and coupled fixed point results for mixed monotone operators in the setting of partially ordered sets, with no additional assumptions on the partial order and with no convergence structure. We define the concept of attractive fixed point with respect to the partial order and obtain several criteria for the existence, uniqueness and order-attractiveness of the fixed points, both in the presence and in the absence of a coupled lower-upper fixed point. As an application, we present a fixed point result for a class of mixed monotone operators in the setting of ordered linear spaces.
Key Words and Phrases: Partially ordered set, mixed monotone operator, monotone iterative method, fixed point, coupled fixed point, coupled lower-upper fixed point, order-attractive point, ordered linear space, cone.
2010 Mathematics Subject Classification: 47H10, 06A06, 06F20.

1. Introduction and preliminaries

A fundamental principle both in mathematics and computer science is iteration. Particularly, fixed point iteration and monotone iterative techniques are the core methods when solving a large class of abstract and applied mathematical problems and play an important part in many algorithms.

Monotone iterative methods (in connection with the method of lower and upper solutions) go back at least to E. Picard [17, 18, 19] in the 1890s, in the study of the Dirichlet problem for nonlinear second order (ordinary and partial) differential equations. Since then, these methods have been further developed in more abstract settings and have been used to solve a wide variety of nonlinear problems arising from various fields of science. In this direction, the class of operators to which these methods were applied has been enlarged to include operators with more general monotonicity-type properties, like the mixed-monotone property.

In this context, most of the abstract fixed point results for the class of mixed monotone operators that make use of monotone iterative techniques were formulated in the framework of ordered topological spaces (particularly, ordered Banach spaces) (e.g., [15, 16, 13, 5, 4, 3]), partially ordered metric spaces (e.g., [2, 11, 22, 6, 12, 20]) and partially ordered cone metric spaces (e.g., [21, 8, 24, 23, 14]). This seems perfectly justified by the need of some convergence structure that is compatible in some way with the partial order, such that one can consistently describe the result of the iterative process.

Following this long line of research, both pure and applied, the aim of this paper is to show that it is still possible to obtain constructive fixed point results by monotone iteration without assuming any convergence structure, in the setting of partially ordered sets and with no additional assumptions on the partial order. In particular, we are interested in obtaining criteria for the existence, uniqueness and attractiveness (in some predefined sense) of the fixed points, exclusively by means of explicit iterative techniques, both in the presence and in the absence of a coupled lower-upper fixed point. Also, we choose to study the class of mixed monotone operators since it contains both the classes of nondecreasing and nonincreasing operators, respectively, in one unified approach, while being large enough to describe a great number of nonlinear problems where usual monotonicity is not present.

Recall that if $(X, \leq)$ is a partially ordered set and $A : X \times X \to X$, then A is said to be *mixed monotone* (or is said to have *the mixed monotone property)* if A is nondecreasing in the first argument and nonincreasing in the second argument, i.e.,

$$x_1, x_2, y_1, y_2 \in X, \quad x_1 \leq x_2,\ y_1 \geq y_2 \Rightarrow A(x_1, y_1) \leq A(x_2, y_2).$$

A pair $(x, y) \in X \times X$ is called *a coupled fixed point of* A if

$$A(x, y) = x, \quad A(y, x) = y,$$

and it is called *a coupled lower-upper fixed point of* A if

$$x \leq y, \quad x \leq A(x, y), \quad y \geq A(y, x).$$

Also, $x \in X$ is called *a fixed point* of A if $A(x, x) = x$, i.e., (x, x) is a coupled fixed point of A. For more details, we refer to [5, 2].

Remark 1.1. While the term "*mixed monotone*" is due to Lakshmikantam and Guo [5], the concept of mixed monotone operator and the corresponding iterative method go back at least to Kurpel′ [9] in the study of two-sided operator inequalities and their applications to approximating the solutions of integral, differential, integro-differential and finite (algebraic and transcendental) equations. We point in this direction to the monograph of Kurpel′ and Šuvar [10]. Later on, Opoĭtsev [15, 16] established the first (to the best of our knowledge) fixed point and coupled fixed point results for this type of operators, in the framework of ordered Banach spaces. In the past three decades, the results of Opoĭtsev have been rediscovered in various forms and have been extended by many authors (we refer to [3] for an overview of the results published on this topic since the 1980s). Regrettably, none of them seems to have been aware of the results of Opoĭtsev, although English translations of his works have been available right after their initial publication in Russian.

In what follows, we will make use of the following notions and notations.

Let $(X, \leq)$ be a partially ordered set. If $x, y \in X$ are such that $x \leq y$, then $[x, y]$ denotes the set of all elements $z \in X$ such that $x \leq z \leq y$. Also, if (u_n) is a sequence in X, then $\sup u_n$ and $\inf u_n$ denote the supremum, i.e., the least upper bound, and the infimum, i.e., the greatest lower bound (when they exist), respectively, of the set $\{u_n : n \in \mathbb{N}\}$, where $\mathbb{N}$ represents the set of all nonnegative integers. We also write $\sup_{n \geq k} u_n$ and $\inf_{n \geq k} u_n$ (for any $k \in \mathbb{N}$) to denote the supremum and the infimum, respectively, of the set $\{u_n : n \geq k\}$.

In order to properly define the iterates of any bivariate operator, we need a composition rule that applies to this class of mappings, hence for any operators $A, B : X \times X \to X$ define (cf. [20]) *the symmetric composition* (or, *the s-composition* for short) of A and B by

$$B * A : X \times X \to X, \quad (B * A)(x, y) = B(A(x, y), A(y, x)) \quad (x, y \in X).$$

The s-composition is associative and the canonical projection

$$P_X : X \times X \to X, \quad P(x, y) = x \quad (x, y \in X)$$

is the identity element, hence one can define the functional powers (i.e., the iterates) of any operator $A : X \times X \to X$ with respect to the s-composition by

$$A^{n+1} = A * A^n = A^n * A \quad (n = 0, 1, ...), \quad A^0 = P_X.$$

When $(X, \leq)$ is a partially ordered set, the s-composition of mixed monotone operators has also the mixed monotone property, hence the iterates of a mixed monotone operator are also mixed monotone. For more details on this topic, we refer to [20].

2. Main results

From this point forward in this Section, it will be assumed that $(X, \leq)$ is a partially ordered set, $A : X \times X \to X$ is a mixed monotone operator and $x_0, y_0 \in X$ are such that $x_0 \leq y_0$. Also, define the sequences (x_n) and (y_n) recursively by

$$x_{n+1} = A(x_n, y_n), \quad y_{n+1} = A(y_n, x_n) \qquad (n \in \mathbb{N}), \tag{2.1}$$

or, equivalently, by

$$x_n = A^n(x_0, y_0), \quad y_n = A^n(y_0, x_0) \qquad (n \in \mathbb{N}).$$

This coupled iteration together with the results contained in the following lemma represent the core of the method of monotone iterations for mixed monotone operators. These ideas are not new and can be found spread throughout the entire literature that studies the (coupled) fixed points for mixed monotone operators, though they are usually considered in a less general setting and are sometimes *hidden* inside proofs. Note that the assumption of (x_0, y_0) being a coupled lower-upper fixed point of A is not essential for obtaining most of the (coupled) fixed point results in this paper, hence it will be considered as a separate assumption, which represents a new approach.

Lemma 2.1. *The following properties take place:*

(1) *For all* $n \in \mathbb{N}$, $x_n \leq y_n$ *and*

$$x, y \in [x_n, y_n] \Rightarrow A(x, y) \in [x_{n+1}, y_{n+1}]. \tag{2.2}$$

(2) *If* $(x, y) \in [x_0, y_0] \times [x_0, y_0]$ *is a coupled fixed point of* A, *then*

$$x, y \in \bigcap_{n \geq 0} [x_n, y_n].$$

(3) *If* (x_0, y_0) *is a coupled lower-upper fixed point of* A, *then* (x_n) *is nondecreasing,* (y_n) *is nonincreasing and* (x_n, y_n) *is a coupled lower-upper fixed point of* A, *for all* $n \in \mathbb{N}$.

Proof.

(1) The proof is by induction on n. Assume that $x_n \leq y_n$ for some $n \in \mathbb{N}$ and consider arbitrary $x, y \in [x_n, y_n]$. By the mixed monotonicity of A,

$$x_{n+1} = A(x_n, y_n) \leq A(x, y) \leq A(y_n, x_n) = y_{n+1},$$

hence $x_{n+1} \leq y_{n+1}$ and $A(x, y) \in [x_{n+1}, y_{n+1}]$, which proves (2.2). Since our assumption is true for $n = 0$, the proof of **1** is complete.

(2) Since $A(x, y) = x$, $A(y, x) = y$ and $x, y \in [x_0, y_0]$, it follows that $x, y \in [x_n, y_n]$ for all $n \in \mathbb{N}$ as a direct consequence of (2.2), by induction on n.

(3) Assume that $x_n \leq x_{n+1}$ and $y_n \geq y_{n+1}$ for some $n \in \mathbb{N}$. Note that this is equivalent to (x_n, y_n) being a coupled lower-upper fixed point of A, since $x_n \leq y_n$ by **1**. Then,

$$x_{n+1} = A(x_n, y_n) \leq A(x_{n+1}, y_{n+1}) = x_{n+2}$$
$$y_{n+1} = A(y_n, x_n) \geq A(y_{n+1}, x_{n+1}) = y_{n+2}.$$

Since our assumption is true for $n = 0$, it follows by induction that $x_n \leq x_{n+1}$ and $y_n \geq y_{n+1}$ (hence (x_n, y_n) is a coupled lower-upper fixed point of A) for all $n \in \mathbb{N}$.

□

The following result is a direct consequence of Lemma 2.1 and provides a negative answer on the existence of (coupled) fixed points.

Corollary 2.2. *If* $\bigcap_{n \geq 0} [x_n, y_n] = \emptyset$, *then* A *has no coupled fixed points in* $[x_0, y_0] \times [x_0, y_0]$ *(hence, no fixed points in* $[x_0, y_0]$*).*

2.1. **Order-attractive points for mixed monotone operators.** Before we formulate and prove the main fixed point theorems, we need to introduce and study some new notions.

Definition 2.3. A point $x^* \in X$ is said to be (x_0, y_0)-*weakly order-attractive for* A if $\bigcap_{n \geq 0} [x_n, y_n] = \{x^*\}$, and we denote this by $(x_0, y_0) \xrightarrow{A} x^*$. Alternatively, we may say that x^* *weakly order-attracts* (x_0, y_0) *through* A, or that (x_0, y_0) *is weakly order-attracted by* x^* *through* A.

Definition 2.4. A point $x^* \in X$ is said to be (x_0, y_0)-*order-attractive for* A if $\sup x_n = \inf y_n = x^*$, and we denote this by $(x_0, y_0) \overset{A}{\rightrightarrows} x^*$. Alternatively, we may say that x^* *order-attracts* (x_0, y_0) *through* A, or that (x_0, y_0) *is order-attracted by* x^* *through* A.

Definition 2.5. A point $x^* \in X$ is said to be *weakly order-attractive for* A *on* $[x_0, y_0]$ if $x^* \in [x_0, y_0]$ and $(u_0, v_0) \overset{A}{\to} x^*$ for all $u_0, v_0 \in [x_0, y_0]$ with $u_0 \leq x^* \leq v_0$, and we denote this by $[x_0, y_0] \overset{A}{\to} x^*$. Alternatively, we may say that x^* *weakly order-attracts* $[x_0, y_0]$ *through* A, or that $[x_0, y_0]$ *is weakly order-attracted by* x^* *through* A.

Definition 2.6. A point $x^* \in X$ is said to be *order-attractive for* A *on* $[x_0, y_0]$ if $x^* \in [x_0, y_0]$ and $(u_0, v_0) \overset{A}{\rightrightarrows} x^*$ for all $u_0, v_0 \in [x_0, y_0]$ with $u_0 \leq x^* \leq v_0$, and we denote this by $[x_0, y_0] \overset{A}{\rightrightarrows} x^*$. Alternatively, we may say that x^* *order-attracts* $[x_0, y_0]$ *through* A, or that $[x_0, y_0]$ *is order-attracted by* x^* *through* A.

Proposition 2.7. *Let $x^* \in X$. The following properties take place:*

(1) *If* $(x_0, y_0) \overset{A}{\to} x^*$, *then* $x^* \in [x_0, y_0]$.
(2) *If* $[x_0, y_0] \overset{A}{\to} x^*$, *then* $(x_0, y_0) \overset{A}{\to} x^*$ *and* $[u_0, v_0] \overset{A}{\to} x^*$ *for all* $u_0, v_0 \in [x_0, y_0]$ *with* $u_0 \leq x^* \leq v_0$.
(3) *If* $[x_0, y_0] \overset{A}{\rightrightarrows} x^*$, *then* $(x_0, y_0) \overset{A}{\rightrightarrows} x^*$ *and* $[u_0, v_0] \overset{A}{\rightrightarrows} x^*$ *for all* $u_0, v_0 \in [x_0, y_0]$ *with* $u_0 \leq x^* \leq v_0$.
(4) $(x_0, y_0) \overset{A}{\rightrightarrows} x^*$ *if and only if* $(x_0, y_0) \overset{A}{\to} x^*$ *and* $\sup x_n, \inf y_n$ *exist.*
(5) *If* $[x_0, y_0] \overset{A}{\rightrightarrows} x^*$, *then* $[x_0, y_0] \overset{A}{\to} x^*$.

Proof. **1**, **2** and **3** are direct consequences of the definitions. Also, **5** follows from **4** and the definitions, hence we only need to prove **4**.

If $(x_0, y_0) \overset{A}{\rightrightarrows} x^*$, then $\sup x_n, \inf y_n$ exist and $\sup x_n = \inf y_n = x^*$, hence $x_n \leq x^* \leq y_n$ for all $n \in \mathbb{N}$. Now, let $x \in X$ such that $x_n \leq x \leq y_n$ for all $n \in \mathbb{N}$. Then

$$x^* = \sup x_n \leq x \leq \inf y_n = x^*,$$

hence $x^* = x$. Concluding, $\bigcap_{n \geq 0} [x_n, y_n] = \{x^*\}$, i.e., $(x_0, y_0) \overset{A}{\to} x^*$.

Conversely, if $(x_0, y_0) \overset{A}{\to} x^*$, then $x_n \leq x^* \leq y_n$ for all $n \in \mathbb{N}$ and since $\sup x_n, \inf y_n$ exist, it follows that $x_n \leq \sup x_n \leq x^* \leq \inf y_n \leq y_n$ for all $n \in \mathbb{N}$, hence

$$\sup x_n, \inf y_n \in \bigcap_{n \geq 0} [x_n, y_n] = \{x^*\},$$

which proves that $(x_0, y_0) \overset{A}{\rightrightarrows} x^*$. □

In the following result we establish the properties of (weakly) ordered-attractive fixed points.

Theorem 2.8. *Let $x^* \in X$. The following equivalences take place:*

(1) $[x_0, y_0] \overset{A}{\to} x^*$ *if and only if* $(x_0, y_0) \overset{A}{\to} x^*$ *and* x^* *is a fixed point of* A.

(2) $[x_0, y_0] \overset{A}{\rightrightarrows} x^*$ *if and only if* $(x_0, y_0) \overset{A}{\rightrightarrows} x^*$ *and* x^* *is a fixed point of* A.

Moreover, in any of the above situations, (x^*, x^*) *is the unique coupled fixed point of* A *in* $[x_0, y_0] \times [x_0, y_0]$ *(hence,* x^* *is the unique fixed point of* A *in* $[x_0, y_0]$*).*

Proof. First, we prove the direct implications.

Assume that $[x_0, y_0] \overset{A}{\to} x^*$. Then $x^* \in [x_0, y_0]$, hence $(x^*, x^*) \overset{A}{\to} x^*$, which ensures that x^* is a fixed point of A. Also, $(x_0, y_0) \overset{A}{\to} x^*$ by Proposition 2.7 and the direct implication in **1** is proved.

Similarly, if $[x_0, y_0] \overset{A}{\rightrightarrows} x^*$, then $(x_0, y_0) \overset{A}{\rightrightarrows} x^*$ and $[x_0, y_0] \overset{A}{\to} x^*$ by Proposition 2.7, hence x^* is a fixed point of A using the direct implication in **1**, and the direct implication in **2** is also proved.

Now we prove the converse implications.

Assume that x^* is a fixed point of A and $(x_0, y_0) \overset{A}{\to} x^*$. We prove that $[x_0, y_0] \overset{A}{\to} x^*$.

Let $u_0, v_0 \in [x_0, y_0]$ such that $u_0 \le x^* \le v_0$ and define the sequences $(u_n), (v_n)$ by

$$u_{n+1} = A(u_n, v_n),\ v_{n+1} = A(v_n, u_n) \quad (n \in \mathbb{N})$$

or, equivalently, by

$$u_n = A^n(u_0, v_0),\ v_n = A^n(v_0, u_0) \quad (n \in \mathbb{N}).$$

Since x^* is a fixed point of A, it follows that x^* is a fixed point of A^n for all $n \in \mathbb{N}$. Also, A^n is mixed monotone for all $n \in \mathbb{N}$, and since

$$x_0 \le u_0 \le x^* \le v_0 \le y_0,$$

it follows that

$$x_n \le u_n \le x^* \le v_n \le y_n \quad \text{for all } n \in \mathbb{N}, \tag{2.3}$$

which ensures that

$$\{x^*\} \subseteq \bigcap_{n\ge 0} [u_n, v_n] \subseteq \bigcap_{n \ge 0} [x_n, y_n] = \{x^*\},$$

hence $(u_0, v_0) \overset{A}{\to} x^*$ and the converse implication in **1** is proved.

Now, assume that x^* is a fixed point of A and $(x_0, y_0) \overset{A}{\rightrightarrows} x^*$. We prove that $[x_0, y_0] \overset{A}{\rightrightarrows} x^*$.

Let $u_0, v_0 \in [x_0, y_0]$ such that $u_0 \le x^* \le v_0$ and let $(u_n), (v_n)$ as previously defined. By using the same argument as before, we obtain (2.3), and since $\sup x_n = \inf y_n = x^*$, it follows that $\sup u_n = \inf v_n = x^*$, i.e., $(u_0, v_0) \overset{A}{\rightrightarrows} x^*$, hence the converse implication in **2** is proved.

Finally, we only need to prove that if x^* is a fixed point of A and $(x_0, y_0) \overset{A}{\to} x^*$ (or $(x_0, y_0) \overset{A}{\rightrightarrows} x^*$), then (x^*, x^*) is the unique coupled fixed point of A in $[x_0, y_0] \times [x_0, y_0]$.

Indeed, if $(x,y) \in [x_0,y_0] \times [x_0,y_0]$ is a coupled fixed point of A, then, by Lemma 2.1, Definition 2.3 (and Proposition 2.7(4)), we have that

$$x, y \in \bigcap_{n\geq 0} [x_n, y_n] = \{x^*\},$$

hence $x = y = x^*$. Clearly, (x^*, x^*) is a coupled fixed point of A in $[x_0,y_0] \times [x_0,y_0]$, and the proof is now complete. □

Remark 2.9. In general, if $(x_0,y_0) \overset{A}{\rightrightarrows} x^*$ (or $(x_0,y_0) \overset{A}{\to} x^*$), then x^* is not necessarily a fixed point of A (though, under additional assumptions, this may be sufficient – see Theorem 2.13). The following elementary example proves this claim by means of a mixed monotone mapping with no (coupled) fixed points that has a (x_0,y_0)-order-attractive point.

Example 2.10. Let $A : \mathbb{R}^2 \to \mathbb{R}$ by defined by $A(x,y) = x + \dfrac{1-\{x\}}{2}$, where $\{x\}$ denotes the fractional part of the real number x. Then A is mixed monotone and $(0,1) \overset{A}{\rightrightarrows} 1$, yet A has no (coupled) fixed points.

First, we prove that A is mixed monotone, which, in this case, is equivalent to A being nondecreasing (with respect to x). Let $x_1, x_2 \in \mathbb{R}$ such that $x_1 \leq x_2$ and let $n = x_2 - \{x_2\}$ be the integer part of x_2. If $x_1 \in [n, n+1)$, then $x_1 = n + \{x_1\}$, hence $\{x_2\} - \{x_1\} = x_2 - x_1$ and

$$A(x_2,y) - A(x_1,y) = x_2 - x_1 - \frac{\{x_2\} - \{x_1\}}{2} = \frac{x_2 - x_1}{2} \geq 0.$$

Else, $x_1 < n \leq x_2 < n+1$, hence

$$A(x_2,y) \geq A(n,y)$$

(from the previous case, by letting $x_1 := n$) and

$$A(n,y) - A(x_1,y) = n - x_1 + \frac{\{x_1\}}{2} > 0,$$

which proves that $A(x_2,y) \geq A(x_1,y)$.

Now, choose $x_0 = 0$ and $y_0 = 1$. It is a simple exercise to show (e.g., by induction) that the corresponding sequences $(x_n), (y_n)$ defined by (2.1) are

$$x_n = 1 - \frac{1}{2^n}, \quad y_n = 2 - \frac{1}{2^n} \qquad (n \in \mathbb{N}),$$

hence

$$\sup x_n = \inf y_n = 1,$$

proving that $x^* = 1$ is (x_0,y_0)-order-attractive for A.

Finally, it can be easily noticed that A has no (coupled) fixed points, since $A(x,y) = x$ if and only if $\{x\} = 1$, which is impossible.

2.2. **Fixed point theorems.** We conclude with the main results. In essence, we prove in each of the following results that for a point $x^* \in X$ to be a weakly ordered-attractive fixed point of A, it is sufficient (under additional assumptions) that x^* is (x_k, y_k)-weakly ordered-attractive for some $k \in \mathbb{N}$. In particular, if there exists $k \in \mathbb{N}$ such that $\sup_{n\geq k} x_n = \inf_{n\geq k} y_n = x^*$, then x^* is an ordered-attractive fixed point of A. In this way, we establish several simple criteria for the existence, uniqueness and (weakly) order-attractiveness of the fixed points of mixed monotone operators.

Theorem 2.11. *Let $k \geq 1$ such that $\bigcap_{n=0}^{k-1} [x_n, y_n]$ is non-empty, and $x^* \in \bigcap_{n=0}^{k-1} [x_n, y_n]$.*

If $(x_k, y_k) \overset{A}{\to} x^*$, then (x^*, x^*) is the unique coupled fixed point of A in $[x_n, y_n] \times [x_n, y_n]$, x^* is the unique fixed point of A in $[x_n, y_n]$ and $[x_n, y_n] \overset{A}{\to} x^*$ for all $n \in \{0, 1, \ldots, k\}$.

Additionally, if $\sup_{n\geq k} x_n$ and $\inf_{n\geq k} y_n$ exist, then $[x_n, y_n] \overset{A}{\rightrightarrows} x^*$ for all $n \in \{0, 1, \ldots, k\}$.

Proof. For each $n \in \mathbb{N}$, let $X_n = \bigcap_{m\geq n} [x_m, y_m]$. It is clear that $(x_n, y_n) \overset{A}{\to} x^*$ if and only if $X_n = \{x^*\}$, hence the hypothesis ensure that $X_k = \{x^*\}$ and

$$X_0 = \left(\bigcap_{n=0}^{k-1} [x_n, y_n]\right) \cap X_k = \{x^*\}.$$

Since, obviously, $X_0 \subseteq X_1 \subseteq \ldots \subseteq X_k \subseteq X_{k+1} \subseteq \ldots$, we conclude that

$$X_0 = X_1 = \ldots = X_k = \{x^*\},$$

hence $(x_n, y_n) \overset{A}{\to} x^*$ for all $n \in \{0, 1, \ldots, k\}$.

Since $x^* \in [x_n, y_n]$ for all $n \in \mathbb{N}$, it follows by (2.2) that $A(x^*, x^*) \in [x_{n+1}, y_{n+1}]$ for all $n \in \mathbb{N}$, hence $A(x^*, x^*) \in X_1 = \{x^*\}$, proving that x^* is a fixed point of A.

The conclusion now follows by applying Theorem 2.8(1) with (x_0, y_0) replaced by (x_n, y_n) $(n \in \{0, 1, \ldots, k\})$.

Additionally, assume that $\sup_{n\geq k} x_n$ and $\inf_{n\geq k} y_n$ exist, hence

$$\sup_{n\geq k} x_n = \inf_{n\geq k} x_n = x^*$$

by Proposition 2.7(4), with (x_0, y_0) replaced by (x_k, y_k). Since $x_m \leq x^* \leq y_m$ for all $m \in \mathbb{N}$ (by $X_0 = \{x^*\}$), it follows that $x^* = \sup_{m\geq n} x_m = \inf_{m\geq n} y_m$ for all $n \in \{0, 1, \ldots, k\}$, i.e., $(x_n, y_n) \overset{A}{\rightrightarrows} x^*$ for all $n \in \{0, 1, \ldots, k\}$ and the proof is complete by further applying Theorem 2.8(2) with (x_0, y_0) replaced by (x_n, y_n) $(n \in \{0, 1, \ldots, k\})$. □

Corollary 2.12. *Let $x^* \in [x_0, y_0]$. If $(x_1, y_1) \overset{A}{\to} x^*$, then (x^*, x^*) is the unique coupled fixed point of A in $[x_0, y_0] \times [x_0, y_0] \cup [x_1, y_1] \times [x_1, y_1]$, x^* is the unique fixed point of A in $[x_0, y_0] \cup [x_1, y_1]$ and $[x_0, y_0] \overset{A}{\to} x^*$, $[x_1, y_1] \overset{A}{\to} x^*$.*

Additionally, if $\sup_{n\geq 1} x_n$ *and* $\inf_{n\geq 1} y_n$ *exist, then* $[x_0, y_0] \overset{A}{\rightrightarrows} x^*$ *and* $[x_1, y_1] \overset{A}{\rightrightarrows} x^*$.

Proof. This follows by Theorem 2.8 with $k = 1$. □

By assuming that (x_0, y_0) is a coupled lower-upper fixed point of A, we obtain the following results.

Theorem 2.13. *Let* $x^* \in X$ *and assume that* (x_0, y_0) *is a coupled lower-upper fixed point of* A*. If* $(x_0, y_0) \overset{A}{\to} x^*$*, then* (x^*, x^*) *is the unique coupled fixed point of* A *in* $[x_0, y_0] \times [x_0, y_0]$*,* x^* *is the unique fixed point of* A *in* $[x_0, y_0]$ *and* $[x_0, y_0] \overset{A}{\to} x^*$.

Additionally, if $\sup x_n$ *and* $\inf y_n$ *exist, then* $[x_0, y_0] \overset{A}{\rightrightarrows} x^*$.

Proof. We use the same notations as in the proof of Theorem 2.11. Since (x_0, y_0) is a coupled lower-upper fixed point of A, it follows by Lemma 2.1 that

$$x_0 \leq x_1 \leq \ldots \leq x_n \leq x_{n+1} \leq \ldots \leq y_{n+1} \leq y_n \leq \ldots \leq y_1 \leq y_0,$$

hence $X_0 = X_1$. Since $(x_0, y_0) \overset{A}{\to} x^*$, we conclude that $X_0 = X_1 = \{x\}$, hence $(x_1, y_1) \overset{A}{\to} x^*$ and $x^* \in [x_0, y_0]$. The conclusion now follows by Corollary 2.12.

Additionally, if $\sup x_n$ and $\inf y_n$ exist, then $(x_0, y_0) \overset{A}{\rightrightarrows} x^*$ by Proposition 2.7(4), hence $[x_0, y_0] \overset{A}{\rightrightarrows} x^*$ by Theorem 2.8(2), which concludes the proof. □

Remark 2.14. In the conditions of Theorem 2.13, $x^* \in [x_n, y_n] \subseteq [x_0, y_0]$ for all $n \in \mathbb{N}$, hence the conclusion of the theorem already contains that (x^*, x^*) is the unique coupled fixed point of A in $[x_n, y_n] \times [x_n, y_n]$ and $[x_n, y_n] \overset{A}{\to} x^*$ for all $n \geq 1$, without explicitly stating it.

In many cases, it is possible that the starting pair of the iterative process is not a coupled lower-upper fixed point, but we arrive to such a pair after several iterations. This situation is studied next.

Theorem 2.15. *Let* $x^* \in [x_0, y_0]$ *and assume there exists* $k \geq 1$ *such that* (x_k, y_k) *is a coupled lower-upper fixed point of* A*. If* $(x_k, y_k) \overset{A}{\to} x^*$*, then* (x^*, x^*) *is the unique coupled fixed point of* A *in* $[x_n, y_n] \times [x_n, y_n]$*,* x^* *is the unique fixed point of* A *in* $[x_n, y_n]$ *and* $[x_n, y_n] \overset{A}{\to} x^*$ *for all* $n \in \{0, 1, \ldots, k\}$.

Additionally, if $\sup_{n\geq k} x_n$ *and* $\inf_{n\geq k} y_n$ *exist, then* $[x_n, y_n] \overset{A}{\rightrightarrows} x^*$ *for all* $n \in \{0, 1, \ldots, k\}$.

Proof. By applying Theorem 2.13, with (x_0, y_0) replaced by (x_k, y_k), it follows that x^* is a fixed point of A and, since $x^* \in [x_0, y_0]$, it follows by Lemma 2.1(2) that $x^* \in \bigcap_{n\geq 0} [x_n, y_n]$, hence $x^* \in \bigcap_{n=0}^{k-1} [x_n, y_n]$. The conclusion now follows by Theorem 2.11. □

Remark 2.16. In the conditions of Theorem 2.15, $x^* \in [x_n, y_n] \subseteq [x_k, y_k]$ for all $n \geq k$, hence the conclusion of the theorem already contains that (x^*, x^*) is the unique coupled fixed point of A in $[x_n, y_n] \times [x_n, y_n]$ and $[x_n, y_n] \overset{A}{\to} x^*$ for all $n \geq k+1$.

3. Application

As an application, we present a fixed point result for a class of mixed monotone operators in the setting of ordered linear spaces. First, recall some notions and results.

3.1. **Some preliminaries on ordered linear spaces.** Let (X, K) be an ordered linear space over $\mathbb{R}$, i.e., X is a real linear space and $K \subseteq X$ a cone in X (i.e., a convex set such that $\lambda K \subseteq K$ for all $\lambda \geq 0$ and $K \cap (-K) = \{\theta\}$, where θ denotes the zero element in X). Then the relation on X defined by $x \leq y \Leftrightarrow y - x \in K$ is a linear order on X, i.e., an order that satisfies:

(i) $x, y, z \in X : x \leq y \Rightarrow x + z \leq y + z$.
(ii) $x, y \in X,\ \lambda \geq 0 : x \leq y \Rightarrow \lambda x \leq \lambda y$.

It is said that K is *Archimedean* if $x \leq \theta$ whenever there exists $y \in X$ such that $nx \leq y$ for all $n \in \mathbb{N}$. It is well known that if K is Archimedean, then for every $x, y \in X$, $\lambda \in \mathbb{R}$ and every nonincreasing sequence (λ_n) convergent to λ:

$$x \leq \lambda_n y \text{ for all } n \in \mathbb{N} \Rightarrow x \leq \lambda y.$$

Two elements x, y in K are said to be *linked* (cf. [25]) if there exists $\lambda \in (0,1)$ such that $\lambda x \leq y$ and $\lambda y \leq x$. This is an equivalence which splits K into disjoint components (called *parts*).

For further details on these topics we refer to, e.g., [7].

In order to state and prove the main result in this Section, we need to consider some new notions.

Definition 3.1. A sequence (x_n) in X is said to be:

(i) *upper self-bounded* if for every $\mu > 1$ exists $k \in \mathbb{N}$ such that $x_n \leq \mu x_k$ for all $n \geq k$;
(ii) *lower self-bounded* if for every $\lambda \in (0,1)$ exists $k \in \mathbb{N}$ such that $\lambda x_k \leq x_n$ for all $n \geq k$.

Example 3.2. Every nondecreasing sequence in K is lower self-bounded. Similarly, every nonincreasing sequence in K is upper self-bounded.

Definition 3.3. K is said to be *self-complete* if every nondecreasing sequence in K that is upper self-bounded has supremum.

Remark 3.4. It is not hard to prove the following equivalence: K is self-complete if and only if every nonincreasing sequence in K that is lower self-bounded has infimum. Since this result is not essential in our arguments, we omit its proof.

Example 3.5. Let $n \in \mathbb{N}$, $n \geq 1$ and

$$\mathbb{R}^n_+ = \{x = (x^1, x^2, \ldots, x^n) \in \mathbb{R}^n : x^i \geq 0 \text{ for all } i \in \{1, 2, \ldots, n\}\}$$

be the nonnegative cone in $\mathbb{R}^n$. Then $\mathbb{R}^n_+$ is Archimedean and self-complete.

Indeed, $\mathbb{R}^n_+$ is Archimedean since for every $x, y \in \mathbb{R}^n$:

$$\begin{aligned} nx \leq y \text{ for all } n \in \mathbb{N} \Leftrightarrow x^i \leq \frac{y^i}{n} \text{ for all } n \in \mathbb{N},\, i \in \{1, 2, \ldots, n\} \\ \Rightarrow x^i \leq 0 \text{ for all } i \in \{1, 2, \ldots, n\} \Leftrightarrow x \leq \theta. \end{aligned}$$

Also, if (x_n) is a nondecreasing sequence in $\mathbb{R}^n_+$ that is upper self-bounded, then for every $i \in \{1, 2, \ldots, n\}$ the sequence (x^i_n) is nondecreasing and bounded (in $\mathbb{R}$), hence has supremum, which concludes the argument.

Example 3.6. Let Q be a compact Hausdorff topological space and $C(Q)$ be the linear space of all real valued continuous functions on Q, while

$$K = \{x \in C(Q) : x(t) \geq 0 \text{ for all } t \in Q\}$$

is the cone of all nonnegative functions in $C(Q)$. Then K is Archimedean and self-complete.

Indeed, K is Archimedean since for every $x, y \in C(Q)$:

$$nx \leq y \text{ for all } n \in \mathbb{N} \Leftrightarrow x(t) \leq \frac{y(t)}{n} \text{ for all } n \in \mathbb{N},\, t \in Q \Rightarrow x(t) \leq 0 \text{ for all } t \in Q.$$

Next, let (x_n) be a nondecreasing sequence in K that is upper self-bounded and let $k \in \mathbb{N}$ be such that $x_n \leq 2x_k$ for all $n \geq k$, hence

$$0 \leq x_n(t) \leq 2x_k(t) \leq 2M \quad \text{for all } t \in Q \text{ and } n \in \mathbb{N}, \tag{3.1}$$

where $M = \sup_t x_k(t)$. Now, let $x : Q \to \mathbb{R}$ be given by

$$x(t) = \sup_n x_n(t) \quad (t \in Q).$$

Clearly, x is correctly defined, i.e., $x(t)$ is finite for all $t \in Q$, by (3.1). In order to show that $x = \sup x_n$ (in the ordered linear space $(C(Q), K)$), we only need to prove that x is continuous.

Let $\varepsilon > 0$ and let n_ε be such that

$$x_n(t) \leq \left(1 + \frac{\varepsilon}{2M}\right) x_{n_\varepsilon}(t) \quad \text{for all } t \in Q \text{ and } n \in \mathbb{N},$$

hence

$$x_n(t) \leq x(t) \leq \left(1 + \frac{\varepsilon}{2M}\right) x_n(t) \quad \text{for all } t \in Q \text{ and } n \geq n_\varepsilon,$$

and by using (3.1), we finally obtain that

$$0 \leq x(t) - x_n(t) \leq \frac{\varepsilon}{2M} x_n(t) \leq \varepsilon \quad \text{for all } t \in Q \text{ and } n \geq n_\varepsilon,$$

which proves that (x_n) uniformly converges to x, hence x is continuous.

3.2. **A fixed point theorem.** We conclude with a result which establishes the existence, uniqueness and order-attractiveness of fixed points for a class of mixed monotone operators, in the context of ordered linear spaces endowed with an Archimedean and self-complete cone. Our result complements and generalizes [1, Cor. 3.2], [4, Th. 1], [16, Th. 2.9], [26, Th. 2.1], [27, Th. 1].

Theorem 3.7. *Let (X, K) be an ordered linear space over $\mathbb{R}$ such that K is Archimedean and self-complete. Let P be a part of K and $A : P \times P \to K$ a mixed monotone operator.*

Assume there exists $\varphi : (0,1) \to (0,1]$ such that $\varphi(\lambda) > \lambda$ for all $\lambda \in (0,1)$ and

$$A(\lambda x, y) \geq \varphi(\lambda) A(x, \lambda y) \quad \text{for all } \lambda \in (0,1) \text{ and } x, y \in P \text{ linearly dependent.} \tag{3.2}$$

If there exists $u \in P$ such that $A(u,u) \in P$, then the following conclusions hold:

1. *for every $x, y \in P$, there exists $(x_0, y_0) \in P \times P$ a coupled lower-upper fixed point of A such that $x, y \in [x_0, y_0]$;*
2. *$A(P \times P) \subseteq P$;*
3. *there exists $x^* \in P$ such that (x^*, x^*) is the unique coupled fixed point of A in $P \times P$, x^* is the unique fixed point of A in P and $[x_0, y_0] \overset{A}{\rightrightarrows} x^*$ for every $(x_0, y_0) \in P \times P$ coupled lower-upper fixed point of A.*

Proof. First, we prove that A has at most a fixed point in P. For that, assume $x^*, y^* \in P$ be two distinct fixed points of A. Let

$$T = \left\{ \lambda > 0 : \lambda x^* \leq y^* \leq \lambda^{-1} x^* \right\}$$

and $\lambda_* = \sup T$. Obviously, T is nonempty since x^*, y^* are in the same part of K, and $\lambda_* \in T \subseteq (0,1)$ since K is Archimedean and $x^* \neq y^*$. Then, by (3.2) and the mixed monotonicity of A,

$$\varphi(\lambda_*) x^* = \varphi(\lambda_*) A(x^*, x^*) \leq A(\lambda_* x^*, \lambda_*^{-1} x^*) \leq A(y^*, y^*) = y^*, \tag{3.3}$$

hence $\varphi(\lambda_*) x^* \leq y^*$. Due to the symmetry, one also has $\varphi(\lambda_*) y^* \leq x^*$ which shows that $\varphi(\lambda_*) \in T$, hence $\varphi(\lambda_*) \leq \lambda_*$, which contradicts the hypothesis on φ. Concluding, A has at most a fixed point in P.

By following the same argument as before, we have that if (x^*, y^*) is a coupled fixed point of A in $P \times P$, then $x^* = y^*$; the only difference from the previous argument is that (3.3) is replaced by:

$$\varphi(\lambda_*) x^* = \varphi(\lambda_*) A(x^*, y^*) \leq A(\lambda_* x^*, \lambda_*^{-1} y^*) \leq A(y^*, x^*) = y^*.$$

The next step in our proof is to claim that φ can be assumed to satisfy

$$\varphi(\lambda)\varphi(\mu) \leq \varphi(\lambda\mu) \quad \text{for all } \lambda, \mu \in (0,1) \tag{3.4}$$

without any loss of generality. In order to prove this, define the set

$$\Phi(\lambda) = \{\eta \in (0,1] : A(\lambda x, y) \geq \eta A(x, \lambda y) \text{ for all } x, y \in P \text{ linearly dependent}\}$$

for every $\lambda \in (0,1)$ and consider the function $\phi : (0,1) \to (0,1]$ given by

$$\phi(\lambda) = \sup \Phi(\lambda) \quad (\lambda \in (0,1)).$$

Since $\varphi(\lambda) \in \Phi(\lambda)$ for all $\lambda \in (0,1)$, then ϕ is correctly defined and $\phi(\lambda) \geq \varphi(\lambda) > \lambda$ for all $\lambda \in (0,1)$. Also, $\phi(\lambda) \in \Phi(\lambda)$ since K is Archimedean, hence

$$A(\lambda x, y) \geq \phi(\lambda) A(x, \lambda y) \quad \text{for all } \lambda \in (0,1) \text{ and } x, y \in P \text{ linearly dependent.}$$

Moreover, for all $\lambda, \mu \in (0,1)$ and $x, y \in P$ linearly dependent,

$$A(\lambda\mu x, y) \geq \phi(\lambda) A(\mu x, \lambda y) \geq \phi(\lambda)\phi(\mu) A(x, \lambda\mu y)$$

which shows that $\phi(\lambda)\phi(\mu) \in \Phi(\lambda\mu)$, hence $\phi(\lambda)\phi(\mu) \leq \phi(\lambda\mu)$. It is clear now that by replacing φ with ϕ, we obtain the desired property (3.4).

Next, since u and $A(u,u)$ are in the same part of K, there exists $\lambda_0 \in (0,1)$ such that

$$\lambda_0 u \leq A(u,u) \leq \lambda_0^{-1} u. \tag{3.5}$$

Also, $\lim_{n\to\infty} \left(\frac{\varphi(\lambda_0)}{\lambda_0}\right)^n = \infty$ since $\frac{\varphi(\lambda_0)}{\lambda_0} > 1$, hence there exists $k_0 \in \mathbb{N}$ such that $\left(\frac{\varphi(\lambda_0)}{\lambda_0}\right)^n \geq \lambda_0^{-1}$ for all $n \geq k_0$, i.e.,

$$\lambda_0^n \leq (\varphi(\lambda_0))^n \lambda_0 \quad \text{for all } n \geq k_0. \tag{3.6}$$

Now, consider arbitrary $x, y \in P$. Since x, y are in the same part of the cone with u, there exists $n_0 \geq k_0$ large enough such that $\lambda_0^{n_0} u \leq x \leq \lambda_0^{-n_0} u$ and $\lambda_0^{n_0} u \leq y \leq \lambda_0^{-n_0} u$. Let $x_0 = \lambda_0^{n_0} u$ and $y_0 = \lambda_0^{-n_0} u$. Clearly, $x_0, y_0 \in P$, $x_0 \leq y_0$ and $x, y \in [x_0, y_0]$. By successively applying (3.2)–(3.6) several times and using the mixed monotonicity of A, we have that

$$\begin{aligned} x_0 &= \lambda_0^{n_0} u \leq (\varphi(\lambda_0))^{n_0} \lambda_0 u \leq \varphi(\lambda_0^{n_0}) A(u,u) \leq A(\lambda_0^{n_0} u, \lambda_0^{-n_0} u) = A(x_0, y_0) \\ &\leq A(x,y) \leq A(y_0, x_0) = A(\lambda_0^{-n_0} u, \lambda_0^{n_0} u) \leq (\varphi(\lambda_0^{n_0}))^{-1} A(u,u) \\ &\leq \left(\lambda_0 (\varphi(\lambda_0))^{n_0}\right)^{-1} u \leq \lambda_0^{-n_0} u = y_0, \end{aligned}$$

which shows that (x_0, y_0) is a coupled lower-upper fixed point of A and

$$A(x,y) \in [x_0, y_0] \subseteq P,$$

hence $A(P \times P) \subseteq P$.

Now, let (x_0, y_0) be any coupled lower-upper fixed point of A. In order to conclude the proof, it is enough to show that there exists $x^* \in [x_0, y_0]$ such that $(x_0, y_0) \overset{A}{\rightrightarrows} x^*$, and the conclusion will follow from Theorem 2.13. In order to achieve this, let $(x_n), (y_n)$ be defined as in (2.1), hence by Lemma 2.1,

$$x_0 \leq x_1 \leq \ldots \leq x_n \leq \ldots \leq y_m \leq \ldots \leq y_1 \leq y_0 \tag{3.7}$$

We break the proof in several steps.

First, it is clear that if $x_k = y_k$ for some k, then $\sup x_n = \inf y_n = x_k$ and the proof is complete, hence one can assume that $x_n \neq y_n$ for all $n \in \mathbb{N}$.

Next, let the sequence (λ_n) be defined by $\lambda_{n+1} = \varphi(\lambda_n)$ for all $n \in \mathbb{N}$, where $\lambda_0 \in (0,1)$ is such that $x_0 \geq \lambda_0 y_0$ (λ_0 exists, since x_0, y_0 are in the same part of K). We show by induction that, for all $n \in \mathbb{N}$,

$$\lambda_n \text{ is correctly defined}, \quad \lambda_n \in (0,1), \quad x_n \geq \lambda_n y_n. \tag{3.8}$$

Clearly, these are satisfied for $n = 0$. Now, assume these properties are true for n. Then $\lambda_{n+1} = \varphi(\lambda_n) \in (0,1]$ is correctly defined (since $\lambda_n \in (0,1)$) and, by (3.2),

$$x_{n+1} = A(x_n, y_n) \geq A(\lambda_n y_n, y_n) \geq \varphi(\lambda_n) A(y_n, \lambda_n y_n) \geq \varphi(\lambda_n) A(y_n, x_n) = \lambda_{n+1} y_{n+1}$$

hence $x_{n+1} \geq \lambda_{n+1} y_{n+1}$. Since $x_{n+1} \neq y_{n+1}$, it also follows from here that $\lambda_{n+1} \neq 1$, hence $\lambda_{n+1} \in (0,1)$, which concludes the inductive proof.

Note also that $\lambda_n < \varphi(\lambda_n) = \lambda_{n+1}$ for all $n \in \mathbb{N}$. Following from here, we conclude that the sequence (λ_n) is increasing, hence convergent to some $\lambda_* \in (0,1]$; we prove that $\lambda_* = 1$. Assume that $\lambda_* \neq 1$. Clearly, $\lambda_n < \lambda_*$ for all $n \in \mathbb{N}$. Then, by (3.4),

$$\lambda_{n+1} = \varphi(\lambda_n) = \varphi\left(\lambda_* \cdot \frac{\lambda_n}{\lambda_*}\right) \geq \varphi(\lambda_*)\varphi\left(\frac{\lambda_n}{\lambda_*}\right) > \varphi(\lambda_*)\frac{\lambda_n}{\lambda_*} \quad \text{for all } n \in \mathbb{N}$$

and by taking $n \to \infty$, we obtain that $\lambda_* \geq \varphi(\lambda_*)$, which is a contradiction. Concluding,

$$\lim_{n\to\infty} \lambda_n = 1. \tag{3.9}$$

We claim now that (x_n) is upper self-bounded. Indeed, let $\mu > 1$ and, by (3.9), let $k \in \mathbb{N}$ such that $\lambda_k \geq \mu^{-1}$. Then, by (3.7) and (3.8),

$$x_n \leq \mu\lambda_k x_n \leq \mu\lambda_k y_k \leq \mu x_k \quad \text{for all } n \in \mathbb{N},$$

which proves our claim.

Next, we use that K is self-complete, hence there exists $x^* = \sup x_n$.

Finally, we show that $x^* = \inf y_n$. Indeed, $x^* \leq y_n$ for all $n \in \mathbb{N}$ (by (3.7)). Also, if $x \in X$ such that $x \leq y_n$ for all $n \in \mathbb{N}$, then

$$x \leq y_n \leq \frac{x_n}{\lambda_n} \leq \frac{x^*}{\lambda_n} \quad \text{for all } n \in \mathbb{N}$$

hence $x \leq x^*$, by (3.9) and using that K is Archimedean. Concluding, $(x_0, y_0) \overset{A}{\rightrightarrows} x^*$ and the proof is now complete. □

Acknowledgement. This work was supported by the Sectoral Operational Programme Human Resources Development 2007-2013 of the Romanian Ministry of Labor, Family and Social Protection through the Financial Agreement POSDRU/ 89/1.5/S/62557.

References

[1] Y. Z. Chen, *Thompson's metric and mixed monotone operators*, J. Math. Anal. Appl., **177**(1993), no. 1, 31–37.

[2] T. Gnana Bhaskar and V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal., **65**(2006), no. 7, 1379–1393.

[3] D. Guo, Y. J. Cho, and J. Zhu, *Partial Ordering Methods in Nonlinear Problems*, Nova Science Publishers Inc., Hauppauge, NY, 2004.

[4] D. J. Guo, *Fixed points of mixed monotone operators with applications*, Appl. Anal., **31**(1988), no. 3, 215–224.

[5] D. J. Guo and V. Lakshmikantham, *Coupled fixed points of nonlinear operators with applications*, Nonlinear Anal., **11**(1987), no. 5, 623–632.

[6] J. Harjani, B. López, and K. Sadarangani, *Fixed point theorems for mixed monotone operators and applications to integral equations*, Nonlinear Anal., **74**(2011), no. 5, 1749–1760.

[7] G. Jameson, *Allied subsets of topological groups and linear spaces*, Proc. London Math. Soc. (3), **18**(1968), 653–690.

[8] E. Karapınar, *Coupled fixed point theorems for nonlinear contractions in cone metric spaces*, Comput. Math. Appl., **59**(2010), no. 12, 3656–3668.

[9] N. S. Kurpel′, *Some methods of constructing two-sided approximations to solutions of operator equations*, in *Problems in the Theory and History of Differential Equations (Russian)*, "Naukova Dumka", Kiev (1968), 131–146.

[10] N. S. Kurpel′ and B. A. Šuvar, *Two-sided Operator Inequalities and their Applications (in Russian)*, Naukova Dumka, Kiev (1980).

[11] V. Lakshmikantham and L. Ćirić, *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Anal., **70**(2009), no. 12, 4341–4349.

[12] N. V. Luong and N. X. Thuan, *Coupled fixed points in partially ordered metric spaces and application*, Nonlinear Anal., **74**(2011), no. 3, 983–992.

[13] J. Moore, *Existence of multiple quasifixed points of mixed monotone operators by iterative techniques*, Appl. Math. Comput., **9**(1981), no. 2, 135–141.

[14] M. Olatinwo, *Coupled fixed point theorems in cone metric spaces*, Ann. Univ. Ferrara Sez. VII Sci. Mat., **57**(2011), 173–180.

[15] V. I. Opoĭtsev, *Heterogeneous and combined-concave operators (in russian)*, Sibirsk. Mat. Ž., **16**(1975), no. 4, 781–792.

[16] V. I. Opoĭtsev, *Generalization of the theory of monotone and concave operators (in Russian)*, Trudy Moskov. Mat. Obshch., **36**(1978), 237–273, English translation in Trans. Moscow Math. Soc. (1979), no. 2, 243–279.

[17] E. Picard, *Mémoire sur la theorie des équations aux dérivées partielles et la méthode des approximations successives*, J. de Math., **6**(1890), 145–210.

[18] E. Picard, *Sur l'application des méthodes d'approximations successives à l'étude de certaines équations différentielles ordinaires*, J. de Math., **9**(1893), 217–272.

[19] E. Picard, *Sur un exemple d'approximations successives divergentes*, Bull. Soc. Math. France, **28**(1900), 137–143.

[20] M.-D. Rus, *Fixed point theorems for generalized contractions in partially ordered metric spaces with semi-monotone metric*, Nonlinear Anal., **74**(2011), no. 5, 1804–1813.

[21] F. Sabetghadam, H. P. Masiha, and A. H. Sanatpour, *Some coupled fixed point theorems in cone metric spaces*, Fixed Point Theory Appl., **2009**(2009), Art. ID 125426, 8 pages.

[22] B. Samet, *Coupled fixed point theorems for a generalized Meir–Keeler contraction in partially ordered metric spaces*, Nonlinear Anal., **72**(2010), no. 12, 4508–4517.

[23] W. Shatanawi, *Partially ordered cone metric spaces and coupled fixed point results*, Comput. Math. Appl., **60**(2010), no. 8, 2508–2515.

[24] W. Shatanawi, *Some common coupled fixed point results in cone metric spaces*, Int. J. Math. Anal. (Ruse), **4**(2010), no. 45-48, 2381–2388.

[25] A. C. Thompson, *On certain contraction mappings in a partially ordered vector space*, Proc. Amer. Math. Soc., **14**(1963), 438–443.

[26] Y. Wu and Z. Liang, *Existence and uniqueness of fixed points for mixed monotone operators with applications*, Nonlinear Anal., **65**(2006), no. 10, 1913–1924.

[27] B. Xu and R. Yuan, *On the positive almost periodic type solutions for some nonlinear delay integral equations*, J. Math. Anal. Appl., **304**(2005), no. 1, 249–268.

Received: June 21, 2012; Accepted: March 19, 2013