

THE GENERALIZED RETRACTION METHODS IN FIXED POINT THEORY FOR NONSELF OPERATORS

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Abstract. Starting from some notions and techniques presented in a paper by R.F. Brown (R.F. Brown, *Retraction methods in Nielsen fixed point theory*, Pacific J. Math., 115(1984), 277-297) we introduce the notion of generalized retract of an operator. Some generic examples of such retracts are given. Using the techniques related to this generalized retract, we present some fixed point theorems for nonself operators on partial ordered sets, metric spaces, generalized metric spaces and Banach spaces. The conjecture of the generalized retracts reads as follows: *Let f be a nonself operator defined on a subset with nonempty boundary. Each boundary condition (Leray-Schauder, Rothe, inwardness, outwardness, ...) on f implies the existence of a generalized retract of f .*

Key Words and Phrases: Partial ordered set, metric space, generalized metric space, Banach space, nonself operator, fixed point, retract, generalized retract, abstract interval, measure of non-compactness, fixed point structure.

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