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THE GENERALIZED RETRACTION METHODS IN FIXED POINT THEORY FOR NONSELF OPERATORS

IOAN A. RUS

Babeş-Bolyai University, Department of Mathematics
Kogălniceanu Street, No. 1, 400084 Cluj-Napoca, Romania
E-mail: iarus@math.ubbcluj.ro

Abstract. Starting from some notions and techniques presented in a paper by R.F. Brown (R.F. Brown, *Retraction methods in Nielsen fixed point theory*, Pacific J. Math., 115(1984), 277-297) we introduce the notion of generalized retract of an operator. Some generic examples of such retracts are given. Using the techniques related to this generalized retract, we present some fixed point theorems for nonself operators on partial ordered sets, metric spaces, generalized metric spaces and Banach spaces. The conjecture of the generalized retracts reads as follows: *Let f be a nonself operator defined on a subset with nonempty boundary. Each boundary condition (Leray-Schauder, Rothe, inwardness, outwardness, ...) on f implies the existence of a generalized retract of f.*
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1. Introduction

The Retraction Operator Principle is a very old and useful technique in fixed point theory of nonself operators ([23], [47], [67], [22], [21], [80], [97], [102], [51], [35], [48], [68], [96], ...). In this paper we introduce the notion of generalized retract of an operator and, using the technique of proof in terms of this notion, we present some fixed point theorems for nonself operators on: partial ordered sets, metric spaces, generalized metric spaces and Banach spaces. Our results are in connection with Problem 5 in [101]. Ours motivating works are, mainly, [22], [29], [48] and [97].

Throughout this paper we follow the terminologies and notations in [100] and [101]. For the convenience of the reader we shall recall some of them:

$\mathbb{N} := \{0, 1, 2, \ldots, n, \ldots\}$,
$\mathbb{N}^* := \{1, 2, \ldots, n, \ldots\}$,
$\mathbb{R} :=$ the set of real numbers,
$\mathbb{R}_+ := \{x \in \mathbb{R} \mid x \geq 0\}$,
$\mathbb{R}_+^* := \{x \in \mathbb{R} \mid x > 0\}$.

Let X be a nonempty set and $f : X \to X$ be an operator. Then:

$P(X) := \{Y \subset X \mid Y \neq \emptyset\}$,
$f^0 := 1_X$, $f^1 := f$, $f^2 := f \circ f$, ... , $f^{n+1} := f \circ f^n$, $n \in \mathbb{N}$,

$F_f := \{x \in X \mid f(x) = x\}$ - the fixed point set of f,

$F_f = \{x^*\}$, means that the operator f has a unique fixed point and we denote this fixed point by x^*.

Let (X, d) be a metric space. Then:

$P_b(X) := \{Y \in P(X) \mid Y$ is a bounded set$\}$,

$P_{cl}(X) := \{Y \in P(X) \mid Y$ is a closed set$\}$,

$P_{cp}(X) := \{Y \in P(X) \mid Y$ is a compact set$\}$.

If X is a set with a convex structure, then

$$P_{co}(X) := \{Y \in P(X) \mid Y \text{ is a convex set}\}.$$

Since this paper is not self-contained, we mention the following general references for fixed point theory in:

(1) partial ordered sets (posets): [2], [12], [31], [16], [49], [58], [69], [96], [102], ...
(2) metric and generalized metric spaces: [67], [46], [10], [98], [102], [103], [40], ...
(3) Banach spaces: [46], [67], [23], [69], [37], [81], [102], [113], [109], ...
(4) linear topological spaces: [47], [102], [109], [113], ...
(5) topological spaces: [23], [47], [102], [109], ...

For the fixed point structure theory and for category theory see: [100], [8], [75], [101], ...

2. Set-theoretic aspects

Let X be a nonempty set, $Y \subset X$ be a nonempty subset of X and $f : Y \to X$ be a nonself operator. By definition a self operator, $\rho_f : Y \to Y$ is called a fixed point self operator of f iff, $F_f = F_{\rho_f}$. If in addition, $x \in Y$, $f(x) \in Y$ imply that $\rho_f(x) = f(x)$, then ρ_f is called a generalized retract of the nonself operator f. Before giving some examples, we present an abstract notion of interval on a nonempty set.

By definition an operator, $[\cdot, \cdot] : X \times X \to P(X)$ is called interval operator if it satisfies the following axioms:

(1) $[x, y] = [y, x]$, for all $x, y \in X$;
(2) $x, y \in [x, y]$, for all $x, y \in X$;
(3) $[x, x] = \{x\}$, for all $x \in X$.

We shall use the following notations:

$]x, y] := [x, y] \setminus \{x\}$,

$[x, y[:= [x, y] \setminus \{y\}$,

$]x, y[:= [x, y] \setminus \{x, y\}$.

Let us give some examples of interval structure on a set X.

Example 2.1. *Let $(X, \leq)$ be a supsemilattice and $x, y \in X$. Then*

$$[x, y] := \{z \in X \mid x \leq z \leq x \vee y\} \cup \{z \in X \mid y \leq z \leq x \vee y\},$$

satisfies the axioms (1), (2) *and* (3).

If $(X, \leq)$ is a poset and $x, y \in X$, $x \leq y$, then $[x, y] := \{z \in X \mid x \leq z \leq y\}$.

We shall denote such an interval by $[x, y]_{\leq}$, i.e., the ordered set interval.

Example 2.2. *Let $(X, +, K)$ be a (real or complex) linear space. Then,*

$$[x, y] := \{(1 - \lambda)x + \lambda y \mid 0 \leq \lambda \leq 1\}$$

satisfies the axioms (1), (2) *and* (3). *We shall denote it by $[x, y]_l$, i.e., the linear space interval.*

Example 2.3. *Let (X, d) be a metric space. Then,*

$$[x, y] := \{z \in X \mid d(x, z) + d(z, y) = d(x, y)\}$$

satisfies the axioms (1), (2) *and* (3). *We shall denote it by $[x, y]_d$, i.e., the metric interval.*

For various notions of interval in a metric space, see: [48], [64], [63], [74], [79], [107], ...

Let $(X, [\cdot, \cdot])$ be a set with an interval structure. Then by definition a subset $Y \subset X$ is a convex subset if, $x, y \in Y$ imply $[x, y] \subset Y$. By definition, $\emptyset$ is a convex subset.

Let $\mathcal{P}_{co}(X) := \{Y \subset X \mid Y \text{ is a convex subset}\}$. We remark that $\mathcal{P}_{co}(X)$ is an abstract convex structure on X, i.e., $\emptyset$ and $X \in \mathcal{P}_{co}(X)$ and $\mathcal{P}_{co}(X)$ is closed under arbitrary intersection (see [74] and [107]). If $Y \in P(X)$, then

$$coY := \bigcap\{Z \mid Y \subset Z,\ Z \text{ - convex subset}\}.$$

Now we shall give some generic examples of fixed point self operators and of generalized retracts of a nonself operator.

Example 2.4 (R.F. Brown [22])**.** *Let $f : Y \to X$ be an operator and $\rho : X \to Y$ be a set retraction, i.e., $\rho\big|_Y = 1_Y$. If X is a structured set (ordered set, topological space, ...), $Y \subset X$, then $\rho : X \to Y$ is a retraction with respect to that structure (ordered set retraction, topological retraction, ...) if ρ is a set retraction and ρ is a morphism (see* [52], [75]*) with respect to that structure (increasing, continuous, ...). By definition $f : Y \to X$ is retractible with respect to the retraction $\rho : X \to Y$ if $F_f = F_{\rho\circ f}$. In this case, $\rho_f := \rho \circ f$ is a (generalized) retract of f. For example:*

(**1**) *Let $(X, +, \mathbb{R}, \|\cdot\|)$ be a Banach space and $\overline{B}(0; 1) := \{x \in X \mid \|x\| \leq 1\}$. Then the operator $\rho : X \to \overline{B}(0; 1)$, defined by*

$$\rho(x) := \begin{cases} x & \text{if } \|x\| \leq 1, \\ \frac{1}{\|x\|}x & \text{if } \|x\| \geq 1, \end{cases}$$

is a topological retraction of X onto $\overline{B}(0; 1)$. One names this retraction, the radial retraction.

Let $f : \overline{B}(0; 1) \to X$ be an operator. If f satisfies the following condition (Leray-Schauder condition)

$$x \in X,\ \|x\| = 1,\ \lambda > 0,\ f(x) = \lambda x \text{ imply } \lambda \leq 1,$$

then f is retractible with respect to ρ.

(**2**) *Let $(X, \|\cdot\|)$ be a Banach space, $0 < r_1 < r_2$ and*

$$Y_{r_1,r_2} := \{x \in X \mid r_1 \leq \|x\| \leq r_2\}.$$

Then the operator $\rho_{r_1,r_2} : X \setminus \{0\} \to Y_{r_1,r_2}$ *defined by*

$$\rho_{r_1,r_2}(x) := \begin{cases} \frac{r_1}{\|x\|} & \text{if } 0 < \|x\| \leq r_1 \\ x & \text{if } r_1 \leq x \leq r_2 \\ \frac{r_2}{\|x\|} & \text{if } \|x\| \geq r_2, \end{cases}$$

is a topological retraction of $X \setminus \{0\}$ *on* Y_{r_1,r_2}.

Let $f : Y_{r_1,r_2} \to X \setminus \{0\}$ *be an operator. If:*

(*i*) $\|x\| = r_1$ *implies* $f(x) \neq \lambda x$, $\forall\ \lambda \in]0, 1[$,

(*ii*) $\|x\| = r_2$ *implies* $f(x) \neq \lambda x$, $\forall\ \lambda > 1$,

then f *is retractible with respect to* ρ_{r_1,r_2}.

For the general theory of retracts see: [15] and [52].

Example 2.5. *Let* $(X, [\cdot,\cdot])$ *be a nonempty set with an interval structure. Let* $f : Y \to X$ *be an operator. We suppose that:*

$$x \in Y,\ f(x) \in X \setminus Y \text{ imply }]x, f(x)] \cap Y \neq \emptyset \tag{GR}$$

We call this condition, generalized retract condition.

Under the condition (GR) *we define the multivalued operator*

$$R_f : Y \to P(Y),\ R_f(x) := \begin{cases} \{f(x)\} & \text{if } f(x) \in Y, \\ ]x, f(x)] & \text{if } f(x) \in X \setminus Y. \end{cases}$$

Let $\rho_f : Y \to Y$ *be a selection of* R_f, *i.e.,* $\rho_f(x) \in R_f(x)$, *for all* $x \in Y$. *Then we have that:*

$$F_f = F_{\rho_f} = F_{R_f}.$$

So, ρ_f *is a generalized retract of* f.

Example 2.6. *Let* $(X, [\cdot,\cdot])$ *be a nonempty set with an interval structure. Let* $f : Y \to X$ *be an operator and* $x_0 \in Y$. *We suppose that:*

$$x \in Y,\ f(x) \in X \setminus Y \text{ imply }]x_0, f(x)] \neq \emptyset \text{ and } x \notin]x_0, f(x)] \tag{GR_{x_0}}$$

Under the condition, (GR_{x_0}), *we define the multivalued operator*

$$R_f : Y \to P(Y),\ R_f(x) := \begin{cases} \{f(x)\} & \text{if } f(x) \in Y, \\ ]x_0, f(x)] & \text{if } f(x) \in X \setminus Y. \end{cases}$$

Let ρ_f *be a selection of* R_f. *Then,*

$$F_f = F_{\rho_f} = F_{R_f}.$$

So, ρ_f *is a generalized retract of the nonself operator* f.

Remark 2.1. *If an operator* $f : Y \to X$ *has a generalized retract,* $\rho_f : Y \to Y$, *such that* $F_{\rho_f} \neq \emptyset$, *then* $F_f \neq \emptyset$. *So, the problem is to find for a nonself operator* f *a generalized retract* ρ_f *such that,* $F_{\rho_f} \neq \emptyset$.

Example 2.7. *Let* $f : Y \to X$ *be an injective operator such that* $Y \subset f(Y)$. *Let us take,* $\rho_f := f^{-1}\big|_Y : Y \to Y$. *Then,* $F_f = F_{\rho_f}$. *So,* ρ_f *is a self fixed point operator for* f.

For more considerations on such operator see: [109](pp. 59-60), [5], [6], [7], [50], [104], ...

Remark 2.2. *Let $f : Y \to X$ be a nonself operator and $\rho : X \to Y$ be a set retraction. The formal boundary of Y with respect to f and ρ is by definition, $\partial_{f,\rho}(Y) := \rho(f(Y) \setminus Y)$ (for other type of formal boundary see* [20]*,* [18]*,* [43] *and* [93]*).*

We have

Lemma 2.1. *The operator f is retractible with respect to ρ if*

$$x \in \partial_{f,\rho}(Y),\ x = \rho(f(x)) \text{ imply } f(x) = x.$$

Let X be a nonempty set, $S(X) \subset P(X)$, $S(X) \neq \emptyset$ and for each $Y \in S(X)$, $M(Y)$ be a family of operators $f : Y \to Y$. By definition (see [100]), the triplet $(X, S(X), M)$ is a large fixed point structure on X iff:

$$Y \in S(X),\ f \in M(Y) \text{ imply } \ F_f \neq \emptyset.$$

Here are some examples of large fixed point structures (l.f.p.s.):

(1) **Tarski's l.f.p.s.** Let $(X, \leq)$ be a poset and consider the sets $S(X) := \{Y \in P(X) | (Y, \leq) \text{ is a complete lattice}\}$ and $M(Y) := \{f : Y \to Y \mid f \text{ is increasing}\}$.

(2) **The l.f.p.s. of progressive operators.** $(X, \leq)$ is a poset, $S(X) := \{Y \in P(X) \mid (Y, \leq) \text{ has at least a maximal element}\}$ and $M(Y) := \{f : Y \to Y \mid f \text{ is progressive, i.e., } x \leq f(x),\ \forall\, x \in Y\}$.

(3) **The l.f.p.s. of contractions.** (X, d) is a complete metric space, $S(X) := P_{cl}(X)$ and $M(Y) := \{f : Y \to Y \mid f \text{ is a contraction}\}$.

(4) **The l.f.p.s. of Brouwer-Schauder-Tychonoff-Cauty.** X is a Hausdorff linear topological space, $S(X) := P_{cp,co}(X)$ and $M(Y) := C(Y, Y) := \{f : Y \to Y \mid f \text{ is continuous}\}$.

(5) **The l.f.p.s. of Browder-Ghöde-Kirk.** X is a uniformly convex Banach space, $S(X) := P_{b,cl,co}(X)$ and $M(Y) := \{f : Y \to Y \mid f \text{ is nonexpansive}\}$.

From Lemma 2.1 we have:

Lemma 2.2. *Let $(X, S(X), M)$ be a l.f.p.s., $Y \in S(X)$, $\rho : X \to Y$ be a set retraction and $f : Y \to X$. We suppose that:*

(i) $\rho \circ f \in M(Y)$*;*

(ii) $x \in \partial_{f,\rho}Y,\ x = \rho(f(x))$ *imply* $f(x) = x$*.*

Then, $F_f \neq \emptyset$.

We also have

Lemma 2.3. *Let $(X, S(X), M)$ be a l.f.p.s., $Y \in S(X)$ and $f : Y \to X$ be an injective operator such that $Y \subset f(Y)$. If $f^{-1}\big|_Y \in M(Y)$, then $F_f \neq \emptyset$.*

Remark 2.3. *Let X be a nonempty set with a convex structure, $Z \subset P(X)$, $Z \neq \emptyset$, such that $A \in Z$, $x \in X$ imply $A \cup \{x\} \in Z$ and $coA \in Z$. Let $\theta : Z \to \mathbb{R}_+$ be such that:*

(*i*) $A, B \in Z$, $A \subset B$ *imply* $\theta(A) \leq \theta(B)$*;*
(*ii*) $\theta(A \cup \{x\}) = \theta(A)$, *for all* $A \in Z$ *and* $x \in X$*;*
(*iii*) $\theta(coA) = \theta(A)$, *for all* $A \in Z$.

We have

Lemma 2.4. *Let* $f : Y \to X$ *satisfying the condition* (GR_{x_0})*. We suppose that there exists a selection* ρ_f *of* R_f *such that,* $A \in Z$ *implies that* $\rho_f(A) \in Z$*. Then*

$$\theta(\rho_f(A)) \leq \theta(f(A)), \text{ for all } A \in P(Y) \cap Z.$$

Proof. We remark that $\rho_f(A) \subset co(f(A) \cup \{x_0\})$. From (*i*), (*ii*) and (*iii*) we have

$$\theta(\rho_f(A)) \leq \theta(co(f(A) \cup \{x_0\})) = \theta(f(A) \cup \{x_0\}) = \theta(f(A)).$$

□

In a similar way we have:

Lemma 2.5. *Let* X *be a Banach space and* $\rho : X \to \overline{B}(0; r)$ *be the radial retraction. Let* θ_k *be the Kuratowski measure of noncompactness on* X*. Then,*

$$\theta_k(\rho(A)) \leq \theta_k(A), \text{ for all } A \in P_b(X).$$

Proof. We remark that, $\rho(A) \subset co(A \cup \{0\})$ and that θ_k satisfies the conditions (*i*), (*ii*) and (*iii*). □

We have a similar result for the Hausdorff measure of noncompactness and, more generally, for a suitable abstract measure of noncompactness on a Banach space.

For the measure of noncompactness see: [100], [37], [38], [41], [77], [80], [83], [88], [89], [90], [94], [95], ...

In what follows we shall give conditions under which a nonself operator has a self fixed point operator, ρ_f, such that, $F_{\rho_f} \neq \emptyset$. In these conditions, we have will get that $F_f \neq \emptyset$.

3. Nonself operators on a poset

The fixed point theory of self operators on a poset is a subject with an intensive development. For the basic results of this topic see [2], [12], [47], [49], [55], [102], ... For the retract theory on a poset see [35], [96], [97], [102], ...

In what follows we need the following results.

Zermelo's Theorem. *Let* $(X, \leq)$ *be a poset and* $f : X \to X$ *be an operator. We suppose that:*

(*i*) *every chain in* X *has an upper bound;*
(*ii*) f *is a progressive operator.*

Then, $F_f \neq \emptyset$.

Tarski's Theorem. *Let* $(X, \leq)$ *be a complete lattice and* $f : X \to X$ *be an increasing operator. Then,*

(*a*) $F_f \neq \emptyset$*;*
(*b*) $(F_f, \leq)$ *is a complete lattice.*

Our results are the following.

Theorem 3.1. *Let* $(X, \leq)$ *be a poset with the least element,* 0 *(i.e.,* $0 \leq x$ *for each* $x \in X$*). Let* $Y \in P(X)$ *and* $f : Y \to X$*. We suppose that:*

(i) $0 \in Y$;

(ii) $(Y, \leq)$ *is a complete lattice;*

(iii) f *is increasing;*

(iv) f *satisfies condition* (GR_0) *with respect to the order interval,* $[\cdot, \cdot]_\leq$.

Then,

(a) $F_f \neq \emptyset$;

(b) $(F_f, \leq)$ *is a complete lattice.*

Proof. Let us consider the self operator

$$\rho_f : Y \to Y,\ \rho_f(x) := \begin{cases} f(x) & \text{if } f(x) \in Y; \\ \sup_Y \big( ]x_0, f(x)]_\leq \cap Y \big) & \text{if } f(x) \in X \setminus Y. \end{cases}$$

The condition (iv) implies that this operator is well defined and that $F_f = F_{\rho_f}$, i.e., ρ_f is a generalized retract of f. On the other hand ρ_f is an increasing operator. Indeed, let us consider, for example, the case: $x_1 \leq x_2$ and $f(x_1), f(x_2) \in X \setminus Y$. It is clear that, since $f(x_1) \leq f(x_2)$ we have that $]0, f(x_1)]_\leq \subset]0, f(x_2)]_\leq$. It follows that $\rho_f(x_1) \leq \rho_f(x_2)$. From Tarski's Theorem we have that $F_{\rho_f} \neq \emptyset$ and $(F_{\rho_f}, \leq)$ is a complete lattice. So, we have (a) and (b). □

Theorem 3.2. *Let* $(X, \leq)$ *be a poset with the least element,* 0*. Let* $Y \in P(X)$ *and* $f : Y \to X$*. We suppose that:*

(i) $0 \in Y$;

(ii) $(Y, \leq)$ *is a sup-complete lattice;*

(iii) f *is a progressive operator;*

(iv) $x \in Y$, $f(x) \in X \setminus Y$ *imply* $]0, f(x)]_\leq \cap Y \neq \emptyset$ *and* $x < \sup_Y \big( ]0, f(x)]_\leq \cap Y \big)$.

Then, $F_f \neq \emptyset$.

Proof. We consider the self operator, $\rho_f : Y \to Y$, defined by,

$$\rho_f(x) := \begin{cases} f(x) & \text{if } f(x) \in Y, \\ \sup_Y \big( ]0, f(x)]_\leq \cap Y \big) & \text{if } f(x) \in X \setminus Y. \end{cases}$$

Condition (iv) implies that ρ_f is a generalized retract of f. On the other hand we observe that ρ_f is a progressive operator. From Zermelo's Theorem it follows that $F_{\rho_f} \neq \emptyset$. Hence $F_f \neq \emptyset$. □

Theorem 3.3. *Let* $(X, \leq)$ *be a poset,* $Y \subset X$ *and* $f : Y \to X$*. We suppose that:*

(i) $(Y, \leq)$ *is a complete lattice;*

(ii) $Y \subset f(Y)$;

(iii) $x_1, x_2 \in Y$, $f(x_1) \leq f(x_2)$ *imply* $x_1 \leq x_2$;

(iv) f *is injective.*

Then,

(a) $F_f \neq \emptyset$;

(*b*) $(F_f, \leq)$ *is a complete lattice.*

Proof. The proof follows from Lemma 2.3 and from Tarski's Theorem. $\square$

In a similar way we have

Theorem 3.4. *Let* $(X, \leq)$ *be a poset,* $Y \subset X$ *and* $f : Y \to X$*. We suppose that:*

(*i*) *the minimal element set of* $(Y, \leq)$*,* $Min(Y, \leq) \neq \emptyset$*;*
(*ii*) $Y \subset f(Y)$*;*
(*iii*) f *is progressive;*
(*iv*) f *is injective.*

Then, $F_f \neq \emptyset$*.*

In order to present our next result we need the following notion (see [31]). Let $(X, \leq)$ be a lattice and $Y \subset X$. An element $c \in Y$ is called a *sup-center* of the subset Y if $\sup\{c, x\} \in Y$, for all $x \in Y$.

Now, let us take, $X := \mathbb{R}^m$. We consider on $\mathbb{R}^m$ the standard order relation (i.e., coordinatewise partial order, $\leq$) and the standard linear space structure. The following result is given in [31](Theorem 1.2, p. 1).

Theorem 3.5. *Let* $Y \subset \mathbb{R}^m$ *be a nonempty subset and* $f : Y \to Y$ *be a self operator. We suppose that:*

(*i*) Y *is closed and bounded with respect to a norm on* $\mathbb{R}^m$*;*
(*ii*) Y *has a sup-center,* $c \in Y$*;*
(*iii*) f *is increasing.*

Then, $F_f \neq \emptyset$*.*

Now we shall give an application of this theorem. Let $m := 2$ and $Y := \{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1\}$, and $f : Y \to \mathbb{R}^2$ be a function. We have

Theorem 3.6. *We suppose that:*

(*i*) f *is increasing;*
(*ii*) $f(x) \in \mathbb{R}^2 \setminus Y$ *implies* $f(x) \in \mathbb{R}_- \times \mathbb{R}_+$ *and* $x \notin]0, f(x)]_l$*.*

Then, $F_f \neq \emptyset$*.*

Proof. First of all we remark that Y is closed, bounded and $c = 0$ is a sup-center of Y. Let $\rho_f : Y \to Y$ be defined by, $\rho_f(x) = f(x)$ if $f(x) \in Y$ and if $f(x) \in \mathbb{R}^2 \setminus Y$, then $\rho_f(x) := y$, where $]0, f(x)]_l \cap \partial Y = \{y\}$. We observe that ρ_f is a generalized retract of f and ρ_f is increasing. From Theorem 3.5 we have that, $F_{\rho_f} \neq \emptyset$. So, $F_f \neq \emptyset$. $\square$

Remark 3.1. *It is a problem to give a nonself correspondent result to Theorem 3.5.*

The following result is well known (see [2]).

Theorem 3.7. *Let* $(X, \leq)$ *be a poset,* $a, b \in X$*,* $a < b$*, and* $f : [a, b]_\leq \to X$ *be an operator. We suppose that:*

(*i*) $([a, b], \leq)$ *is a complete lattice;*
(*ii*) f *is increasing;*
(*iii*) $a \leq f(a)$*,* $f(b) \leq b$*.*

Then,

(a) $F_f \neq \emptyset$;

(b) $(F_f, \leq)$ *is a complete lattice.*

Proof. Condition (iii) implies that $f([a,b]) \subset [a,b]$. So, f is a self operator and the proof follows from Tarski's Theorem. □

Remark 3.2 (see [96]). *By definition, a poset $(X, \leq)$ has the fixed point property if every increasing operator $f : X \to X$ has at least a fixed point. Let $Y \subset X$ and $\rho : X \to Y$ be a set retraction. If in addition ρ is increasing, then by definition ρ is a poset retraction and Y is a poset retract of X. It is well known that:* If $(X, \leq)$ is a poset with fixed point property, then each poset retract of X has the fixed point property. *So, an important problem of the fixed point theory in a poset is to give generic examples of increasing retracts. We have a similar problem for generalized retracts.*

4. Nonself operators on a metric space

In what follows we need the following fixed point theorems for self operators.

Caristi-Kirk's Theorem. *Let (X, d) be a complete metric space and $f : X \to X$ be an operator which satisfies*

$$d(x, f(x)) \leq \varphi(x) - \varphi(f(x))$$

for all $x \in X$, where $\varphi : X \to \mathbb{R}_+$ is a lower semicontinuous functional. Then, $F_f \neq \emptyset$.

Jachymski's Theorem. *Let (X, d) be a complete metric space, $\varphi : X \to \mathbb{R}_+$ be a lower semicontinuous functional and $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing and subadditive function, continuous at 0 and such that $\eta^{-1}(0) = \{0\}$. If $f : X \to X$ is such that*

$$\eta(d(x, f(x))) \leq \varphi(x) - \varphi(f(x)), \text{ for all } x \in X,$$

then, $F_f \neq \emptyset$.

For the above results and for other generalizations of Caristi-Kirk's Theorem see: [29], [30], [66], [54], [55], [1], [16], [61], [63], [92], [114], ...

Let (X, d) be a metric space, $Y \in P(X)$ and $f : Y \to X$. We suppose that f satisfies the condition (GR) with respect to the metric interval, i.e.,

$$x \in Y,\ f(x) \in X \setminus Y \text{ imply }]x, f(x)[_d \cap Y \neq \emptyset \qquad \text{(GR)}$$

Let us define the multivalued operator

$$R_f : Y \to P(Y),\ R_f(x) := \begin{cases} f(x) & \text{if } f(x) \in Y, \\ ]x, f(x)[_d \cap Y & \text{if } f(x) \in X \setminus Y \end{cases}$$

Definition 4.1. *An operator $f : Y \to X$ is a (GR)-directional contraction if it satisfies the (GR)-condition and there exists a selection ρ_f of R_f and $\alpha \in]0, 1[$ such that*

$$d(f(x), f(\rho_f(x))) \leq \alpha d(x, \rho_f(x)),\ \forall\ x \in Y.$$

Definition 4.2. *An operator $f : Y \to X$ is a (GR)-φ-directional contraction if $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function (see* [98], [10]*) and there exists a selection ρ_f of R_f such that*

$$d(f(x), f(\rho_f(x))) \le \varphi(d(x, \rho_f(x))),\ \forall\ x \in Y.$$

Definition 4.3. *An operator $f : Y \to X$ is a (GR)-separate contraction if there exists two functions, $\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying:*

(i) ψ is strictly increasing and $\psi(0) = 0$;
(ii) $\psi(t) \le t - \varphi(t)$, for $t > 0$;
(iii) $d(f(x), f(\rho_f(x))) \le \varphi(d(x, \rho_f(x)))$, $\forall\ x \in Y$.

Example 4.1. *If $f : Y \to X$ is a contraction and satisfies the (GR)-condition, then f is a (GR)-directional contraction with respect to each selection ρ_f of R_f.*

Example 4.2. *If $f : Y \to X$ is a φ-contraction and satisfies the (GR)-condition then f is a (GR)-φ-directional contraction with respect to each selection ρ_f of R_f*

Example 4.3. *If $f : Y \to X$ satisfies the (GR)-condition and is a separate contraction (see* [73]*), then f is a (GR)-separate contraction.*

Example 4.4. *If f satisfies the (GR)-condition and it is a large contraction (see* [26] *and* [73]*), then f is a (GR)-separate contraction.*

For more considerations on generalized self contractions see: [56], [11], [26], [57], [65], [73], ...

Our results are the following.

Theorem 4.1. *Let (X, d) be a complete metric space, $Y \in P_{cl}(X)$ and $f : Y \to X$ be a continuous (GR)-α-directional contraction. Then, $F_f \ne \emptyset$.*

Proof. Let ρ_f be a selection of R_f. Then,

$$d(x, \rho_f(x)) + d(\rho_f(x), f(x)) = d(x, f(x)),\ \forall\ x \in Y.$$

We have

$$\begin{aligned} d(\rho_f(x), f(\rho_f(x))) &\le d(\rho_f(x), f(x)) + d(f(x), f(\rho_f(x))) \le \\ &\le d(x, f(x)) - d(x, \rho_f(x)) + \alpha d(x, \rho_f(x)) \le \\ &\le d(x, f(x)) - (1-\alpha) d(x, \rho_f(x)). \end{aligned}$$

This implies that:

$$d(x, \rho_f(x)) \le \varphi(x) - \varphi(\rho_f(x)),$$

where $\varphi(x) := \frac{1}{1-\alpha} d(x, f(x))$.

From Caristi-Kirk's Theorem it follows that $F_{\rho_f} \ne \emptyset$. Since ρ_f is a generalized retract of f, we have that, $F_f \ne \emptyset$. □

Corollary 4.1 (Caristi [29])**.** *Let (X, d) be a complete metric space, $Y \in P_{cl}(X)$ and $f : Y \to X$. We suppose that:*

(i) f is an α-contraction;
(ii) f satisfies the condition (GR).

Then, $F_f = \{x^*\}$.

Proof. From Theorem 4.1 we have that, $F_f \neq \emptyset$. Since f is a contraction it follows that $F_f = \{x^*\}$. □

Remark 4.1. *Let us consider the following notations:*

$(MI)_f$ *- the maximal invariant subset of* f,

$(AB)_f(x^*) := \{x \in Y \mid f^n(x)$ *is defined for all* $n \in \mathbb{N}$ *and* $f^n(x) \to x^* \in F_f\}$ *- the attraction basin of the fixed point* x^* *with respect to* f.

By definition (see [33], [99]*) an operator* $f : Y \to X$ *is said to be Picard operator* (PO) *if*

(a) $F_f = \{x^*\}$

(b) $(MI)_f = (AB)_f(x^*)$.

An operator $f : Y \to X$ *is said to be weakly Picard operator* (WPO) *if*

(a) $F_f \neq \emptyset$;

(b) $(MI)_f = (AB)_f := \bigcup_{x \in F_f} (AB)_f(x)$.

It is clear that if f *is as in Corollary 4.1, then* f *is* PO. *For the theory of such operators, see* [33].

Theorem 4.2. *Let* (X, d) *be a complete metric space,* $Y \in P_{cl}(X)$ *and* $f : Y \to X$. *We suppose that:*

(i) f *is a* (GR)*-*φ*-directional contraction;*

(ii) φ *is superadditive;*

(iii) f *is continuous,*

Then, $F_f \neq \emptyset$.

Proof. From

$$d(\rho_f(x), f(\rho_f(x))) \leq d(x, f(x)) - d(x, \rho_f(x)) + d(f(x), f(\rho_f(x)))$$

we have that

$$(1_{\mathbb{R}_+} - \varphi)(d(x, \rho_f(x))) \leq d(x, f(x)) - d(\rho_f(x), f(\rho_f(x))).$$

From Jachymski's Theorem we have that, $F_{\rho_f} \neq \emptyset$. So, $F_f \neq \emptyset$. □

Corollary 4.2. *Let* (X, d) *be a complete metric space,* $Y \in P_{cl}(X)$ *and* $f : Y \to X$. *We suppose that:*

(i) f *is a* φ*-contraction, i.e.,* φ *is a comparison function (see* [98]*) and*

$$d(f(x), f(y)) \leq \varphi(d(x, y)), \ \forall \ x, y \in X;$$

(ii) f *satisfies the* (GR)*-condition;*

(iii) φ *is superadditive.*

Then, $F_f = \{x^*\}$.

Theorem 4.3. *Let* (X, d) *be a complete metric space,* $Y \in P_{cl}(X)$ *and* $f : Y \to X$. *We suppose that:*

(i) f *is a* (GR)*-*(φ, ψ)*-separate contraction;*

(ii) ψ *is subadditive;*

(iii) f *is continuous.*

Then, $F_f \neq \emptyset$.

Proof. The proof is similar with that of Theorem 4.2. □

Corollary 4.3. *Let* (X, d) *be a complete metric space,* $Y \in P_{cl}(X)$ *and* $f : Y \to X$. *We suppose that:*

(i) f *is a* (φ, ψ)*-separate contraction;*
(ii) f *satisfies the* (GR)*-condition;*
(iii) ψ *is subadditive.*

Then, $F_f \neq \emptyset$.

Remark 4.2. *Let* (X, d) *be a complete* $\mathbb{R}^m_+$*-metric space (i.e.,* $d(x, y) \in \mathbb{R}^m_+$ *and satisfies the standard axioms of metric). For such a metric space, from a general result of Eisenfeld-Laksmikantham (see* [42]*; see also* [78]*), we have the following result.*

Theorem 4.4. *Let* (X, d) *be a complete* $\mathbb{R}^m_+$*-metric space and* $f : X \to X$ *be an operator. We suppose that there exists a lower semicontinuous functional* $\varphi : X \to \mathbb{R}^m_+$ *such that*

$$d(x, f(x)) \leq \varphi(x) - \varphi(f(x)), \ \forall \ x \in X.$$

Then, $F_f \neq \emptyset$.

For the metric space with vectorial metric see: [112], [78], [13], [40], [42], [62], [102], [114], ...

Definition 4.4. *An operator* $f : Y \to X$ *is a* (GR)*-*S*-directional contraction if* f *satisfies the* (GR)*-condition and there exists a selection* ρ_f *of* R_f *such that*

$$d(f(x), f(\rho_f(x))) \leq Sd(x, \rho_f(x)), \ \forall \ x \in Y,$$

where $S \in \mathbb{R}^{m \times m}_+$ *is a matrix convergent to zero (see* [98] *and* [102]*).*

We have

Theorem 4.5. *Let* (X, d) *be a complete* $\mathbb{R}^m_+$*-metric space,* $Y \in P_{cl}(X)$ *and* $f : Y \to X$. *We suppose that:*

(i) f *is a* (GR)*-*S*-directional contraction;*
(ii) f *is continuous.*

Then, $F_f \neq \emptyset$.

Proof. Let ρ_f be a selection of R_f. Then,

$$d(\rho_f(x), f(\rho_f(x))) \leq d(x, f(x)) - d(x, \rho_f(x)) + Sd(x, \rho_f(x)).$$

This implies that

$$d(x, \rho_f(x)) \leq (I - S)^{-1} d(x, f(x)) - (I - S)^{-1} d(\rho_f(x), f(\rho_f(x))).$$

The proof follows from Theorem 4.4. □

Corollary 4.4. *Let* (X, d) *be a complete* $\mathbb{R}^m_+$*-metric space,* $Y \in P_{cl}(X)$ *and* $f : Y \to X$. *We suppose that:*

(i) f *satisfies the* (GR)*-condition;*

(*ii*) *there exists a matrix* $S \in \mathbb{R}_+^{m\times m}$*, convergent to zero, such that*

$$d(f(x), f(y)) \leq Sd(x,y),\ \forall\ x,y \in Y.$$

Then, $F_f = \{x^*\}$.

The following result is well known.

Theorem 4.6. *Let* (X,d) *be a complete metric space,* $x_0 \in X$ *and* $f : \tilde{B}(x_0;r) \to X$ *be an operator. We suppose that:*

(*i*) f *is an* α*-contraction;*

(*ii*) $d(x_0, f(x_0)) \leq (1-\alpha)r$.

Then, $F_f = \{x^*\}$.

Proof. We observe that, $f(\tilde{B}(x_0;r)) \subset \tilde{B}(x_0;r)$. So, f is a self operator and the proof follows from the contraction principle. □

Remark 4.3. *For a less restrictive condition as* (*ii*) *see* [32].

5. Operators on Banach spaces: retraction technique

Let X be a Banach space and $\theta : P_b(X) \to \mathbb{R}_+$. By definition the set-functional θ is an abstract measure of noncompactness if it satisfies the following axioms:

(*i*) $\theta(A) = 0$ implies $\overline{A} \in P_{cp}(X)$;

(*ii*) $A \subset B$ implies $\theta(A) \leq \theta(B)$;

(*iii*) $\theta(\overline{A}) = \theta(A)$;

(*iv*) $\theta(coA) = \theta(A)$;

(*v*) $x \in X$, $A \in P_b(X)$ imply $\theta(A \cup \{x\}) = \theta(A)$.

It is clear that the Kuratowski measure of noncompactness, θ_K, and the Hausdorff measure of noncompactness, θ_H, satisfy the above axioms. For these measures of noncompactness and for various notions of abstract measure of noncompactness see [100], [37], [47], [71], [83], [102], ...

We have

Theorem 5.1. *Let* X *be a Banach space and* θ *be an abstract measure of noncompactness on* X*. Let* $f : \overline{B}(0;r) \to X$ *be a continuous operator and* $\rho : X \to \overline{B}(0;r)$ *be the radial retraction. We suppose that:*

(*i*) *there exists a comparison function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $\theta(f(A)) \leq \varphi(\theta(A))$*, for all* $A \in P_b(X)$ *such that* $\rho(f(A)) \subset A$*;*

(*ii*) f *is retractible with respect to the radial retraction* ρ*.*

Then,

(*a*) $F_f \neq \emptyset$*;*

(*b*) $\theta(F_f) = 0$.

Proof. First we observe that from axiom (*v*) of an abstract measure of noncompactness it follows that $\theta\big|_{P_{b,cl}(X)}$ is a functional with intersection property (see [100], p. 50). From condition (*ii*) it follows that the operator $\rho \circ f : \overline{B}(0;r) \to \overline{B}(0;r)$ is a retract of f and we have that $F_f = F_{\rho\circ f}$. Now the proof follows from Theorem 5.3.1 in [100]. □

Theorem 5.2. *Let X be a Banach space and $\theta : P_b(X) \to \mathbb{R}_+$ be a set-functional which satisfies the axioms (i)-(iv) and*

(v') $\theta\big|_{P_{b,cl}(X)}$ *is a functional with intersection property.*

Let $f : \overline{B}(0;r) \to X$ be a continuous operator and $\rho : X \to \overline{B}(0;r)$ be an l-Lipschitz retraction. We suppose that:

(i) *there exists $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that: $\theta(f(A)) \leq \varphi(\theta(A))$, for all $A \in P_b(X)$ such that $\rho(f(A)) \subset A$;*

(ii) *φ is a comparison function;*

(iii) *f is retractible with respect to ρ.*

Then:

(a) $F_f \neq \emptyset$;

(b) $\theta(F_f) = 0$.

Proof. See the proof of Theorem 5.1. □

Remark 5.1. *In Theorem 5.2, instead of θ we can put, θ_K, θ_H and the diameter functional, $\delta_{\|\cdot\|}$. In the case of $\delta_{\|\cdot\|}$ we shall have $\delta_{\|\cdot\|}(F_f) = 0$, i.e., $F_f = \{x^*\}$.*

Remark 5.2. *Each of the following conditions implies the condition (ii) in Theorem 5.1 (see* [100]*, p. 115):*

(1) *(Leray-Schauder)* $x \in \partial\overline{B}(0;r)$, $f(x) = \lambda x$ *imply* $\lambda \leq 1$.

(2) *(E. Rothe)* $f(\partial\overline{B}(0;r)) \subset \overline{B}(0;r)$.

(3) *(M. Altman)* $\|f(x) - x\|^2 \geq \|f(x)\|^2 - \|x\|^2$ *for all* $x \in \partial\overline{B}(0;R)$.

Corollary 5.1. *Let X be a Banach space and $f : \overline{B}(0;r) \to X$ be an l-Lipschitz operator with $l < \frac{1}{2}$. If f is retractible with respect to the radial retraction, then:*

(i) $F_f = \{x^*\}$

(ii) $\rho \circ f : \overline{B}(0;r) \to \overline{B}(0;1)$ *is a Picard operator, where ρ is the radial retraction.*

Proof. We take in Theorem 5.2, ρ the radial retraction. □

Remark 5.3. *For the radial retraction technique in the fixed point theory of nonself operator see also:* [17], [21], [22], [25], [70], [77], [64], [80], [88]-[90], [94], [95], [111], ...

Remark 5.4. *For the topological retraction see:* [4], [14], [15], [23], [24], [39], [41], [44], [45], [48], [51] ,[52], [59], [68], [70], [79], [91], ...

6. Operators on Banach spaces: generalized retraction technique

Let X be a Banach space and $Y \in P_{cl,co}(X)$. Let $f : Y \to X$ be a nonself operator. Let us consider the following subsets of Y:

$$Y_0 := \{x \in Y \mid f(x) \in Y\} \text{ and } Y_1 := \{x \in Y \mid f(x) \in X \setminus Y\}.$$

If $Y_0 = \emptyset$, then $F_f = \emptyset$ and if $Y_1 = \emptyset$, then f is a self operator. So, in what follows we suppose that, $Y_0 \neq \emptyset$ and $Y_1 \neq \emptyset$. Under this condition, we have the following partition of Y with respect to f: $Y = Y_0 \cup Y_1$.

Let us suppose that f satisfies the (GR) condition with respect to linear space interval, $[\cdot,\cdot]_l$, i.e.

$$x \in Y_1 \text{ implies }]x, f(x)]_l \cap Y \neq \emptyset \tag{GR}$$

Let $R_f : Y \to P(Y)$ be defined by

$$R_f(x) := \begin{cases} f(x) & \text{if } x \in Y_0, \\ ]x, f(x)]_l \cap Y & \text{if } x \in Y_1. \end{cases}$$

In what follows we suppose that f is continuous.

Now we consider

$$\Lambda_f := \{\lambda : Y_1 \to]0,1] \mid (1-\lambda(x))x + \lambda(x)f(x) \in Y,\ x \in Y_1\}$$

and

$$\lambda_m : Y_1 \to]0,1[\text{ be defined by } \lambda_m(x) := \sup_{\lambda \in \Lambda_f} \lambda(x).$$

Since Y is a closed convex subset of X it follows that

$$(1-\lambda_m(x))x + \lambda_m(x)f(x) \in Y, \text{ for all } x \in Y_1.$$

Let $\lambda_f : Y \to]0,1]$ be defined by

$$\lambda_f(x) := \begin{cases} 1 & \text{if } x \in Y_0, \\ \lambda_m(x) & \text{if } x \in Y_1. \end{cases}$$

Then, $\rho_f := (1-\lambda_f)1_Y + \lambda_f f$ is a generalized retract of f.

In what follows we suppose that f and Y are such that λ_f is a continuous function, i.e.,

$$\lambda_f \in C(Y,]0,1]) \tag{TGR}$$

The (TGR) condition implies that ρ_f is continuous, i.e., is a topological generalized retract of f.

The following problem is fundamental in this theory.

Problem 6.1. *Let X be a Banach space, $Y \in P_{cl,co}(X)$ and $f : Y \to X$ be a continuous operator. In which conditions on f and Y the function λ_f, defined above, is continuous ?*

We have the following fixed point theorem in terms of a topological general retract of an operator.

Theorem 6.1. *Let X be a Banach space, $Y \in P_{cp,co}(X)$ and $f : Y \to X$. We suppose that:*

(*i*) *f is continuous;*
(*ii*) *f satisfies condition* (TGR).

Then, $F_f \neq \emptyset$.

Proof. Condition (ii) implies that ρ_f defined by λ_f is a topological generalized retract of f. From Schauder's fixed point theorem we have that $F_{\rho_f} \neq \emptyset$. So, $F_f \neq \emptyset$. □

Example 6.1. *Let $a, b \in \mathbb{R}$, $a < b$ and $f : [a, b] \to \mathbb{R}$. Let*

$$Y_0 := \{x \in [a, b] \mid f(x) \in \ [a, b]\ \}$$

and

$$Y_- := \{x \in [a, b] \mid f(x) \in \,]-\infty, a[\ \},\ Y_+ := \{x \in [a, b] \mid f(x) \in \,]b, +\infty[\ \}.$$

We suppose that $a \in Y_+$ and $b \in Y_-$ and there exists $a_1 > a$ and $b_1 < b$ such that $Y_- = [b_1, b]$, $Y_+ = [a, a_1]$. Then f satisfies the (GR) condition. Now let f be a continuous function. Then λ_f is a continuous function.
Indeed, we have

$$\lambda_m(x) = \frac{b - x}{f(x) - x},\ x \in [a, a_1] \text{ and } \lambda_m(x) = \frac{a - x}{f(x) - x},\ x \in [b_1, b]$$

which are continuous and $\lambda_m(a_1) = 1$ and $\lambda_m(b_1) = 1$.

So, f satisfies the (TGR) *condition.*

For example the function, $f : [-2, 2] \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} -x + 1 & , x \in [-2, 0] \\ -2x + 1 & , x \in [0, 2] \end{cases}$$

satisfies the (TGR) *condition. In this case $a_1 = -1$, $b_1 = \frac{3}{2}$ and*

$$\lambda_f(x) = \begin{cases} \frac{2-x}{1-2x} & , x \in [-2, -1] \\ 1 & , x \in [-1, \frac{3}{2}] \\ \frac{-2-x}{-3x+1} & , x \in [\frac{3}{2}, 2]. \end{cases}$$

The above considerations give rise to

Conjecture. *Let X be a Banach space, $\Omega \subset X$ be a bounded open convex subset of X and $Y := \overline{\Omega}$. Let $f : Y \to X$ be a continuous operator. If f satisfies the (GR) condition and ∂Y is smooth then f satisfies the* (TGR) *condition.*

7. A problem

The following result was given by Ky Fan, in [43]:

Theorem 7.1. *Let Y be a nonempty compact convex subset of a normed linear space X. Then, for any continuous operator $f : Y \to X$, there exists a point $y_0 \in Y$ such that*

$$\|y_0 - f(y_0)\| = \underset{y \in Y}{Min}\ \|y - f(y_0)\|$$

The problem is to use this result in order to give fixed point theorems for a nonself operator $f : Y \to X$.

For example, the following result was given by L. Pasicki in [84]:

Theorem 7.2. *Let X be a Banach space, $Y \subset X$ be a nonempty compact convex subset and $f : Y \to X$ be a continuous operator. We suppose that:*

$$x \in Y,\ f(x) \in X \setminus Y \text{ imply } \|x - f(x)\| > \underset{y \in Y}{Min}\ \|y - f(x)\|.$$

Then, $F_f \neq \emptyset$.

We observe that Theorem 7.1 implies Theorem 7.2.
References: [43], [72], [28], [105], [106], ...

8. A conjecture

The above considerations give rise to the following:

The conjecture of the generalized retracts. *Let X be a Banach space, $Y \subset X$ be a subset with nonempty boundary and $f : Y \to X$ be a nonself operator. Then, each boundary condition (Leray-Schauder, Rothe, inwardness, outwardness, ...) on f implies the existence of a generalized retract of f.*

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