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SOME REMARKS ON MULTIDIMENSIONAL FIXED POINT THEOREMS

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Abstract. In this paper, we show that most of the multidimensional (including coupled, tripled, quadrupled) fixed point theorems in the context of (ordered) metric spaces are, in fact, immediate consequences of well-known fixed point theorems in the literature.

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1. Introduction

In the last decades, there has been a great interest in searching for fixed point theorems in ordered metric spaces involving a contractivity condition which holds for all points that are related by the partial order (see e.g. [1]-[44]). One of the most interesting papers in this trend was reported by Bhaskar and Lakshmikantham [25] in 2006. In this pioneer paper, in order to solve some certain type of periodic boundary value problems, the authors proved the existence and uniqueness of a coupled fixed point of a certain class of operators in partially ordered metric spaces by introducing the *mixed monotone property.* Following this result, a number of papers have been published dealing with the notions of *coupled fixed/coincidence point* and their applications (see e.g. [1, 2, 4, 5, 6, 8, 9, 10, 11, 15, 16, 17, 18, 22, 23, 24, 32, 33, 34, 35, 36, 41, 42, 43, 44]). In 2011, Berinde and Borcut [12] introduced the concept of *tripled fixed point* and proved some related theorems (see also [7, 13, 14]). The last remarkable result of this trend was given by Roldán *et al.* [39] by introducing the notion of *multidimensional fixed point* which covers the concepts of coupled, tripled, quadruple fixed point etc. Regarding the wide potential application of multidimensional fixed point results in various branches of Mathematics, many authors have attracted attention to this subject and have reported some interesting results.

Very recently, Samet *et al.* [40] and Agarwal *et al.* [3] have proved that coupled fixed point results can be obtained as easy consequences of fixed point results in dimension one in the setup of metric spaces and G-metric spaces, respectively. In this paper, following their techniques, we present different contractivity conditions in order to guarantee the existence (and, in some cases, uniqueness) of multidimensional fixed points. We show that our results extend, generalize and improve very recent theorems in the related literature on the theory of multidimensional (coupled, tripled, quadruple, etc.) fixed point. This paper can be considered as a continuation of the papers [3] and [40].

2. Preliminaries

Preliminaries and notation about coincidence points can also be found in [39]. We use abbreviation MS for metric spaces. Let n be a positive integer. Henceforth, X will denote a non-empty set and X^n will denote the product space $X \times X \times \overset{n}{\cdots} \times X$. Throughout this manuscript, m and k will denote non-negative integers and $i, j, s \in \{1, 2, \ldots, n\}$. Unless otherwise stated, "for all m" will mean "for all $m \geq 0$" and "for all i" will mean "for all $i \in \{1, 2, \ldots, n\}$ ".

Definition 2.1. ([25]) An ordered MS $(X, d, \preccurlyeq)$ is said to have the *sequential monotone property* if it verifies:

- **(i):** If $\{x_m\}$ is a non-decreasing sequence and $\{x_m\} \overset{d}{\to} x$, then $x_m \preccurlyeq x$ for all m.
- **(ii):** If $\{y_m\}$ is a non-increasing sequence and $\{y_m\} \overset{d}{\to} y$, then $y_m \succcurlyeq y$ for all m.

Henceforth, fix a partition $\{\mathsf{A}, \mathsf{B}\}$ of $\Lambda_n = \{1, 2, \ldots, n\}$, that is, $\mathsf{A} \cup \mathsf{B} = \Lambda_n$ and $\mathsf{A} \cap \mathsf{B} = \varnothing$ such that A and B are non-empty sets. We will denote:

$$\Omega_{\mathsf{A},\mathsf{B}} = \{\sigma : \Lambda_n \to \Lambda_n : \sigma(\mathsf{A}) \subseteq \mathsf{A} \text{ and } \sigma(\mathsf{B}) \subseteq \mathsf{B}\} \quad \text{and}$$
$$\Omega'_{\mathsf{A},\mathsf{B}} = \{\sigma : \Lambda_n \to \Lambda_n : \sigma(\mathsf{A}) \subseteq \mathsf{B} \text{ and } \sigma(\mathsf{B}) \subseteq \mathsf{A}\}.$$

If $(X, \preccurlyeq)$ is a partially ordered space, $x, y \in X$ and $i \in \Lambda_n$, we will use the following notation:

$$x \preccurlyeq_i y \quad \Leftrightarrow \quad \begin{cases} x \preccurlyeq y, & \text{if } i \in \mathsf{A}, \\ x \succcurlyeq y, & \text{if } i \in \mathsf{B}. \end{cases}$$

Consider on the product space X^n the following partial order:
for $\mathsf{X} = (x_1, x_2, \ldots, x_n),\ \mathsf{Y} = (y_1, y_2, \ldots, y_n) \in X^n$,

$$\mathsf{X} \sqsubseteq \mathsf{Y} \quad \Leftrightarrow \quad x_i \preccurlyeq_i y_i, \text{ for all } i. \tag{2.1}$$

We say that two points X and Y are *comparable* if $\mathsf{X} \sqsubseteq \mathsf{Y}$ or $\mathsf{X} \sqsupseteq \mathsf{Y}$.

Proposition 2.2. *If* $\mathsf{X} \sqsubseteq \mathsf{Y}$*, it follows that*

$$(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) \sqsubseteq (y_{\sigma(1)}, y_{\sigma(2)}, \ldots, y_{\sigma(n)}) \quad \text{if } \sigma \in \Omega_{\mathsf{A},\mathsf{B}},$$
$$(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) \sqsupseteq (y_{\sigma(1)}, y_{\sigma(2)}, \ldots, y_{\sigma(n)}) \quad \text{if } \sigma \in \Omega'_{\mathsf{A},\mathsf{B}}.$$

Proof. Suppose that $x_i \preccurlyeq_i y_i$ for all i. Hence $x_{\sigma(i)} \preccurlyeq_{\sigma(i)} y_{\sigma(i)}$ for all i. Fix $\sigma \in \Omega_{\mathsf{A},\mathsf{B}}$. If $i \in \mathsf{A}$, then $\sigma(i) \in \mathsf{A}$, so $x_{\sigma(i)} \preccurlyeq_{\sigma(i)} y_{\sigma(i)}$ implies that $x_{\sigma(i)} \preccurlyeq y_{\sigma(i)}$, which means that $x_{\sigma(i)} \preccurlyeq_i y_{\sigma(i)}$. If $i \in \mathsf{B}$, then $\sigma(i) \in \mathsf{B}$, so $x_{\sigma(i)} \preccurlyeq_{\sigma(i)} y_{\sigma(i)}$ implies that $x_{\sigma(i)} \succcurlyeq y_{\sigma(i)}$, which means that $x_{\sigma(i)} \preccurlyeq_i y_{\sigma(i)}$. In any case, if $\sigma \in \Omega_{\mathsf{A},\mathsf{B}}$, then $x_{\sigma(i)} \preccurlyeq_i y_{\sigma(i)}$ for all i. It follows that $(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) \sqsubseteq (y_{\sigma(1)}, y_{\sigma(2)}, \ldots, y_{\sigma(n)})$.

Now fix $\sigma \in \Omega'_{\mathsf{A},\mathsf{B}}$. If $i \in \mathsf{A}$, then $\sigma(i) \in \mathsf{B}$, so $x_{\sigma(i)} \preccurlyeq_{\sigma(i)} y_{\sigma(i)}$ implies that $x_{\sigma(i)} \succcurlyeq y_{\sigma(i)}$, which means that $x_{\sigma(i)} \succcurlyeq_i y_{\sigma(i)}$. If $i \in \mathsf{B}$, then $\sigma(i) \in \mathsf{A}$, so $x_{\sigma(i)} \preccurlyeq_{\sigma(i)} y_{\sigma(i)}$ implies that $x_{\sigma(i)} \preccurlyeq y_{\sigma(i)}$, which means that $x_{\sigma(i)} \succcurlyeq_i y_{\sigma(i)}$. □

Let $F : X^n \to X$ be a mapping.

Definition 2.3. ([39]) Let $(X, \preccurlyeq)$ be a partially ordered space. We say that F has the *mixed monotone property (w.r.t.* $\{\mathsf{A}, \mathsf{B}\}$*)* if F is monotone non-decreasing in arguments of A and monotone non-increasing in arguments of B, i.e., for all $x_1, x_2, \ldots, x_n, y, z \in X$ and all i,

$$y \preccurlyeq z \quad \Rightarrow \quad F(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) \preccurlyeq_i F(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n).$$

Henceforth, let $\sigma_1, \sigma_2, \ldots, \sigma_n : \Lambda_n \to \Lambda_n$ be n mappings from Λ_n into itself and let Υ be the n-tuple $(\sigma_1, \sigma_2, \ldots, \sigma_n)$. The main aim of this paper is to study the following class of multidimensional fixed points.

Definition 2.4. ([39]) A point $(x_1, x_2, \ldots, x_n) \in X^n$ is called a Υ-*fixed point of the mapping* F if

$$F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \ldots, x_{\sigma_i(n)}) = x_i \qquad \text{for all } i. \tag{2.2}$$

If one represents a mapping $\sigma : \Lambda_n \to \Lambda_n$ throughout its ordered image, i.e., $\sigma = (\sigma(1), \sigma(2), ..., \sigma(n))$, then

- Gnana-Bhaskar and Lakshmikantham's coupled fixed points occur when $n = 2$, $\sigma_1 = \tau = (1, 2)$ and $\sigma_2 = (2, 1)$;
- Berinde and Borcut's tripled fixed points are associated to $n = 3$, $\sigma_1 = \tau = (1, 2, 3), \sigma_2 = (2, 1, 2)$ and $\sigma 2 = (3, 2, 1)$;
- Karapınar's quadruple fixed points are considered when $n = 4$, $\sigma_1 = \tau = (1, 2, 3, 4), \sigma_2 = (2, 3, 4, 1), \sigma_3 = (3, 4, 1, 2)$ and $\sigma_4 = (4, 1, 2, 3)$.
- Berzig and Samet's multidimensional fixed points are given when $\mathsf{A} = \{1, 2, \ldots, m\}$ and $\mathsf{B} = \{m + 1, m + 2, \ldots, n\}$.

For more details see [39].

2.1. **Unidimensional fixed point results.** Based on the classical Banach theorem, in which a non-negative constant less than one plays a key role, many results have been proved replacing such constant by an appropriate mapping, depending on the contractivity condition. Some of these kinds of families are the following.

$\Phi_1 = \left\{\phi : [0,\infty) \to [0,\infty) : \phi \text{ is continuous, non-decreasing and } \phi^{-1}(\{0\}) = \{0\}\right\}.$

$\Phi_2 = \{\phi : [0,\infty) \to [0,\infty) : \phi(t) < t \text{ and } \lim_{r\to t^+} \phi(r) < t \text{ for all } t > 0\}.$

$\Phi_3 = \{\phi : [0,\infty) \to [0,\infty) \ : \phi \text{ is continuous and } \phi(t) < t \text{ for all } t > 0\}.$

$\Phi_4 = \{\phi : [0,\infty) \to [0,\infty) : \lim_{r\to t} \phi(r) > 0 \text{ for all } t > 0 \text{ and } \lim_{t\to 0^+} \varphi(t) = 0\}.$

$\Phi_5 = \{\phi \in \Phi_1 : \phi(s+t) \le \phi(s) + \phi(t) \text{ for all } s,t \in [0,\infty)\}$

Remark 2.5. If $\psi \in \Phi_2$ and $N > 0$, define ψ_N by $\psi_N(t) = N \cdot \psi(t/N)$ for all $t \ge 0$. Then $\psi_N \in \Phi_2$.

Remark 2.6. If $\phi \in \Phi_3$ and $N > 0$, define ϕ_N by $\phi_N(t) = N \cdot \phi(t/N)$ for all $t \ge 0$. Then $\phi_N \in \Phi_3$.

Proposition 2.7. *If $\phi \in \Phi_3$, then there exists a nondecreasing mapping $\varphi \in \Phi_3$ such that $\phi \le \varphi$.*

In [38], Ran and Reurings proved the following fixed point theorem.

Theorem 2.8 (Ran and Reurings [38])**.** *Let $(X, \preccurlyeq)$ be an ordered set endowed with a metric d and $T : X \to X$ be a given mapping. Suppose that the following conditions hold:*

- **(a):** *(X, d) is complete.*
- **(b):** *T is nondecreasing (w.r.t. $\preccurlyeq$).*
- **(c):** *T is continuous.*
- **(d):** *There exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$.*
- **(e):** *There exists a constant $k \in (0,1)$ such that for all $x, y \in X$ with $x \succcurlyeq y$,*

$$d(Tx, Ty) \le kd(x, y).$$

Then T has a fixed point. Moreover, if for all $(x,y) \in X^2$ there exists $z \in X$ such that $x \preccurlyeq z$ and $y \preccurlyeq z$, we obtain uniqueness of the fixed point.

Nieto and Rodríguez-López [37] slightly modified the hypothesis of the previous result obtaining the following theorem.

Theorem 2.9 (Nieto and Rodríguez-López [37])**.** *Let $(X, \preccurlyeq)$ be an ordered set endowed with a metric d and $T : X \to X$ be a given mapping. Suppose that the following conditions hold:*

- **(a):** *(X, d) is complete.*
- **(b):** *T is nondecreasing (w.r.t. $\preccurlyeq$).*
- **(c):** *If a nondecreasing sequence $\{x_m\}$ in X converges to a some point $x \in X$, then $x_m \preccurlyeq x$ for all m.*

(d): *There exists* $x_0 \in X$ *such that* $x_0 \preccurlyeq Tx_0$.

(e): *There exists a constant* $k \in (0,1)$ *such that for all* $x, y \in X$ *with* $x \succcurlyeq y$,

$$d(Tx, Ty) \leq kd(x, y).$$

Then T *has a fixed point. Moreover, if for all* $(x, y) \in X^2$ *there exists* $z \in X$ *such that* $x \preccurlyeq z$ *and* $y \preccurlyeq z$, *we obtain uniqueness of the fixed point.*

3. Auxiliary results

In this section we show some properties that we will use in the proofs of our main results.

Lemma 3.1. *Let* (X, d) *be a MS and define* $D_n, \Delta_n : X^n \times X^n \to [0, \infty)$, *for all* $A = (a_1, a_2, \ldots, a_n), B = (b_1, b_2, \ldots, b_n) \in X^n$, *by*

$$D_n(A, B) = \max_{1 \leq i \leq n} d(a_i, b_i) \quad \text{and} \quad \Delta_n(A, B) = \sum_{i=1}^{n} d(a_i, b_i).$$

Then D_n *and* Δ_n *are complete metrics on* X^n.

Actually, both metrics are equivalent since $D_n \leq \Delta_n \leq nD_n$.

Theorem 3.2. *Let* $(X, d, \preccurlyeq)$ *be a partially ordered MS and let* $F : X^n \to X$ *be a mapping. Let* $\Upsilon = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ *be a* n*-tuple of mappings from* $\{1, 2, \ldots, n\}$ *into itself verifying* $\sigma_i \in \Omega_{\mathsf{A},\mathsf{B}}$ *if* $i \in \mathsf{A}$ *and* $\sigma_i \in \Omega'_{\mathsf{A},\mathsf{B}}$ *if* $i \in \mathsf{B}$. *Define* $F_\Upsilon : X^n \to X^n$, *for all* $x_1, x_2, \ldots, x_n \in X$, *by*

$$\begin{aligned} F_\Upsilon(x_1, x_2, \ldots, x_n) =& (F(x_{\sigma_1(1)}, x_{\sigma_1(2)}, \ldots, x_{\sigma_n(n)}), F(x_{\sigma_2(1)}, x_{\sigma_2(2)}, \ldots, x_{\sigma_2(n)}), \\ & \ldots, F(x_{\sigma_n(1)}, x_{\sigma_n(2)}, \ldots, x_{\sigma_n(n)})). \end{aligned}$$

(1) *If* F *has the mixed monotone property, then* F_Υ *is monotone nondecreasing w.r.t. the partial order* $\sqsubseteq$ *on* X^n *given by (2.1).*
(2) *If* F *is continuous (w.r.t.* D_n *or* Δ_n*), then* F_Υ *is also continuous (w.r.t.* D_n *or* Δ_n*).*
(3) *A point* $(x_1, x_2, \ldots, x_n) \in X^n$ *is a* Υ*-fixed point of* F *if, and only if,* $(x_1, x_2, \ldots, x_n)$ *is a fixed point of* F_Υ.

Proof. **(1)** Suppose that $(gx_1, gx_2, \ldots, gx_n) \leq (gy_1, gy_2, \ldots, gy_n)$, that is, $gx_i \preccurlyeq_i gy_i$ for all i. Since F has the mixed g-monotone property, it is not difficult to prove that, for all $a_1, a_2, \ldots, a_n \in X$

$$\begin{aligned} F(a_1, a_2, \ldots, a_{j-1}, \overset{(j)}{x_i}, a_{j+1}, \ldots, a_n) &\preccurlyeq F(a_1, a_2, \ldots, a_{j-1}, \overset{(j)}{y_i}, a_{j+1}, \ldots, a_n), \\ & \text{if } i, j \in \mathsf{A} \text{ or } i, j \in \mathsf{B}; \\ F(a_1, a_2, \ldots, a_{j-1}, \overset{(j)}{x_i}, a_{j+1}, \ldots, a_n) &\succcurlyeq F(a_1, a_2, \ldots, a_{j-1}, \overset{(j)}{y_i}, a_{j+1}, \ldots, a_n), \\ & \text{if } i \in \mathsf{A}, j \in \mathsf{B} \text{ or } i \in \mathsf{B}, j \in \mathsf{A}. \end{aligned}$$

Suppose that $i \in \mathsf{A}$. Therefore $\sigma_i \in \Omega_{\mathsf{A},\mathsf{B}}$, that is, $\sigma_i(\mathsf{A}) \subseteq \mathsf{A}$ and $\sigma_i(\mathsf{B}) \subseteq \mathsf{B}$. Therefore $j \in \mathsf{A}$ if and only if $\sigma_i(j) \in \mathsf{A}$ and the same holds swapping A by B. In this case,

$$\begin{aligned} F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, x_{\sigma_i(3)}, \ldots, x_{\sigma_i(n)}) \preccurlyeq \quad & (\text{either } 1, \sigma_i(1) \in \mathsf{A} \text{ or } 1, \sigma_i(1) \in \mathsf{B}) \\ \preccurlyeq F(y_{\sigma_i(1)}, x_{\sigma_i(2)}, x_{\sigma_i(3)}, \ldots, x_{\sigma_i(n)}) \preccurlyeq \quad & (\text{either } 2, \sigma_i(2) \in \mathsf{A} \text{ or } 2, \sigma_i(2) \in \mathsf{B}) \\ \preccurlyeq F(y_{\sigma_i(1)}, y_{\sigma_i(2)}, x_{\sigma_i(3)}, \ldots, x_{\sigma_i(n)}) \preccurlyeq \quad & (\text{either } 3, \sigma_i(3) \in \mathsf{A} \text{ or } 3, \sigma_i(3) \in \mathsf{B}) \\ \preccurlyeq \ldots \preccurlyeq & \\ \preccurlyeq F(y_{\sigma_i(1)}, y_{\sigma_i(2)}, y_{\sigma_i(3)}, \ldots, y_{\sigma_i(n)}), & \end{aligned}$$

that is, $F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, x_{\sigma_i(3)}, \ldots, x_{\sigma_i(n)}) \preccurlyeq_i F(y_{\sigma_i(1)}, y_{\sigma_i(2)}, y_{\sigma_i(3)}, \ldots, y_{\sigma_i(n)})$. Now suppose that $i \in \mathsf{B}$. Therefore $\sigma_i \in \Omega'_{\mathsf{A},\mathsf{B}}$, that is, $\sigma_i(\mathsf{A}) \subseteq \mathsf{B}$ and $\sigma_i(\mathsf{B}) \subseteq \mathsf{A}$. Therefore $j \in \mathsf{A}$ if and only if $\sigma_i(j) \in \mathsf{B}$, and viceversa. In this case,

$$\begin{aligned} F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, x_{\sigma_i(3)}, \ldots, x_{\sigma_i(n)}) \succcurlyeq \quad & (\text{either } 1 \in \mathsf{A}, \sigma_i(1) \in \mathsf{B} \text{ or } 1 \in \mathsf{B}, \sigma_i(1) \in \mathsf{A}) \\ \succcurlyeq F(y_{\sigma_i(1)}, x_{\sigma_i(2)}, x_{\sigma_i(3)}, \ldots, x_{\sigma_i(n)}) \succcurlyeq & \\ & (\text{either } 2 \in \mathsf{A}, \sigma_i(2) \in \mathsf{B} \text{ or } 2 \in \mathsf{B}, \sigma_i(2) \in \mathsf{A}) \\ \succcurlyeq F(y_{\sigma_i(1)}, y_{\sigma_i(2)}, x_{\sigma_i(3)}, \ldots, x_{\sigma_i(n)}) \succcurlyeq & \\ & (\text{either } 3 \in \mathsf{A}, \sigma_i(3) \in \mathsf{B} \text{ or } 3 \in \mathsf{B}, \sigma_i(3) \in \mathsf{A}) \\ \succcurlyeq \ldots \succcurlyeq F(y_{\sigma_i(1)}, y_{\sigma_i(2)}, y_{\sigma_i(3)}, \ldots, y_{\sigma_i(n)}), & \end{aligned}$$

that is, $F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, x_{\sigma_i(3)}, \ldots, x_{\sigma_i(n)}) \preccurlyeq_i F(y_{\sigma_i(1)}, y_{\sigma_i(2)}, y_{\sigma_i(3)}, \ldots, y_{\sigma_i(n)})$ also holds. Hence,

$$F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, x_{\sigma_i(3)}, \ldots, x_{\sigma_i(n)}) \preccurlyeq_i F(y_{\sigma_i(1)}, y_{\sigma_i(2)}, y_{\sigma_i(3)}, \ldots, y_{\sigma_i(n)}) \quad \text{for all } i,$$

and, consequently, $F_\Upsilon(x_1, x_2, \ldots, x_n) \sqsubseteq F_\Upsilon(y_1, y_2, \ldots, y_n)$.

(2) It is an straightforward exercise.

(3) $(x_1, x_2, \ldots, x_n) \in X^n$ is a Υ-coincidence point of F and g if, and only if, $x_i = F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, x_{\sigma_i(3)}, \ldots, x_{\sigma_i(n)})$ for all i, that is, $F_\Upsilon(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_n)$. □

4. Main results

Throughout this section, let $(X, d, \preccurlyeq)$ be an ordered MS, let $\Upsilon = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ be a n-tuple of mappings from $\{1, 2, \ldots, n\}$ into itself such that $\sigma_i \in \Omega_{\mathsf{A},\mathsf{B}}$ if $i \in \mathsf{A}$ and $\sigma_i \in \Omega'_{\mathsf{A},\mathsf{B}}$ if $i \in \mathsf{B}$, and let $F : X^n \to X$ be a mapping. Consider the following conditions:

(I): (X, d) is complete.

(II): F has the mixed monotone property.

(III): F is continuous or $(X, d, \preccurlyeq)$ has the sequential monotone property.

(IV): There exist $x_0^1, x_0^2, \ldots, x_0^n \in X$ verifying

$$x_0^i \preccurlyeq_i F(x_0^{\sigma_i(1)}, x_0^{\sigma_i(2)}, \ldots, x_0^{\sigma_i(n)}) \quad \text{for all } i.$$

In some cases, we will also use an additional hypothesis.

(V): For all i, the mapping σ_i is a permutation of $\{1, 2, \ldots, n\}$.

This last condition implies that, for all i and all $\mathsf{X}, \mathsf{Y} \in X^n$,

$$\max_{1 \leq j \leq n} d(x_{\sigma_i(j)}, y_{\sigma_i(j)}) = \max_{1 \leq j \leq n} d(x_j, y_j) = D_n(\mathsf{X}, \mathsf{Y}) \quad \text{and}$$
$$\sum_{j=1}^{n} d(x_{\sigma_i(j)}, y_{\sigma_i(j)}) = \sum_{j=1}^{n} d(x_j, y_j) = \Delta_n(\mathsf{X}, \mathsf{Y}). \tag{4.1}$$

4.1. **Roldán, Martínez and Roldán's multidimensional fixed point results.** In 2012, Roldán *et al.* proved the following theorem in order to show sufficient conditions to ensure the existence of Φ-coincidence points (we particularize it in the case of Υ-coincidence points taking τ as the identity mapping on $\{1, 2, \ldots, n\}$ and g as the identity mapping on X).

Theorem 4.1 (Roldán *et al.* [39])**.** *Under hypothesis (I)-(IV), assume that there exists $k \in [0, 1)$ verifying:*

$$d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq k \max_{1 \leq i \leq n} d(x_i, y_i) \tag{4.2}$$

for which $x_i \preccurlyeq_i y_i$ for all i. Then F has, at least, one Υ-fixed point.

We shall prove the following result.

Theorem 4.2. *Theorem 4.1 follows from Theorems 2.8 and 2.9.*

Proof. Consider the product space $Y = X^n$ provided with the metric D_n (as in Lemma 3.1) and the partial order $\sqsubseteq$ on Y given by (2.1). Then $(Y, D_n, \sqsubseteq)$ is a complete ordered MS. Since F has the mixed monotone property, item 1 of Theorem 3.2 shows that $F_\Upsilon : Y \to Y$ is nondecreasing w.r.t. $\sqsubseteq$. By item 2 of Theorem 3.2, if F is continuous, then F_Υ is also continuous. If $\mathsf{x}_0 = (x_0^1, x_0^2, \ldots, x_0^n) \in Y$, then condition (IV) is equivalent to $\mathsf{x}_0 \sqsubseteq F_\Upsilon \mathsf{x}_0$. Recall that, by Proposition 2.2, given $\mathsf{X} = (x_1, x_2, \ldots, x_n), \mathsf{Y} = (y_1, y_2, \ldots, y_n) \in Y$ such that $\mathsf{X} \sqsubseteq \mathsf{Y}$, the points $(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \ldots, x_{\sigma_i(n)})$ and $(y_{\sigma_i(1)}, y_{\sigma_i(2)}, \ldots, y_{\sigma_i(n)})$ are comparable by $\sqsubseteq$. Therefore, (4.2) can be applied to these points, and it follows that

$$\begin{aligned}
D_n(F_\Upsilon \mathsf{X}, F_\Upsilon \mathsf{Y}) &= \max_{1 \leq i \leq n} d\left(F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \ldots, x_{\sigma_i(n)}), F(y_{\sigma_i(1)}, y_{\sigma_i(2)}, \ldots, y_{\sigma_i(n)})\right) \\
&\leq \max_{1 \leq i \leq n} \left[k \max_{1 \leq j \leq n} d(x_{\sigma_i(j)}, y_{\sigma_i(j)})\right] = k \max_{1 \leq j \leq n} \left[\max_{1 \leq i \leq n} d(x_{\sigma_i(j)}, y_{\sigma_i(j)})\right] \\
&\leq k \max_{1 \leq j \leq n} \left[\max_{1 \leq i \leq n} d(x_i, y_i)\right] = k \max_{1 \leq i \leq n} d(x_i, y_i) = k\, D_n(\mathsf{X}, \mathsf{Y}).
\end{aligned}$$

Theorems 2.8 and 2.9 imply that F_Υ has a fixed point, which is a Υ-fixed point of F by item 3 of Theorem 3.2. □

Uniqueness of the fixed point also follows from Theorems 2.8 and 2.9.

Theorem 4.3. *Under the hypothesis of Theorem 4.1, assume that for all $A, B \in X^n$ there exists $U \in X^n$ such that $A \sqsubseteq U$ and $B \sqsubseteq U$. Then F has a unique Υ-fixed point.*

In order to assure the uniqueness of the fixed point, notice that the previous result shows a sufficient condition which is only related to the partial order $\preccurlyeq$ on X (and its extension $\sqsubseteq$ to X^n). It is not difficult to prove that a similar property of uniqueness may be included in the results we will present throughout this paper. However, for brevity, we will not write this part.

4.2. **Bhaskar and Lakshmikantham's coupled fixed point results.** We present the following multidimensional extension of the main result of Bhaskar and Lakshmikantham [25] using a similar argument of which we have showed in the previous subsection.

Corollary 4.4. *Under hypothesis (I)-(IV), assume that there exists $k \in [0,1)$ verifying:*

$$d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \frac{k}{n} \sum_{i=1}^{n} d(x_i, y_i)$$

for which $x_i \preccurlyeq_i y_i$ for all i. Then F has, at least, one Υ-fixed point.

Proof. It is obvious since

$$\begin{aligned} d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) &\leq \frac{k}{n} \sum_{i=1}^{n} d(x_i, y_i) \leq \frac{k}{n} \sum_{i=1}^{n} \max_{1 \leq j \leq n} d(x_j, y_j) \\ &= k \max_{1 \leq j \leq n} d(x_j, y_j). \end{aligned}$$

□

4.3. **Berinde's coupled fixed point results.** We extend [Theorem 3, Berinde [8]] in the following sense.

Theorem 4.5. *Under hypothesis (I)-(IV), assume that there exists $k \in [0,1)$ verifying:*

$$\sum_{i=1}^{n} d(F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \ldots, x_{\sigma_i(n)}), F(y_{\sigma_i(1)}, y_{\sigma_i(2)}, \ldots, y_{\sigma_i(n)})) \leq k \sum_{j=1}^{n} d(x_j, y_j)$$

for which $x_i \preccurlyeq_i y_i$ for all i. Then F has, at least, one Υ-fixed point.

Proof. It follows repeating the argument of Theorems 4.2 using the complete metric Δ_n and taking into account that, for all $\mathsf{X}, \mathsf{Y} \in X^n$ such that $\mathsf{X} \sqsubseteq \mathsf{Y}$, we have that

$$\begin{aligned} \Delta_n(F_\Upsilon \mathsf{X}, F_\Upsilon \mathsf{Y}) &= \sum_{i=1}^{n} d\left(F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \ldots, x_{\sigma_i(n)}), F(y_{\sigma_i(1)}, y_{\sigma_i(2)}, \ldots, y_{\sigma_i(n)})\right) \\ &\leq k \sum_{j=1}^{n} d(x_j, y_j) = k\ \Delta_n(\mathsf{X}, \mathsf{Y}). \end{aligned}$$

□

Notice that this result implies that [Theorem 3, Berinde [8]] follows from Theorems 2.8 and 2.9.

4.4. **Berzig and Samet's multidimensional fixed point results and Berinde and Borcut's tripled fixed point results.** Berzig and Samet [15] extended the main result of Berinde and Borcut [12] in the setting of multidimensional mappings.

Both results are special cases of the following theorem, which was also established in [39, Corollary 16].

Theorem 4.6. *Under hypothesis (I)-(IV), assume that there exists* $\alpha_1, \alpha_2, \ldots, \alpha_n \in [0,1)$ *such that* $\alpha_1 + \alpha_2 + \ldots + \alpha_n < 1$ *verifying:*

$$d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \le \sum_{j=1}^{n} \alpha_j d(x_j, y_j)$$

for which $x_i \preccurlyeq_i y_i$ *for all* i*. Then* F *has, at least, one* Υ*-fixed point.*

Proof. Let $k = \alpha_1 + \alpha_2 + \ldots + \alpha_n \in [0,1)$. Then for all $\mathsf{X}, \mathsf{Y} \in X^n$ such that $\mathsf{X} \sqsubseteq \mathsf{Y}$, we have that

$$D_n(F_\Upsilon \mathsf{X}, F_\Upsilon \mathsf{Y}) = \max_{1 \le i \le n} d\left(F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \ldots, x_{\sigma_i(n)}), F(y_{\sigma_i(1)}, y_{\sigma_i(2)}, \ldots, y_{\sigma_i(n)})\right)$$

$$\le \max_{1 \le i \le n} \sum_{j=1}^{n} \alpha_j d(x_{\sigma_i(j)}, y_{\sigma_i(j)}) \le \sum_{j=1}^{n} \left(\alpha_j \max_{1 \le i \le n} d(x_{\sigma_i(j)}, y_{\sigma_i(j)})\right)$$

$$\le \sum_{j=1}^{n} \left(\alpha_j \; D_n(\mathsf{X}, \mathsf{Y})\right) = \left(\sum_{j=1}^{n} \alpha_j\right) D_n(\mathsf{X}, \mathsf{Y}) = k \; D_n(\mathsf{X}, \mathsf{Y}).$$ □

4.5. **Ćirić, Cakić, Rajović and Ume's multidimensional fixed point results.** The following theorem is a multidimensional version of Theorem 2.2 in [19].

Theorem 4.7. *Under hypothesis (I)-(IV), assume that there exists* $\phi \in \Phi_3$ *verifying:*

$$d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \le \phi\left(\max_{1 \le i \le n} d(x_i, y_i)\right)$$

for which $x_i \preccurlyeq_i y_i$ *for all* i*. Then* F *has, at least, one* Υ*-fixed point.*

Proof. By Proposition 2.7, there exists a nondecreasing mapping $\varphi \in \Phi_3$ such that $\phi \le \varphi$. Since φ is non-decreasing, $\varphi(\max(s_1, s_2, \ldots, s_n)) = \max(\varphi(s_1), \varphi(s_2), \ldots, \varphi(s_n))$ for all $s_1, s_2, \ldots, s_n \ge 0$. Then

$$\begin{aligned} D_n(F_\Upsilon \mathsf{X}, F_\Upsilon \mathsf{Y}) &= \max_{1 \le i \le n} d\left(F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \ldots, x_{\sigma_i(n)}), F(y_{\sigma_i(1)}, y_{\sigma_i(2)}, \ldots, y_{\sigma_i(n)})\right) \\ &\le \max_{1 \le i \le n} \phi\left(\max_{1 \le j \le n} d(x_{\sigma_i(j)}, y_{\sigma_i(j)})\right) \le \max_{1 \le i \le n} \varphi\left(\max_{1 \le j \le n} d(x_{\sigma_i(j)}, y_{\sigma_i(j)})\right) \\ &\le \max_{1 \le i \le n} \varphi\left(D_n(\mathsf{X}, \mathsf{Y})\right) = \varphi\left(D_n(\mathsf{X}, \mathsf{Y})\right). \end{aligned}$$

Theorem 2.2 in [19] guarantees that F has a Υ-fixed point. □

Corollary 4.8. *Under hypothesis (I)-(IV), assume that there exists $\phi \in \Phi_3$ verifying:*

$$d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \phi\left(\frac{1}{n}\sum_{i=1}^{n} d(x_i, y_i)\right)$$

for which $x_i \preccurlyeq_i y_i$ for all i. Then F has, at least, one Υ-fixed point.

Proof. It follows from the previous theorem since

$$d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \phi\left(\frac{1}{n}\sum_{i=1}^{n} d(x_i, y_i)\right)$$

$$\leq \varphi\left(\frac{1}{n}\sum_{i=1}^{n} d(x_i, y_i)\right) \leq \varphi\left(\max_{1\leq i\leq n} d(x_i, y_i)\right). \qquad \square$$

Corollary 4.9. *Under hypothesis (I)-(IV), assume that there exists $\phi \in \Phi_3$ verifying:*

$$d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \frac{1}{n}\ \phi\left(\sum_{i=1}^{n} d(x_i, y_i)\right)$$

for which $x_i \preccurlyeq_i y_i$ for all i. Then F has, at least, one Υ-fixed point.

Proof. It follows from Remark 2.6 and Theorem 4.7 since

$$d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \frac{1}{n}\ \phi\left(\sum_{i=1}^{n} d(x_i, y_i)\right) \leq \frac{1}{n}\ \varphi\left(\sum_{i=1}^{n} d(x_i, y_i)\right)$$

$$\leq \frac{1}{n}\ \varphi\left(n\ \max_{1\leq i\leq n} d(x_i, y_i)\right) = \varphi_n\left(\max_{1\leq i\leq n} d(x_i, y_i)\right). \qquad \square$$

4.6. **Lakshmikantham and Ciric's coupled fixed point results.** We shall prove the following result as a generalization of the main result of Lakshmikantham and Ćirić [34].

Theorem 4.10. *Under hypothesis (I)-(V), assume that there exists $\psi \in \Phi_2$ verifying:*

$$d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \psi\left(\frac{1}{n}\sum_{j=1}^{n} d(x_j, y_j)\right)$$

for which $x_i \preccurlyeq_i y_i$ for all i. Then F has, at least, one Υ-fixed point.

Proof. By Remark 2.5, the mapping ψ_n, given by $\psi_n(t) = n\psi(t/n)$ for all $t \geq 0$, is in Φ_2. Therefore, applying (4.1), for all $\mathsf{X}, \mathsf{Y} \in X^n$ such that $\mathsf{X} \sqsubseteq \mathsf{Y}$ we have that

$$\Delta_n(F_\Upsilon \mathsf{X}, F_\Upsilon \mathsf{Y}) = \sum_{i=1}^{n} d\left(F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \ldots, x_{\sigma_i(n)}), F(y_{\sigma_i(1)}, y_{\sigma_i(2)}, \ldots, y_{\sigma_i(n)})\right)$$

$$\leq \sum_{j=1}^{n} \psi\left(\frac{1}{n}\sum_{j=1}^{n} d(x_{\sigma_i(j)}, y_{\sigma_i(j)})\right) = n\ \psi\left(\frac{1}{n}\ \Delta_n(\mathsf{X}, \mathsf{Y})\right) = \psi_1\left(\Delta_n(\mathsf{X}, \mathsf{Y})\right). \qquad \square$$

4.7. **Harjani, López and Sadarangani's coupled fixed point results.** We shall prove the following result which is an extention of the main result of Harjani *et al.*[26].

Theorem 4.11. *Under hypothesis (I)-(V), assume that there exist $\psi,\varphi\in\Phi_1$ verifying:*

$$\psi(d(F(x_1,x_2,\ldots,x_n),F(y_1,y_2,\ldots,y_n)))\leq\psi\left(\max_{1\leq i\leq n}d(x_i,y_i)\right)-\varphi\left(\max_{1\leq i\leq n}d(x_i,y_i)\right)$$

for which $x_i\preccurlyeq_i y_i$ for all i. Then F has, at least, one Υ-fixed point.

Proof. Applying (4.1), for all $\mathsf{X},\mathsf{Y}\in X^n$ such that $\mathsf{X}\sqsubseteq\mathsf{Y}$ we have that

$$\begin{aligned}
D_n(F_\Upsilon\mathsf{X},F_\Upsilon\mathsf{Y})&=\max_{1\leq i\leq n}d\left(F(x_{\sigma_i(1)},x_{\sigma_i(2)},\ldots,x_{\sigma_i(n)}),F(y_{\sigma_i(1)},y_{\sigma_i(2)},\ldots,y_{\sigma_i(n)})\right)\\
&\leq\max_{1\leq i\leq n}\left[(\psi-\varphi)\left(\max_{1\leq j\leq n}d(x_{\sigma_i(j)},y_{\sigma_i(j)})\right)\right]=\max_{1\leq i\leq n}\left[(\psi-\varphi)\left(D_n(\mathsf{X},\mathsf{Y})\right)\right]\\
&=\psi\left(D_n(\mathsf{X},\mathsf{Y})\right)-\varphi\left(D_n(\mathsf{X},\mathsf{Y})\right).
\end{aligned}$$

Thesis follows from [Theorem 2.1, Harjani and Sadarangani [27]]. □

4.8. **Karapınar's quadruple fixed point results.** We prove the following theorem as an extension of the main result of Karapınar-Luong [28].

Theorem 4.12. *Under hypothesis (I)-(V), assume that there exist $\phi\in\Phi_4$ and $\psi\in\Phi_5$ verifying:*

$$\psi(d(F(x_1,x_2,\ldots,x_n),F(y_1,y_2,\ldots,y_n)))\leq\frac{1}{n}\ \psi\left(\sum_{j=1}^{n}d(x_j,y_j)\right)-\varphi\left(\sum_{j=1}^{n}d(x_j,y_j)\right)$$

for which $x_i\preccurlyeq_i y_i$ for all i. Then F has, at least, one Υ-fixed point.

Proof. Applying (4.1) and the contractivity condition, for all $\mathsf{X},\mathsf{Y}\in X^n$ such that $\mathsf{X}\sqsubseteq\mathsf{Y}$ we have that

$$\begin{aligned}
&\psi(\Delta_n(F_\Upsilon\mathsf{X},F_\Upsilon\mathsf{Y}))\\
&\quad=\psi\left(\sum_{i=1}^{n}d\left(F(x_{\sigma_i(1)},x_{\sigma_i(2)},\ldots,x_{\sigma_i(n)}),F(y_{\sigma_i(1)},y_{\sigma_i(2)},\ldots,y_{\sigma_i(n)})\right)\right)\\
&\quad\leq\sum_{i=1}^{n}\ \psi(d\left(F(x_{\sigma_i(1)},x_{\sigma_i(2)},\ldots,x_{\sigma_i(n)}),F(y_{\sigma_i(1)},y_{\sigma_i(2)},\ldots,y_{\sigma_i(n)})\right))\\
&\quad\leq\sum_{i=1}^{n}\left(\frac{1}{n}\psi-\varphi\right)\left(\sum_{j=1}^{n}d(x_{\sigma_i(j)},y_{\sigma_i(j)})\right)=\sum_{i=1}^{n}\left(\frac{1}{n}\psi-\varphi\right)(\Delta_n(\mathsf{X},\mathsf{Y}))\\
&\quad=(\psi-n\varphi)\left(\Delta_n(\mathsf{X},\mathsf{Y})\right)=\psi(\Delta_n(\mathsf{X},\mathsf{Y}))-n\varphi\left(\Delta_n(\mathsf{X},\mathsf{Y})\right).
\end{aligned}$$

Since $n\varphi\in\Phi_4$, the result immediately follows from [Theorem 2.1, Harjani and Sadarangani [27]]. □

Finally, we remark that the techniques used in this paper might be applied in order to prove other coupled, tripled, quadrupled, n-tupled fixed point theorems in the setup of various abstract spaces, e.g., partial metric spaces, cone metric spaces, fuzzy metric spaces, b-metric spaces, etc.

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