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LEVITIN-POLYAK WELL-POSEDNESS OF THE SYSTEM OF WEAK GENERALIZED VECTOR EQUILIBRIUM PROBLEMS

JIAN-WEN PENG* AND JEN-CHIH YAO**

*School of Mathematics, Chongqing Normal University
Chongqing 400047, P. R. China
E-mail: jwpeng2008@gmail.com

**Center for General Education, Kaohsiung Medical University
Kaohsiung 80708, Taiwan, ROC
E-mail: yaojc@math.nsysu.edu.tw

Abstract. In this paper, we introduce two types of the Levitin-Polyak well-posedness for the system of weak generalized vector equilibrium problems. By using the gap function of the system of weak generalized vector equilibrium problems, we establish the equivalent relationship between the two types of Levitin-Polyak well-posedness of the system of weak generalized vector equilibrium problems and the corresponding well-posednesses of the minimization problems. We also present some metric characterizations for the two types of the Levitin-Polyak well-posedness of the system of weak generalized vector equilibrium problems. The results in this paper are new and extend some known results in the literature.

Key Words and Phrases: System of weak generalized vector variational inequalities, Levitin-Polyak well-posedness, Levitin-Polyak approximating solution sequence, gap function, metric characterizations.
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1. Introduction

Well-posedness plays a crucial role in the stability theory for optimization problems, which guarantees that, for an approximating solution sequence, there exists a subsequence which converges to a solution. The study of well-posedness for scalar minimization problems started from Tykhonov [37] and Levitin and Polyak [25]. Since then, various notions of well-posedness for scalar minimization problems have been defined and studied in [10, 13, 19, 23, 30, 39] and the references therein. Recent studies on various notions of well-posedness for vector optimization problems can be found in [4, 9, 17, 18, 21, 29, 31]. It is worth noting that the recent study for various types of well-posedness have been generalized to variational inequalities [11, 12, 22, 27, 32], generalized variational inequalities [6, 20], generalized vector variational inequalities [38], equilibrium problems [28], vector equilibrium problems [26], [35], generalized

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vector equilibrium problems [34], system of vector quasi-equilibrium problems [36] and many other problems.

On the other hand, Pang [33], Cohen and Chaplais [8], Bianchi [5] and Ansari and Yao [2] considered a system of scalar variational inequalities, which is related to the traffic equilibrium problem, the spatial equilibrium problem, the Nash equilibrium, and the general equilibrium programming problem. Inspired by the study of vector variational inequalities by Giannessi [14], Ansari, Schaible and Yao [1] considered a system of weak vector equilibrium problems and a system of vector variational inequality problems and obtained their existence results. Hou, Yu and Chen [15] considered a system of weak generalized vector equilibrium problems and obtained its existence results.

In this paper, we are interested in investigating two types of the Levitin-Polyak well-posedness for a system of weak generalized vector equilibrium problems which contains those mathematical models in [1, 2, 4-6, 8-14, 17-23, 25-39] as special cases. The paper is organized as follows: In section 2, we introduce the definitions of two types of Levitin-Polyak well-posedness for the system of weak generalized vector equilibrium problems. In section 3, the lower semi-continuous property of the gap functions of the system of weak generalized vector equilibrium problems and the equivalent relationship between two types of the Levitin-Polyak well-posednesses of the system of weak generalized vector equilibrium problems and the corresponding well-posednesses of the minimization problems are established. In section 4, some metric characterizations for the Levitin-Polyak well-posedness for the system of weak generalized vector equilibrium problems are obtained. The results in this paper generalize and extend some known results in [20, 22, 26, 28, 34, 35, 38] and the references therein.

2. Preliminaries

Throughout this paper, without other specification, let I be a countable index set and for each $i \in I$, let (E_i, d_i) be a metric space, X_i be a nonempty closed subset of E_i, Z_i be a nonempty subset of a topological vector spaces F_i, let Y_i be a locally convex Hausdorff topological vector space. Let $E = \prod_{i\in I} E_i$, $X = \prod_{i\in I} X_i$, $E_{-i} = \prod_{j\in I\setminus i} E_i$ and $X_{-i} = \prod_{j\in I\setminus i} X_i$. For each fixed $i \in I$ and $x \in E$, we write $x = (x_i, x_{-i}) = (x_i)_{i\in I}$,$z = (z_i, z_{-i}) = (z_i)_{i\in I}$, where x_i and x_{-i} denote the projection of x onto E_i and E_{-i}, respectively. Let $d(x, y) = \sup_{i\in I} \frac{d_i(x_i,y_i)}{1+d_i(x_i,y_i)}$ for all $x, y \in E$. It is clear that (E, d) is a metric space. For each $i \in I$, let $C_i : E \to 2^{Y_i}$ be a set-valued map such that for any $x \in E$, $C_i(x)$ is a proper, pointed, closed and convex cone in Y_i with nonempty interior $intC_i(x)$, $e_i : E \to Y_i$ be a continuous vector-valued map and satisfies that for any $x \in E$, $e_i(x) \in intC_i(x)$, $T_i : E \to 2^{Z_i}$ be a set-valued map, $f_i : E \times Z_i \times E_i \to Y_i$ and $\varphi_i : E \times E_i \to Y_i$ be continuous vector-valued function. We are interesting in the following system of weak generalized vector equilibrium problems (in short, (SWGVEP)) introduced and studied by Hou, Yu and Chen [15]: Finding $\bar{x} \in X$ such that for each $i \in I$,

$$\exists \bar{z}_i \in T_i(\bar{x}) : \ f_i(\bar{x}, \bar{z}_i, y_i) \notin -intC_i(\bar{x}), \forall\ y_i \in X_i.$$

The following problems are special cases of (SWGVEP):

(1) If for each $i \in I$, $T_i(x) = \{\bar{z}_i\}$ for all $x \in X$, define a function $\varphi_i : E \times E_i \to Y_i$ as $\varphi_i(x, y_i) = f_i(x, \bar{z}_i, y_i)$, $\forall (x, y_i) \in E \times E_i$, then (SWGVEP) reduces to the system of weak vector equilibrium problems (in short, (SWVEP)): Finding $\bar{x} \in X$ such that for each $i \in I$,

$$\varphi_i(\bar{x}, y_i) \notin -intC_i(\bar{x}), \forall\ y_i \in X_i.$$

If for each $i \in I$, $Y_i \equiv Y$ and $C_i(x) \equiv C$ for all $x \in X$, then (SWVEP) becomes the system of vector equilibrium problems introduced by Ansari, Schaible and Yao [1], which contains the system of scalar variational inequalities in [2, 5, 8, 33], the system of vector variational inequalities, the system of vector optimization problems, the Nash equilibrium problem with vector-valued functions in [1] as special cases.

(2) If the index set I is singleton, then (SWGVEP) and (SWVEP), respectively, reduces to the generalized vector equilibrium problem (in short, (GVEP)) studied in [34] and the vector equilibrium problem (in short, (GVEP)) studied in [26, 35].

We denote by Ω and Ω_1, respectively, the set of solutions of (SWGVEP) and (SWVEP). Let (P, d) be a metric space, $P_1 \subset P$ and $x \in P$. We denote by $d(x, P_1) = \inf\{d(x, p) : p \in P_1\}$ the distance function from the point $x \in P$ to the set P_1.

Definition 2.1. (i) A sequence $\{x^n\} \subset E$ is called a type I Levitin-Polyak (LP in short) approximating solution sequence of (SWGVEP) if there exists a sequence $\{\epsilon^n\} \subseteq \mathbb{R}_+ = \{r \in \mathbb{R} : r \geq 0\}$ with $\epsilon^n \to 0$ and for each $i \in I$, there exists $z_i^n \in T_i(x^n)$ such that

$$d_i(x_i^n, X_i) \leq \epsilon^n, \tag{2.1}$$

and

$$f_i(x^n, z_i^n, y_i) + \epsilon^n e_i(x^n) \notin -intC_i(x^n), \forall y_i \in X_i. \tag{2.2}$$

(ii) A sequence $\{x^n\} \subset E$ is called a type II LP approximating solution sequence of (SWGVEP) if there exists a sequence $\{\epsilon^n\} \subseteq \mathbb{R}_+$ with $\epsilon^n \to 0$ and for each $i \in I$, there exists $z_i^n \in T_i(x^n)$ such that (2.1) and (2.2) hold; and there exists $i_0 \in I$, for any $z_{i_0} \in T_{i_0}(x^n)$, $\exists \omega_{i_0}(n, z_{i_0}) \in X_{i_0}$, such that

$$f_{i_0}(x^n, z_{i_0}, w_{i_0}(n, z_{i_0})) - \epsilon^n e_{i_0}(x^n) \in -C_{i_0}(x^n). \tag{2.3}$$

(iii) A sequence $\{x^n\} \subset E$ is called a type I LP approximating solution sequence of (SWVEP) if there exists a sequence $\{\epsilon^n\} \subseteq \mathbb{R}_+$ with $\epsilon^n \to 0$ such that for each $i \in I$, (2.1) holds and

$$\varphi_i(x^n, y_i) + \epsilon^n e_i(x^n) \notin -intC_i(x^n), \forall y_i \in X_i. \tag{2.4}$$

(iv) A sequence $\{x^n\} \subset E$ is called a type II LP approximating solution sequence of (SWVEP) if there exists a sequence $\{\epsilon^n\} \subseteq \mathbb{R}_+$ with $\epsilon^n \to 0$ such that for each $i \in I$, (2.1) and (2.4) hold; and there exist $i_0 \in I$, and $\omega_{i_0}^n \in X_{i_0}$, such that

$$\varphi_{i_0}(x^n, w_{i_0}^n) - \epsilon^n e_{i_0}(x^n) \in -C_{i_0}(x^n). \tag{2.5}$$

Definition 2.2. (SWGVEP) is said to be type I (resp. type II) LP well-posed if $\Omega \neq \emptyset$ and for every type I (resp. type II) LP approximating solution sequence $\{x^n\}$ for (SWGVEP), there exists a subsequence $\{x^{n_j}\}$ of $\{x^n\}$ and $\bar{x} \in \Omega$ such that $x^{n_j} \to \bar{x}$.

Definition 2.3. (SWVEP) is said to be type I (resp. type II) LP well-posed if $\Omega_1 \neq \emptyset$ and for every type I (resp. type II) LP approximating solution sequence

$\{x^n\}$ for (SWVEP), there exists a subsequence $\{x^{n_j}\}$ of $\{x^n\}$ and $\bar{x} \in \Omega_1$ such that $x^{n_j} \to \bar{x}$.

Remark 2.1. (i) Both type I LP well-posedness and type II LP well-posedness for (SWGVEP) imply that the solution set Ω is nonempty and compact.

(ii) It is clear that any type II LP approximating solution sequence of (SWGVEP) is a type I LP approximating solution sequence of (SWGVEP), thus the type I LP well-posedness of (SWGVEP) implies the type II LP well-posedness of (SWGVEP).

(iii) If the index set I is singleton, then by (i) and (ii) of Definition 2.1, Definition 2.2, respectively, we can easily obtain the definitions of type I (type II) approximating solution sequence and the type I (type II) LP well-posedness of (GVEP) in [34]; by (iii) and (iv) of Definition 2.1, Definition 2.3, respectively, we can easily obtain the definitions of type I (type II) approximating solution sequence and the type I (type II) LP well-posedness of (VEP) in [35, 26]. Thus, Definitions 2.1-2.3 generalize, extend and unify the corresponding one in [6, 26, 28, 34, 35, 38] and the references therein.

Definition 2.5. [3, 16, 24] Let Z_1, Z_2 be two metric spaces. A set-valued map F from Z_1 to 2^{Z_2} is

(i) closed, on $Z_3 \subseteq Z_1$, if for any sequence $\{x_n\} \subseteq Z_3$ with $x_n \to x$ and $y_n \in F(x_n)$ with $y_n \to y$, one has $y \in F(x)$;

(ii) lower semicontinuous (*l.s.c.* in short) at $x \in Z_1$, if $\{x_n\} \subseteq Z_1, x_n \to x$, and $y \in F(x)$ imply that there exists a sequence $\{y_n\} \subseteq Z_2$ satisfying $y_n \to y$ such that $y_n \in F(x_n)$ for n sufficiently large. If F is *l.s.c.* at each point of Z_1, we say that F is *l.s.c.* on Z_1.

(iii) upper semicontinuous (*u.s.c.* in short) at $x \in Z_1$, if for any neighborhood V of $F(x)$, there exists a neighborhood U of x such that $F(z) \subseteq V, \forall z \in U$. If F is *u.s.c.* at each point of Z_1, we say that F is *u.s.c.* on Z_1.

(iv) continuous at $x \in Z_1$, if it is both *u.s.c.* and *l.s.c.* at x. If F is continuous at each point of Z_1, we say that F is continuous on Z_1.

From the proof of Theorem 2.1 in [7], we can obtain the following result:

Lemma 2.1. Let X and Y be two locally convex Hausdorff topological vector spaces, $C : X \to 2^Y$ a set-valued map such that, for any $x \in X$, $C(x)$ is proper, pointed, closed and convex cone in Y with nonempty interior $intC(x)$. Let $e : X \to Y$ be a continuous vector-valued map and satisfies that for any $x \in X$, $e(x) \in intC(x)$. Define a set-valued map $W : X \to 2^Y$ by $W(x) = Y / - intC(x)$, for all $x \in X$. The nonlinear scalarization function $\xi_e : X \times Y \to \mathbb{R}$ is defined as follows

$$\xi_e(x, y) =: \inf\{\lambda \in \mathbb{R} : y \in \lambda e(x) - C(x)\}.$$

Then

(i) If W is *u.s.c.*, then $\xi_e(.,.)$ is upper semi-continuous on $X \times Y$;

(ii) If C is *u.s.c.*, then $\xi_e(.,.)$ is lower semi-continuous on $X \times Y$.

3. The equivalent relations

In this section, the lower semi-continuous property of the gap functions of (SWGVEP) and the equivalent relationship of the two types of LP well-posedness of (SWGVEP) and the corresponding well-posednesses of minimization problems will be presented.

Definition 3.1. A function $\phi : E \to \mathbb{R} \cup \{+\infty\}$ is said to be a gap function for (SWGVEP), if

(i) $\phi(x) \geq 0, \forall x \in X$;

(ii) $\phi(x^*) = 0$ and $x^* \in X$ if and only if $x^* \in \Omega$.

Theorem 3.1. Assume that for each $i \in I$,

(i) the set-valued map T_i is compact-valued on X;

(ii) for each $x \in X$ and $z_i \in T_i(x)$, $f_i(x, z_i, x_i) \in -\partial C_i(x)$, where ∂C is the topological boundary of C.

(iii) for any $(x, y_i) \in E \times E_i$, the vector-valued function $z_i \mapsto f_i(x, z_i, y_i)$ is continuous.

Then the function $\phi : E \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\phi(x) = \sup_{i \in I} \inf_{z_i \in T_i(x)} \sup_{y_i \in X_i} \{-\xi_{e_i}(x, f_i(x, z_i, y_i))\}. \tag{3.1}$$

is a gap function of (SWGVEP).

Proof. It follows from Proposition 2.3 in [7] and (ii) that for each $i \in I$, for any $x \in X$ and $z_i \in T_i(x)$, $\xi_{e_i}(x, f_i(x, z_i, x_i)) = 0$ and so $\sup_{y_i \in X_i} \{-\xi_{e_i}(x, f_i(x, z_i, y_i))\} \geq 0$. Hence, for any $x \in X$ and $i \in I$,

$$\inf_{z_i \in T_i(x)} \sup_{y_i \in X_i} \{-\xi_{e_i}(x, f_i(x, z_i, y_i))\} \geq 0,$$

and it follows from (3.1) that

$$\phi(x) \geq 0, \forall x \in X. \tag{3.2}$$

If $\phi(\bar{x}) = 0$ and $\bar{x} \in X$, then for each $i \in I$,

$$\inf_{z_i \in T_i(\bar{x})} \sup_{y_i \in X_i} \{-\xi_{e_i}(\bar{x}, f_i(\bar{x}, z_i, y_i))\} \leq 0.$$

Then, for each $i \in I$, there exist $0 < \epsilon_n \to 0$ and $z_i^n \in T_i(\bar{x})$ such that

$$\sup_{y_i \in X_i} \{-\xi_{e_i}(\bar{x}, f_i(\bar{x}, z_i^n, y_i))\} \leq \epsilon_n,$$

which implies that

$$\xi_{e_i}(\bar{x}, f_i(\bar{x}, z_i^n, y_i)) \geq -\epsilon_n, \forall y_i \in X_i.$$

It follows from Proposition 2.3 in [7] that for each $i \in I$, $z_i^n \in T_i(\bar{x})$ and

$$f_i(\bar{x}, z_i^n, y_i) + \epsilon_n e_i(\bar{x}) \notin -intC_i(\bar{x}), \forall y_i \in X_i. \tag{3.3}$$

By the compactness of $T_i(\bar{x})$, there exist a sequence $\{z_i^{n_j}\}$ of $\{z_i^n\}$ and some $\bar{z}_i \in T_i(\bar{x})$ such that $z_i^{n_j} \to \bar{z}_i$. It follows from (iii) and (3.3) that for each $i \in I$, $f_i(\bar{x}, \bar{z}_i, y_i) \notin -intC_i(\bar{x}), \forall y_i \in X_i$. and thus, $\bar{x} \in \Omega$.

Conversely, if $\bar{x} \in \Omega$, then $\bar{x} \in X$ such that for each $i \in I$, $\bar{z}_i \in T_i(\bar{x})$ and $f_i(\bar{x}, \bar{z}_i, y_i) \notin -intC_i(\bar{x}), \forall y_i \in X_i$. It follows from Proposition 2.3 in [7] that for each $i \in I$, $\bar{z}_i \in T_i(\bar{x})$ and $\sup_{y_i \in X_i} \{-\xi_{e_i}(\bar{x}, f_i(\bar{x}, \bar{z}_i, y_i))\} \leq 0$. And so for each $i \in I$,

$$\inf_{z_i \in T_i(\bar{x})} \sup_{y_i \in X_i} \{-\xi_{e_i}(\bar{x}, f_i(\bar{x}, \bar{z}_i, y_i))\} \leq 0.$$

It follows from (3.1) that

$$\phi(\bar{x}) \leq 0. \tag{3.4}$$

Now (3.2) and (3.4) imply that $\phi(\bar{x}) = 0$. This completes the proof.

Now we present an important property of the gap function for (SWGVEP) as follows:

Lemma 3.1. Assume that for each $i \in I$,

(i) the set-valued map T_i is $u.s.c$ and compact-valued on E;

(ii) the set-valued map $W_i : E \to 2^{Y_i}$ defined by $W_i(x) = Y_i \setminus -intC_i(x)$ is $u.s.c.$;

(iii) for any $y_i \in E_i$, the vector-valued function $(x, z_i) \mapsto f_i(x, z_i, y_i)$ is continuous.

Then the function ϕ defined by (3.1) is lower semi-continuous from E to $\mathbb{R} \cup +\{\infty\}$.

Proof. First, it is obvious that $\phi(x) > -\infty, \forall x \in E$. Otherwise, suppose that there exists $x_0 \in E$ such that $\phi(x_0) = -\infty$. Then, for each $i \in I$, there exist $z_i^n \in T_i(x_0)$ and $\{M_n\} \subset \mathbb{R}_+$ with $M_n \to +\infty$ such that

$$\sup_{y_i \in X_i} \{-\xi_{e_i}(x_0, f_i(x_0, z_i^n, y_i))\} \leq -M_n. \tag{3.5}$$

Hence,

$$\xi_{e_i}(x_0, f_i(x_0, z_i^n, y_i)) \geq M_n, \forall y_i \in X_i.$$

By the compactness of $T_i(x_0)$, there exist a sequence $\{z_i^{n_j}\}$ of $\{z_i^n\}$ and some $z_i \in T_i(x_0)$ such that $\{z_i^{n_j}\} \to z_i$. It follows from (ii) and Lemma 2.1 that ξ_{e_i} is upper semi-continuous, and so

$$\xi_{e_i}(x_0, f_i(x_0, z_i, y_i)) \geq \limsup_{j \to +\infty} \xi_{e_i}(x_0, f_i(x_0, z_i^{n_j}, y_i)) = +\infty, \forall y_i \in X_i$$

which is impossible, since $\xi_{e_i}(.,.)$ is a finite function on $E \times Y_i$.

Second, we show that ϕ is lower semi-continuous on E. Let $t \in \mathbb{R}$, suppose that $\{x^n\} \subset E$ satisfies $\phi(x^n) \leq t, \quad \forall n$ and $x^n \to x_0$. Then, for each $i \in I$ and $\forall n$,

$$\inf_{z_i \in T_i(x^n)} \sup_{p_i \in X_i} \{-\xi_{e_i}(x^n, f_i(x^n, z_i, p_i))\} \leq t \tag{3.6}$$

Then, for each $i \in I$, there exist $0 < \epsilon_n \to 0$ and $z_i^n \in T_i(x^n)$ such that

$$\sup_{p_i \in X_i} \{-\xi_{e_i}(x^n, f_i(x^n, z_i^n, p_i))\} \leq t + \epsilon_n,$$

which implies that

$$\xi_{e_i}(x^n, f_i(x^n, z_i^n, p_i)) \geq -t - \epsilon_n, \forall p_i \in X_i.$$

It follows from proposition 2.3 in [7] that for each $i \in I$, there exists $z_i^n \in T_i(x^n)$ such that

$$f_i(x^n, z_i^n, p_i) + (t + \epsilon_n)e_i(x^n) \notin -intC_i(x^n), \forall p_i \in X_i. \tag{3.7}$$

Moreover, by the upper semi-continuity and compactness of T_i, there exist a sequence $\{z_i^{n_j}\}$ of $\{z_i^n\}$ and some $z_i \in T_i(x_0)$ such that $z_i^{n_j} \to z_i$. By taking the limit in (3.7) (with n replaced by n_j), we know that for each $i \in I$,

$$f_i(x_0, z_i, y_i) + te_i(x_0) \notin -intC_i(x_0), \forall y_i \in X_i.$$

It follows from proposition 2.3 in [7] that for each $i \in I$, $z_i \in T_i(x_0)$ and

$$\xi_{e_i}(x_0, f_i(x_0, z_i, y_i)) \geq -t, \forall y_i \in X_i. \tag{3.8}$$

It follows from (3.8) that for each $i \in I$, $\inf_{z_i \in T_i(x_0)} \sup_{y_i \in X_i} \{-\xi_{e_i}(x_0, f_i(x_0, z_i, y_i))\} \leq t$. Let ϕ be defined by (3.1), then $\phi(x_0) \leq t$. Thus ϕ is lower semi-continuous on X. This completes the proof.

In order to relate the LP well-posedness of (SWGVEP) with that of constrained minimization problems, we consider the LP well-posedness of the following general constrained program:

$$(P) \begin{cases} \min \phi(x) \\ s.t.\ x \in X, \end{cases}$$

where $\phi : E \to \mathbb{R} \cup \{\infty\}$ is proper and lower semicontinuous. The optimal set and optimal value of (P) are denoted by $\bar{\Omega}$ and $\bar{v}$, respectively.

Definition 3.2. [19] A sequence $\{x^n\} \subset E$ is called a type I LP minimizing sequence for (P) if

$\limsup_{n \to +\infty} \phi(x^n) \leq \bar{v}$, (3.9)

and for each $i \in I$,

$d_i(x_i^n, X_i) \to 0$. (3.10)

Definition 3.3. [19] A sequence $\{x^n\} \subset E$ is called a type II LP minimizing sequence for (P) if

$$\lim_{n \to +\infty} \phi(x^n) = \bar{v} \tag{3.11}$$

and for each $i \in I$, (3.10) holds.

Definition 3.4. [19] (P) is said to be type I LP well-posed if $\bar{\Omega} \neq \emptyset$, and for any type I LP minimizing sequence $\{x^n\}$ for (P), there exists a subsequence $\{x^{n_j}\}$ of $\{x^n\}$ and $\bar{x} \in \bar{\Omega}$ such that $x^{n_j} \to \bar{x}$.

Definition 3.5. [19] (P) is said to be type II LP well-posed if $\bar{\Omega} \neq \emptyset$, and for any type II LP minimizing sequence $\{x^n\}$ for (P), there exists a subsequence $\{x^{n_j}\}$ of $\{x^n\}$ and $\bar{x} \in \bar{\Omega}$ such that $x^{n_j} \to \bar{x}$.

The following results reveals the relationship between the two types of LP well-posedness of (SWGVEP) and those of (P).

Theorem 3.2. Assume that for each $i \in I$,

(i) the set-valued map T_i is compact-valued on E;

(ii) for each $x \in X$, $z_i \in T_i(x)$, $f_i(x, z_i, x_i) \in -\partial C_i(x)$;

(iii) the set-valued map $W_i : E \to 2^{Y_i}$ defined by $W_i(x) = Y_i \setminus -intC_i(x)$ is $u.s.c$;

(iv) for any $(x, y_i) \in E \times E_i$, the vector-valued function $z_i \mapsto f_i(x, z_i, y_i)$ is continuous.

Then, (SWGVEP) is type I LP well-posed if and only if (P) is type I LP well-posed with $\phi(x)$ defined by (3.1).

Proof. Let $\phi(x)$ be defined by (3.1). From Theorem 3.1 $\bar{x} \in X$ is a solution of (SWGVEP) if and only if $\bar{x}$ is an optimal solution of (P) with $\bar{v} = \phi(\bar{x}) = 0$.

We first prove the sufficency. Assume that $\{x^n\}$ is a type I LP approximating solution sequence of (SWGVEP), then there exists $\{\epsilon^n\} \subseteq R_+$ with $\epsilon^n \to 0$ such that (2.1) and (2.2) hold, for each $i \in I$, it follows from (2.1) that (3.10) holds. By (2.2) and Proposition 2.3 in [7], we know that for each $i \in I$, there exists $z_i^n \in T_i(x^n)$ such that $\xi_{e_i}(x^n, f_i(x_n, z_i^n, y_i)) \geq -\epsilon^n, \forall y_i \in X_i$. It follows from (3.1) that $\phi(x^n) \leq \epsilon^n$, which implies that (3.9) holds with $\bar{v} = 0$. Hence, $\{x^n\}$ is a type I LP approximating solution sequence of (P). It follows from the type I LP well-posedness of (P) that (SWGVEP) is type I LP well-posed.

Now we show the converse, let $\{x^n\}$ is a type I LP approximating solution sequence of (P), then for each $i \in I$, (3.9) and (3.10) hold. It follows from (3.10) that there exists $\{\epsilon^n\} \subseteq \mathbb{R}_+$ with $\epsilon^n \to 0$ such that for each $i \in I$, (2.1) holds. Furthermore, by (3.9), we have that

$$\phi(x^n) = \sup_{i\in I} \inf_{z_i\in T_i(x^n)} \sup_{y_i\in X_i} \{-\xi_{e_i}(x^n, f_i(x^n, z_i, y_i))\} \leq \epsilon^n.$$

Thus, for each $i \in I$, we have

$$\inf_{z_i\in T_i(x^n)} \sup_{y_i\in X_i} \{-\xi_{e_i}(x^n, f_i(x^n, z_i, y_i))\} \leq \epsilon^n.$$

By (iii) and Lemma 2.1, we know that $\xi_{e_i}(.,.)$ is upper semi-continuous. It follows from the compactness of $T_i(x^n)$ that for each $i \in I$, $\exists z_i^n \in T_i(x^n)$, such that

$$\xi_{e_i}(x^n, f_i(x^n, z_i^n, y_i)) \geq -\epsilon^n, \forall y_i \in X_i,$$

which implies that (2.2) holds. Thus, $\{x^n\}$ is a type I LP approximating solution sequence of (SWGVEP). It follows from the type I LP well-posedness of (SWGVEP) that (P) is type I LP well-posed. This completes the proof.

Theorem 3.3. Assume that all conditions in Theorem 3.2 are satisfied, then,

(a) The type II LP well-posedness of (P) with $\phi(x)$ defined by (3.1) implies the type II LP well-posedness of (SWGVEP).

(b) Moreover, if I is a finite index set and (SWGVEP) is type II LP well-posed, then (P) is type II LP well-posed with $\phi(x)$ defined by (3.1).

Proof. Let $\phi(x)$ be defined by (3.1). From Theorem 3.1 $\bar{x} \in X$ is a solution of (SWGVEP) if and only if $\bar{x}$ is an optimal solution of (P) with $\bar{v} = \phi(\bar{x}) = 0$.

We first prove (a). Assume that $\{x^n\}$ is type II LP approximating solution sequence of (SWGVEP), then there exists $\{\epsilon^n\} \subseteq \mathbb{R}_+$ with $\epsilon^n \to 0$ such that for each $i \in I$, $z_i^n \in T_i(x^n)$, (2.1) and (2.2) hold, and there exist $i_0 \in I$, for any $z_{i_0} \in T_{i_0}(x^n)$, $\exists \omega_{i_0}(n, z_{i_0}) \in X_{i_0}$ such that (2.3) holds. From (2.1), (2.2) and the proof of Theorem 3.2, we know that (3.9) holds with $\bar{v} = 0$ and for each $i \in I$, (3.10) holds. By (2.3) and Proposition 2.3 in [7], we get $\xi_{e_{i_0}}(x^n, f_{i_0}(x^n, z_{i_0}, \omega_{i_0}(n, z_{i_0}))) \leq \epsilon^n$. Thus,

$$\inf_{z_{i_0}\in T_{i_0}(x^n)} \sup_{\nu_{i_0}(n,z_{i_0})\in X_{i_0}} \{-\xi_{e_{i_0}}(x^n, f_{i_0}(x^n, z_{i_0}, \nu_{i_0}(n, z_{i_0})))\} \geq -\epsilon^n.$$

It follows from (3.1) that $\phi(x^n) \geq -\epsilon^n$, which implies that

$$\liminf_{n\to+\infty} \phi(x^n) \geq 0. \quad (3.12)$$

Combining (3.9) together with (3.12), we know that (3.11) holds with $\bar{v} = 0$. Hence, $\{x^n\}$ is a type II LP approximating solution sequence of (P). It follows from the type II LP well-posedness of (P) that (SWGVEP) is type II LP well-posed.

Now we prove (b). Assume that $\{x^n\}$ is a type II LP approximating solution sequence of (P), then (3.11) holds and for each $i \in I$, (3.10) holds. Hence, there exist $\{\epsilon^n\} \subseteq \mathbb{R}_+$ with $\epsilon^n \to 0$ such that for each $i \in I$, (2.1) holds, and for n sufficiently large, the following formula holds

$$-\frac{\epsilon^n}{2} \le \phi(x^n) \le \frac{\epsilon^n}{2}. \ (3.13)$$

By the right side of (3.13) and the proof of Theorem 3.2, we know that for each $i \in I$, $\exists z_i^n \in T_i(x^n)$, such that

$$f_i(x^n, z_i^n, y_i) + \epsilon^n e_i(x^n) \notin -intC_i(x^n), \forall y_i \in X_i.$$

The left side of (3.13) can be rewritten as,

$$\phi(x^n) = \sup_{i\in I} \inf_{z_i\in T_i(x^n)} \sup_{y_i\in X_i} \{-\xi_{e_i}(x^n, f_i(x^n, z_i, y_i))\} \ge -\frac{\epsilon^n}{2}.$$

Since I is a finite index sets, there exists $i_0 \in I$, such that

$$\inf_{z_{i_0}\in T_{i_0}(x^n)} \sup_{y_{i_0}\in X_{i_0}} \{-\xi_{e_{i_0}}(x^n, f_{i_0}(x^n, z_{i_0}, y_{i_0}))\} \ge -\frac{\epsilon^n}{2}.$$

Thus, for all $z_{i_0} \in T_{i_0}(x^n)$, we have

$$\beta := \sup_{y_{i_0}\in X_{i_0}} \{-\xi_{e_{i_0}}(x^n, f_{i_0}(x^n, z_{i_0}, y_{i_0}))\} \ge -\frac{\epsilon^n}{2}.$$

It follows from the definition of supremum that there exists $\omega_{i_0}(n, v_{i_0}) \in X_{i_0}$ such that

$$-\xi_{e_{i_0}}(x^n, f_{i_0}(x^n, z_{i_0}, \omega_{i_0}(n, z_{i_0}))) \ge \beta - \frac{\epsilon^n}{2} \ge -\epsilon^n.$$

It follows from proposition 2.3 in [7] that $f_{i_0}(x^n, z_{i_0}, \omega_{i_0}(n, z_{i_0})) - \epsilon^n e_{i_0}(x^n) \in -C_{i_0}(x^n)$. Thus, $\{x^n\}$ is a type II LP approximating solution sequence of (SWGVEP). It follows from the type II LP well-posedness of (SWGVEP) that (P) is the type II LP well-posed. This completes the proof.

If for each $i \in I$, $T_i(x) = \{\bar{z_i}\}$ for all $x \in E$, $\varphi_i(x, y_i) = f_i(x, \bar{z}, y_i)$, $\forall(x, y_i) \in E \times E_i$, then by Theorem 3.1 and Lemma 3.1, Theorems 3.2 and 3.3, respectively, we can obtain the following new results:

Corollary 3.1. (a) If for any $x \in X$ and for each $i \in I$, $\varphi_i(x, x_i) \in -\partial C_i(x)$, then the function $\phi_1 : E \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\phi_1(x) = \sup_{i\in I} \sup_{y_i\in X_i} \{-\xi_{e_i}(x, \varphi_i(x, y_i))\}. \tag{3.14}$$

is a gap function of (SWVEP).

(b) If for each $i \in I$, the set-valued map $W_i : E \to 2^{Y_i}$ defined by $W_i(x) = Y_i \setminus -intC_i(x)$ is *u.s.c.*, and for any $y_i \in E_i$, the vector-valued function $x \mapsto \varphi_i(x, y_i)$ is continuous, then the function ϕ_1 defined by (3.14) is lower semi-continuous from E to $\mathbb{R} \cup +\{\infty\}$.

Corollary 3.2. Assume that for each $i \in I$, the set-valued map T_i is compact-valued on E; for each $x \in X$, $\varphi_i(x, x_i) \in -\partial C_i(x)$; the set-valued map $W_i : E \to 2^{Y_i}$ defined by $W_i(x) = Y_i \setminus -intC_i(x)$ is *u.s.c.* Then,

(a) (SWVEP) is type I LP well-posed if and only if (P) is type I LP well-posed with $\phi(x)$ replaced by $\phi_1(x)$ defined by (3.14).

(b) The type II LP well-posedness of (P) with ϕ replaced by ϕ_1 defined by (3.14) implies the type II LP well-posedness of (SWVEP).

(c) Moreover, if I is a finite index set and (SWVEP) is type II LP well-posed, then (P) is type II LP well-posed with ϕ replaced by ϕ_1 defined by (3.14).

Remark 3.1. If the index set I is singleton, then by Theorem 3.1, Lemma 3.1, Theorems 3.2 and 3.3, respectively, we recover Propositions 2.2, 2.3 and Theorem 2.1 in [34]; by Corollary 3.2 and (a) and (b) of Corollary 3.1, respectively, we recover Theorem 4.1, Propositions 4.1 and 4.2 in [26].

4. Metric characterizations for the LP well-posedness of (SWGVEP)

In this section, we give some metric characterizations for the two types of LP well-posedness of (SWGVEP).

Definition 4.1. (SWGVEP) is said to be type I generalized Tykhonov well-set (resp. type II LP well-set) if $\Omega \neq \emptyset$ and for any type I LP approximating solution sequence (resp. type II LP approximating solution sequence) $\{x^n\}$ for (SWGVEP), we have $d(x^n, \Omega) \to 0$ as $n \to \infty$.

We can easily obtain the equivalent relations between the two types of generalized Tykhonov well-posedness and the corresponding types of LP well set of (SWGVEP) as follows:

Proposition 4.1. (SWGVEP) is type I LP well-posed (resp. type II LP well-posed) if and only if (SWGVEP) is type I LP well-set (resp. type II LP well-set) and Ω is compact.

Now we consider the Kuratowski measure of noncompactness for a nonempty subset A of X (see [24]) defined by

$\alpha(A) = \inf\{\epsilon > 0 : A \subset \cup_{i=1}^{n} A_i, \text{ for every } A_i, diamA_i < \epsilon\}$,

where diamA_i is the diameter of A_i defined by

$$\text{diam}A_i = \sup\{d(x_1, x_2) : x_1, x_2 \in A_i\}.$$

Given two nonempty subsets A and B of X, the excess of A to B is defined by

$$e(A, B) = \sup\{d(a, B) : a \in A\},$$

and the Hausdorff distance between A and B is defined by

$$H(A, B) = \max\{e(A, B), e(B, A)\}.$$

For $\epsilon > 0$, two types of LP approximating solution set for (SWGVEP) are defined, respectively, by

$\Theta_1(\epsilon) := \{x \in X : \forall i \in I, d_i(x_i, X_i) \leq \epsilon \text{ and } \exists z_i \in T_i(x), s.t. f_i(x, z_i, y_i) + \epsilon e_i(x) \notin -intC_i(x), \quad \forall y_i \in X_i\}$.

$\Theta_2(\epsilon) := \{x \in X : \forall i \in I, d_i(x_i, X_i) \leq \epsilon \text{ and } \exists z_i \in T_i(x), s.t. f_i(x, z_i, y_i) + \epsilon e_i(x) \notin -intC_i(x), \quad \forall y_i \in X_i; \text{ and } \exists i_0 \in I, \ \forall z_{i_0} \in T_{i_0}(x), \ \exists \omega_{i_0}(z_{i_0}) \in X_{i_0}, s.t. f_{i_0}(x, z_{i_0}, \omega_{i_0}(z_{i_0})) - \epsilon e_{i_0}(x) \in -C_{i_0}(x)\}$.

Theorem 4.1. (a) (SWGVEP) is a type I LP well-posed if and only if the solution set Ω is nonempty, compact and

$$e(\Theta_1(\epsilon), \Omega) \to 0 \text{ as } \epsilon \to 0;$$

(b) (SWGVEP) is a type II LP well-posed if and only if the solution set Ω is nonempty, compact and

$$e(\Theta_2(\epsilon), \Omega) \to 0 \text{ as } \epsilon \to 0 \tag{4.1}$$

Proof. We only prove (b). The proof of (a) is similar and is omitted here.

Let (SWGVEP) be type II LP well-posed. Then Ω is nonempty and compact. Now we show that (4.1) holds. Suppose to the contrary that there exist $M > 0$, $\epsilon^n > 0$ with $\epsilon^n \to 0$ and $v^n \in \Theta_2(\epsilon^n)$ such that

$$d(v^n, \Omega) \geq M \tag{4.2}$$

Since $\{v^n\} \subseteq \Theta_2(\epsilon^n)$, we know that $\{v^n\}$ is a type II LP approximating solution sequence for (SWGVEP). By the type II LP well-posedness of (SWGVEP), there exists a subsequence $\{v^{n_j}\}$ of $\{v^n\}$ converging to some element of Ω. This contradicts (4.2). Hence, (4.1) holds.

Conversely, suppose that Ω is nonempty, compact and (4.1) holds. Let $\{x^n\}$ be type II LP approximating solution sequence of (SWGVEP). Then, there exists a sequence $\{\epsilon^n\} \subseteq \mathbb{R}_+$ with $\epsilon^n \to 0$ such that for each $i \in I$, $\exists z_i^n \in T_i(x^n)$, (2.1) and (2.2) hold; and there exists $i_0 \in I$, for any $z_{i_0} \in T_{i_0}(x^n)$, $\exists \omega_{i_0}(n, z_{i_0}) \in X_{i_0}$, such that (2.3) holds. Thus, $\{x^n\} \subseteq \Theta_2(\epsilon^n)$. It follows from (4.1) that there exist a sequence $\{z^n\} \subseteq \Omega$ such that $d(x^n, z^n) = d(x^n, \Omega) \leq e(\Theta_2(\epsilon^n), \Omega) \to 0$. Since Ω is compact, there exists a subsequence $\{z^{n_j}\}$ of $\{z^n\}$ converging to $x_0 \in \Omega$. And so the corresponding subsequence $\{x^{n_j}\}$ of $\{x^n\}$ converging to x_0. Therefore, (SWGVEP) is type II LP well-posed. This completes the proof.

Theorem 4.2. Assume that for each $i \in I$,

(i) the set-valued map $W_i : E \to 2^{Y_i}$ defined by $W_i(x) = Y_i \setminus -intC_i(x)$ is $u.s.c.$;

(ii) the set-valued map T_i is $u.s.c$ and compact-valued on E,

(iii) for any $y_i \in E_i$, the vector-valued function $(x, z_i) \mapsto f_i(x, z_i, y_i)$ is continuous.

Then, (SWGVEP) is type I LP well-posed if and only if

$$\Theta_1(\epsilon) \neq \emptyset, \forall \epsilon > 0 \text{ and } \lim_{\epsilon \to 0} \alpha(\Theta_1(\epsilon)) = 0. \tag{4.3}$$

Proof. Assume that (4.3) holds. Then, for any $\epsilon > 0$, $Cl(\Theta_1(\epsilon))$ is nonempty closed and increasing with $\epsilon > 0$. By (4.3), $\lim_{\epsilon \to 0} \alpha(Cl(\Theta_1(\epsilon))) = \lim_{\epsilon \to 0} \alpha(\Theta_1(\epsilon)) = 0$, where $Cl(\Theta_1(\epsilon))$ is the closure of $\Theta_1(\epsilon)$. By the generalized Cantor theorem (P. 412 in [24]), we know that

$$H(Cl(\Theta_1(\epsilon)), \triangle_1) \to 0, \text{ as } \epsilon \to 0. \tag{4.4}$$

where $\triangle_1 = \cap_{\epsilon > 0} Cl(\Theta_1(\epsilon))$ is nonempty compact.

Now we show that

$$\Omega = \triangle_1. \tag{4.5}$$

We first show that

$$\triangle_1 \subseteq \Omega. \tag{4.6}$$

Indeed, let $\bar{x} \in \Delta_1$. Then $d(\bar{x}, \Theta_1(\epsilon)) = 0$, for every $\epsilon > 0$. Given $\epsilon^n > 0$, $\epsilon^n \to 0$, for every n there exists $u^n \in \Theta_1(\epsilon^n)$ such that $d(\bar{x}, u^n) < \epsilon^n$. Hence, $u^n \to \bar{x}$ and for each $i \in I$,

$$d_i(u_i^n, X_i) \leq \epsilon^n, \tag{4.7}$$

and there exists

$$z_i^n \in T_i(u^n), \tag{4.8}$$

such that

$$f_i(u^n, z_i^n, y_i) + \epsilon^n e_i(u^n) \notin -intC_i(u^n), \quad \forall y_i \in X_i. \tag{4.9}$$

(4.7) and $u^n \to \bar{x}$ imply that for each $i \in I$, there exists $w_i^n \in X_i$ such that $w_i^n \to \bar{x}_i$. It follows from the closedness of X_i that for each $i \in I$, $\bar{x}_i \in X_i$.

It follows from the assumption (ii) and (4.8) that there exists a subsequence $\{z_i^{n_j}\}$ of $\{z_i^n\}$ and some $\bar{z}_i \in T_i(\bar{x})$ such that $\{z_i^{n_j}\} \to \bar{z}_i$.

By the continuity of f_i, the closedness of W_i, and (4.9), we know that for each $i \in I$, $f_i(\bar{x}, \bar{z}_i, y_i) \notin -intC_i(\bar{x}), \forall\ y_i \in X_i$. That is, $\bar{x} \in \Omega$, which implies that (4.6) holds. It is obvious that $\Omega \subseteq \triangle_1$. Thus, (4.5) holds.

By (4.4) and (4.5), we know that $e(\Theta_1(\epsilon), \Omega) \to 0$ as $\epsilon \to 0$. It follows from Theorem 4.1(a) that (SWGVEP) is type I LP well-posed.

Conversely, let (SWGVEP) be type I LP well-posed. Then Ω is nonempty and compact. It follows that $\Theta_1(\epsilon) \neq \emptyset, \forall \epsilon > 0$. Observe that for every $\epsilon > 0$,

$$H(\Theta_1(\epsilon), \Omega) = max\{e(\Theta_1(\epsilon), \Omega), e(\Omega, \Theta_1(\epsilon))\} = e(\Theta_1(\epsilon), \Omega)$$

Hence,

$$\alpha(\Theta_1(\epsilon)) \leq 2H(\Theta_1(\epsilon), \Omega) + \alpha(\Omega) = 2e(\Theta_1(\epsilon), \Omega). \tag{4.10}$$

where $\alpha(\Omega) = 0$ since Ω is compact. By Theorem 4.1(a), we get that $e(\Theta_1(\epsilon), \Omega) \to 0$ as $\epsilon \to 0$. It follows from (4.10) that (4.3) holds. This completes the proof.

Theorem 4.3. Assume that for each $i \in I$,

(i) the set-valued map $W_i : E \to 2^{Y_i}$ defined by $W_i(x) = Y_i \setminus -intC_i(x)$ is *u.s.c.*;

(ii) the set-valued map T_i is *u.s.c* and compact-valued on E;

(iii) for any $y_i \in E_i$, the vector-valued function $(x, z_i) \mapsto f_i(x, z_i, y_i)$ is continuous;

(iv) Suppose that for any $x \in \Omega$, there exist $i_0 \in I$, for any $z_{i_0} \in T_{i_0}(x)$, $\exists \omega_{i_0}(z_{i_0}) \in X_{i_0}$ such that $f_{i_0}(x, z_{i_0}, \omega_{i_0}(z_{i_0})) \in -\partial C_{i_0}(x)$.

Then, (SWGVEP) is type II LP well-posed if and only if

$$\Theta_2(\epsilon) \neq \emptyset, \forall \epsilon > 0, \text{ and } \lim_{\epsilon \to 0} \alpha(\Theta_2(\epsilon)) = 0 \tag{4.11}$$

Proof. Let (4.11) holds. Then, for any $\epsilon > 0$, $Cl(\Theta_2(\epsilon))$ is nonempty closed and increasing with $\epsilon > 0$. By (4.11), $\lim_{\epsilon \to 0} \alpha(Cl(\Theta_2(\epsilon))) = \lim_{\epsilon \to 0} \alpha(\Theta_2(\epsilon)) = 0$, where $Cl(\Theta_2(\epsilon))$ is the closure of $\Theta_2(\epsilon)$.By the generalized Cantor theorem (P.412 in [24]). We know that

$$H(Cl(\Theta_2(\epsilon)), \triangle_2) \to 0, \text{ as } \epsilon \to 0 \tag{4.12}$$

where $\triangle_2 = \cap_{\epsilon > 0} Cl(\Theta_2(\epsilon))$ is nonempty compact.

Now we show that

$$\triangle_2 = \Omega \tag{4.13}$$

Let $\bar{x} \in \Omega$, then for each $i \in I$, $\bar{x}_i \in X_i$, $\bar{z}_i \in T_i(\bar{x})$ and $f_i(\bar{x}, \bar{z}_i, y_i) \notin -intC_i(\bar{x}), \forall y_i \in X_i$. Then for $\epsilon > 0$, we have $d_i(\bar{x}_i, X_i) \leq \epsilon$ and $\bar{z}_i \in T_i(\bar{x})$, $f_i(\bar{x}, \bar{z}_i, y_i) + \epsilon e_i(\bar{x}) \notin -intC_i(\bar{x})$, $\forall y_i \in X_i$. It follows from (iii) that there exists $i_0 \in I$, for any $z_{i_0} \in T_{i_0}(\bar{x})$, $\exists \omega_{i_0}(z_{i_0}) \in X_{i_0}$ such that $f_{i_0}(\bar{x}, z_{i_0}, \omega_{i_0}(z_{i_0})) \in -\partial C_{i_0}(\bar{x})$. Hence, $f_{i_0}(\bar{x}, z_{i_0}, \omega_{i_0}(z_{i_0})) - \epsilon e_{i_0}(\bar{x}) \in -\partial C_{i_0}(\bar{x}) - intC_{i_0}(\bar{x}) \subseteq -intC_{i_0}(\bar{x}) \subseteq -C_{i_0}(\bar{x})$, which implies that $\Omega \subseteq \triangle_2$. It follows from the proof of Theorem 4.1 that

$\triangle_2 \subseteq \triangle_1 \subseteq \Omega$, thus (4.13) holds. Combining (4.12) together with (4.13), we get that $e(\Theta_2(\epsilon), \Omega) \to 0$ as $\epsilon \to 0$. It follows from Theorem 4.1(b) that (SWGVEP) is type II LP well-posed.

The converse of the proof is similar with that of the proof of Theorem 4.2 and it is omitted here. This completes the proof.

Now we present some sufficient conditions for the two types of LP well-posedness of (SWGVEP).

Theorem 4.4. Let E be finite dimensional. Assume that for each $i \in I$,

(i) the set-valued map T_i is *u.s.c* and compact-valued on E;

(ii) the set-valued map $W_i : E \to 2^{Y_i}$ defined by $W_i(x) = Y_i \setminus -intC_i(x)$ is *u.s.c*;

(iii) for any $y_i \in E_i$, the vector-valued function $(x, z_i) \mapsto f_i(x, z_i, y_i)$ is continuous;

(iv) Ω is nonempty and there exists $\epsilon_0 > 0$ such that $\Theta_1(\epsilon_0)$ (resp., $\Theta_2(\epsilon_0)$) is bounded.

Then (SWGVEP) is type I (resp., type II) LP well-posed.

Proof. We only prove the sufficiency for the type I LP well-posedness of (SWGVEP). The proof of the sufficiency for the type II LP well-posedness for (SWGVEP) is similar and is omitted here.

Let $\{x^n\}$ be any type I LP approximating solution sequence of (SWGVEP). Then there exist $\epsilon^n > 0$ with $\epsilon^n \to 0$ such that for each $i \in I$, $z_i^n \in T_i(x^n)$, (2.1) and (2.2) hold. Thus $x^n \in \Theta_1(\epsilon^n)$. Clearly, $\Theta_1(.)$ is increasing with $\epsilon > 0$. Without loss of generality, we can assume that $\{x^n\} \subseteq \Theta_1(\epsilon_0)$. Hence, $\{x^n\}$ is bounded. Since E is finite dimensional, let $\{x^{n_j}\}$ be any subsequence of $\{x^n\}$ such that $x^{n_j} \to \bar{x} \in X$. From (2.1) and (2.2), we can get for each $i \in I$, $\exists z_i^{n_j} \in T_i(x^{n_j})$,

$$d_i(x_i^{n_j}, X_i) \leq \epsilon^n. \tag{4.14}$$

and

$$f_i(x^{n_j}, z_i^{n_j}, y_i) + \epsilon^{n_j} e_i(x^{n_j}) \notin -intC_i(x^{n_j}), \forall y_i \in X_i. \tag{4.15}$$

For each $i \in I$, (4.14) and the closedness of X_i imply that $\bar{x}_i \in X_i$.

It follows from (i) that there exist a subsequence of $\{z_i^{n_j}\}$, denoted by $\{z_i^{n_{j_k}}\}$ and some $\bar{z}_i \in T_i(\bar{x})$ such that $z_i^{n_{j_k}} \to \bar{z}_i$. Taking the limit in (4.15) (with n_j replaced by n_{j_k}), we have $f_i(\bar{x}, \bar{z}_i, y_i) \in W_i(\bar{x})$, $\forall y_i \in X_i$. Hence, $\bar{x} \in \Omega$, and (SWGVEP) is type I Tykhonov well-posed. This completes the proof.

Proposition 4.2. Assume that for each $i \in I$,

(i) the set-valued map T_i is *u.s.c* and compact-valued on E;

(ii) the set-valued map $W_i : E \to 2^{Y_i}$ defined by $W_i(x) = Y_i \setminus -intC_i(x)$ is *u.s.c*;

(iii) for any $y_i \in E_i$, the vector-valued function $(x, z_i) \mapsto f_i(x, z_i, y_i)$ is continuous;

(iv) Ω is nonempty and there exists $\epsilon_0 > 0$ such that $\Theta_1(\epsilon_0)$ (resp., $\Theta_2(\epsilon_0)$) is compact.

Then (SWGVEP) is type I (resp., type II) LP well-posed.

Proof. The proof is similar to Theorem 4.4 and is omitted.

For $\epsilon > 0$, two types of LP approximating solution set for (SWVEP) are defined, respectively, by

$\Theta_3(\epsilon) := \{x \in X : \forall i \in I, d_i(x_i, X_i) \leq \epsilon$ and $\varphi_i(x, y_i) + \epsilon e_i(x) \notin -intC_i(x), \ \ \forall y_i \in X_i\}$.

$\Theta_4(\epsilon) := \{x \in X : \forall i \in I, d_i(x_i, X_i) \leq \epsilon$ and $\varphi_i(x, y_i) + \epsilon e_i(x) \notin -intC_i(x), \quad \forall y_i \in X_i$; and $\exists i_0 \in I, \exists \omega_{i_0} \in X_{i_0}, s.t. \varphi_{i_0}(x, \omega_{i_0}) - \epsilon e_{i_0}(x) \in -C_{i_0}(x)\}$.

Let $Z = \{\bar{z}\}$ and for each $i \in I$, $T_i(x) = \{\bar{z}_i\}$ for all $x \in X$, $\varphi_i(x, y_i) = f_i(x, \bar{z}, y_i)$, $\forall (x, y_i) \in X \times X_i$, then by Theorems 4.1-4.4 and Proposition 4.2, respectively, we obtain the following new results:

Corollary 4.1. (a) (SWVEP) is a type I LP well-posed if and only if the solution set Ω_1 is nonempty, compact and

$$e(\Theta_3(\epsilon), \Omega_1) \to 0 \text{ as } \epsilon \to 0;$$

(b) (SWVEP) is a type II LP well-posed if and only if the solution set Ω_1 is nonempty, compact and

$$e(\Theta_4(\epsilon), \Omega_1) \to 0 \text{ as } \epsilon \to 0.$$

Corollary 4.2. If for each $i \in I$, the set-valued map $W_i : E \to 2^{Y_i}$ defined by $W_i(x) = Y_i \setminus -intC_i(x)$ is $u.s.c.$, and for any $y_i \in E_i$, the vector-valued function $x \mapsto \varphi_i(x, y_i)$ is continuous, then (SWVEP) is type I LP well-posed if and only if

$$\Theta_3(\epsilon) \neq \emptyset, \forall \epsilon > 0 \text{ and } \lim_{\epsilon \to 0} \alpha(\Theta_3(\epsilon)) = 0. \tag{4.3}$$

Corollary 4.3. Assume that for each $i \in I$, the set-valued map $W_i : E \to 2^{Y_i}$ defined by $W_i(x) = Y_i \setminus -intC_i(x)$ is $u.s.c.$; for any $y_i \in E_i$, the vector-valued function $x \mapsto \varphi_i(x, y_i)$ is continuous; and for any $x \in \Omega_1$, there exist $i_0 \in I$, $\omega_{i_0} \in X_{i_0}$ such that $\varphi_{i_0}(x, \omega_{i_0}) \in -\partial C_{i_0}(x)$, then, (SWVEP) is type II LP well-posed if and only if

$$\Theta_4(\epsilon) \neq \emptyset, \forall \epsilon > 0, \text{ and } \lim_{\epsilon \to 0} \alpha(\Theta_4(\epsilon)) = 0.$$

Corollary 4.4. Let E be finite dimensional. Assume that for each $i \in I$, the set-valued map $W_i : E \to 2^{Y_i}$ defined by $W_i(x) = Y_i \setminus -intC_i(x)$ is $u.s.c$; for any $y_i \in E_i$, the vector-valued function $x \mapsto \varphi_i(x, y_i)$ is continuous; Ω_1 is nonempty and there exists $\epsilon_0 > 0$ such that $\Theta_3(\epsilon_0)$ (resp., $\Theta_4(\epsilon_0)$) is bounded. Then (SWVEP) is type I (resp., type II) LP well-posed.

Corollary 4.5. Assume that for each $i \in I$, the set-valued map $W_i : E \to 2^{Y_i}$ defined by $W_i(x) = Y_i \setminus -intC_i(x)$ is $u.s.c$; for any $y_i \in E_i$, the vector-valued function $x \mapsto \varphi_i(x, y_i)$ is continuous; Ω_1 is nonempty and there exists $\epsilon_0 > 0$ such that $\Theta_3(\epsilon_0)$ (resp., $\Theta_4(\epsilon_0)$) is compact. Then (SWVEP) is type I (resp., type II) LP well-posed.

Remark 4.1. If the index set I is singleton, then by Theorem 4.1(a), Theorem 4.2 and 4.4, Propositions 4.1 and 4.2, respectively, we recover (a) and (b) of Theorem 3.1, Theorem 3.2, Propositions 2.1 and Corollary 3.1 in [34]; by Theorem 4.1(b) and Theorem 4.3, respectively, we can obtain some new metric characterization of the type II well-posedness for (GVEP); by Corollaries 3.2-3.5, respectively, we recover Theorem 3.1-3.3 and Corollary 3.1 in [26].

Remark 4.2. If for each $i \in I$, $A_i(x) \equiv X_i$ for all $x \in X$ in [36], we can easily obtain some results involving the Tykhonov well-posedness for (SWVEP) but not any types of LP well-posedness for (SWVEP) .

Remark 4.3. It is easy to see that the results in this paper generalize, extend and unify those results in [20, 22, 26, 28, 34, 35, 38] and the references therein.

5. Concluding remarks

We introduce the new and interesting notions of type I LP well-posedness and type II LP well-posedness for (SWGVEP) with a countable index set. We show that the type I (resp., type II) LP well-posedness of (SWGVEP) is equivalent to the limit of the excess of the type I (resp., type II) approximating solution set for (SWGVEP) and the solution set of (SWGVEP) is zero. Under some suitable conditions, we also show that the type I (resp., type II) Levitin-Polyak well-posedness of (SWGVEP) is equivalent to one of the following conditions:

i) the type I (resp., type II) Levitin-Polyak well-posedness of a minimization problem.

ii) the type I (resp., type II) approximating solution set for (SWGVEP) is nonempty and the limit of the Kuratowski measure of noncompactness of the type I (resp., type II) approximating solution set for (SWGVEP) is zero.

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