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APPROXIMATING FIXED POINTS OF 2-GENERALIZED HYBRID MAPPINGS IN BANACH SPACES AND CAT(0) SPACES

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Abstract. In this paper, we first prove weak and strong convergence theorems for Ishikawa and Halpern iterations of 2-generalized hybrid mappings in uniformly convex Banach spaces and we apply our method to provide an affirmative answer to an open problem raised by Hojo, Takahashi and Termwuttipong [Strong convergence theorems for 2-generalized hybrid mappings in Hilbert spaces, Nonlinear Analysis, 75 (2012) 2166-2176]. We then extend the results to CAT(0) spaces, which include especially simply connected complete Riemannian manifolds with nonpositive sectional curvature. Our results improve and generalize some known results in the current literature.
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1. Introduction

Throughout this paper, we denote the set of real numbers, the set of nonnegative real numbers, the set of negative real numbers and the set of positive integers by $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{R}^-$ and $\mathbb{N}$, respectively. Let E be a (real) Banach space and let C be a nonempty subset of E. Let $T : C \to E$ be a mapping. We denote by $F(T)$ the set of fixed points of T, *i.e.*, $F(T) = \{x \in C : Tx = x\}$. A mapping $T : C \to E$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A mapping $T : C \to E$ is said to be *quasi-nonexpansive* if $F(T) \neq \emptyset$ and $\|Tx - y\| \leq \|x - y\|$ for all $x \in C$ and $y \in F(T)$.

The concept of nonexpansivity plays an important role in the study of Mann-type iteration for finding fixed points of a mapping $T : C \to C$. Recall that the *Mann-type iteration* is given by the following formula

$$x_{n+1} = \gamma_n T x_n + (1 - \gamma_n) x_n, \qquad x_1 \in C. \tag{1.1}$$

Here, $\{\gamma_n\}_{n\in\mathbb{N}}$ is a real sequence in $[0,1]$ satisfying some appropriate conditions. A more general iteration scheme is the *Ishikawa iteration*, given by

$$\begin{cases} x_1 \in C \quad \text{chosen arbitrarily}, \\ y_n = \lambda_n T x_n + (1-\lambda_n) x_n, \\ x_{n+1} = \gamma_n T y_n + (1-\gamma_n) x_n, \end{cases} \tag{1.2}$$

where the sequences $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{\gamma_n\}_{n\in\mathbb{N}}$ satisfy some appropriate conditions. In particular, when all $\lambda_n = 0$, the Ishikawa iteration (1.2) becomes the standard Mann iteration (1.1). Recall that the *one-step Halpern iteration* is given by the following formula

$$x_{n+1} = \alpha_n u + (1-\alpha_n) T x_n, \qquad u \in C,\ x_1 \in C. \tag{1.3}$$

Here, $\{\alpha_n\}_{n\in\mathbb{N}}$ is a real sequence in $[0,1]$ satisfying some appropriate conditions. A more general iteration scheme of one-step Halpern iteration is the *two-step Halpern iteration* given by

$$\begin{cases} u \in C,\ x_1 \in C \quad \text{chosen arbitrarily}, \\ y_n = (1-\beta_n) x_n + \beta_n T x_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n) y_n, \end{cases} \tag{1.4}$$

where the sequences $\{\beta_n\}_{n\in\mathbb{N}}$ and $\{\alpha_n\}_{n\in\mathbb{N}}$ satisfy some appropriate conditions. In particular, when all $\beta_n = 1$, the Halpern iteration (1.4) becomes the standard Halpern iteration (1.3). When all $\alpha_n = 0$, the Halpern iteration (1.4) becomes the standard Mann iteration (1.1).

The construction of fixed points of nonexpansive mappings via Mann's algorithm [22] has been extensively investigated recently in the current literature (see, for example, [26] and the references therein). Numerous results have been proved on Mann and Halpern's iterations for nonexpansive mappings in Hilbert and Banach spaces (see, e.g., [16, 33, 28]).

In 1998, Takahashi and Kim [30] proved the following interesting result.

Theorem 1.1. *Let C be a nonempty, closed and convex subset of a uniformly convex Banach space E which satisfies the Opial property, and let T be a nonexpansive mapping of C into itself. Let, for any initial data x_1 in C, the iterates $\{x_n\}_{n\in\mathbb{N}}$ be defined by (1.2) such that $\lambda_n \in [0,b]$ and $\gamma_n \in [a,b]$, or $\lambda_n \in [a,b]$ and $\gamma_n \in [a,1]$, for some a, b with $0 < a \le b < 1$. Then the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges weakly to a fixed point of T.*

Let C be a nonempty subset of a real Banach space E. Following Hojo, Takahashi and Termwuttipong [17], a mapping $T : C \to E$ is said to be
(1) *generalized hybrid* if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \le \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2, \quad \forall x, y \in C.$$

(2) 2-*generalized hybrid* or $(\alpha_1, \alpha_2, \beta_1, \beta_2)$-*generalized hybrid* if there exist α_1, α_2, β_1, $\beta_2 \in \mathbb{R}$ such that

$$\begin{aligned} &\alpha_1\|T^2x - Ty\|^2 + \alpha_2\|Tx - Ty\|^2 + (1-\alpha_1-\alpha_2)\|x - Ty\|^2 \\ &\le \beta_1\|T^2x - y\|^2 + \beta_2\|Tx - y\|^2 + (1-\beta_1-\beta_2)\|x - y\|^2, \quad \forall x, y \in C. \end{aligned}$$

Clearly, $(0,1,1,1)$-generalized hybrid maps are exactly nonexpansive maps.

Let C be a nonempty, closed and convex subset of a Hilbert space H and $x \in H$. Then there exists a unique nearest point $z \in C$ such that $\|x-z\| = \inf_{y\in C} \|x-y\|$. We denote such correspondence by $z = P_C x$. The mapping P_C is called *metric projection* of H onto C.
Recently, Hojo, Takahashi and Termwuttipong [17] proved the following fixed point theorem for 2-generalized hybrid mappings in a Hilbert space.

Theorem 1.2. *([17]) Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $T : C \to C$ be a 2-generalized hybrid mapping with $F(T) \neq \emptyset$. Suppose that $\{x_n\}_{n\in\mathbb{N}}$ is a sequence generated by $x_1 = x \in C$, $u \in C$ and*

$$x_{n+1} = \gamma_n u + (1-\gamma_n)\frac{1}{n}\sum_{k=0}^{n-1} T^k x_n, \qquad \forall n \in \mathbb{N}, \tag{1.4}$$

where $0 \leq \gamma_n \leq 1$, $\lim_{n\to\infty} \gamma_n = 0$ and $\sum_{n=1}^{\infty} \gamma_n = \infty$. Then $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $P_{F(T)}u$, where $P_{F(T)}$ is metric projection of H onto $F(T)$.

Hojo, Takahashi and Termwuttipong investigated strong convergence theorems for 2-generalized hybrid mappings and posed the following open question in their final remark of [17, Section 3].

Question 1.1. Is there any strong convergence theorem of Halpern's type for 2-generalized hybrid mappings in a real Hilbert space H?

We know that the assumption $\gamma_n \to 0$ as $n \to \infty$ in iterations (1.1) and (1.2) will weaken the action of operator T. So, we are interested in imposing other assumptions on the parameter γ_n so that the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to a fixed point of T. This is the main motivation of this paper to study fixed point theorems of 2-generalized hybrid mappings in the framework of Banach spaces. In the present paper, we first prove weak and strong convergence theorems for Ishikawa and Halpern iterations of 2-generalized hybrid mappings in uniformly convex Banach spaces and we apply our method to provide an affirmative answer to question 1.1. We then extend the results to CAT(0) spaces, which include as an important special case the simply connected complete Riemannian manifolds with nonpositive sectional curvature. Our results improve and generalize some known results in the current literature, see for example [17, 30].

2. Preliminaries

Let E be a Banach space with the norm $\|.\|$ and the dual space E^*. The modulus δ of convexity of E is denoted by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon\right\}$$

for every ϵ with $0 \leq \epsilon \leq 2$. A Banach space E is said to be *uniformly convex* if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. Let $S_E = \{x \in E : \|x\| = 1\}$. The norm of E is said to be *Gâteaux differentiable* if for each $x, y \in S_E$, the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.1}$$

exists. In this case, E is called *smooth.* If the limit (2.1) is attained uniformly in $x, y \in S_E$, then E is called *uniformly smooth.* The Banach space E is said to be *strictly convex* if $\|\frac{x+y}{2}\| < 1$ whenever $x, y \in S_E$ and $x \neq y$. It is well known that E is uniformly convex if and only if E^* is uniformly smooth. It is also known that if E is reflexive, then E is strictly convex if and only if E^* is smooth; for more details, see [29]. Let E be a Banach space with the norm $\|.\|$ and the dual space E^*. When $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in E, we denote the strong convergence of $\{x_n\}_{n\in\mathbb{N}}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. For any sequence $\{x_n^*\}_{n\in\mathbb{N}}$ in E^*, we denote the strong convergence of $\{x_n^*\}_{n\in\mathbb{N}}$ to $x^* \in E^*$ by $x_n^* \to x^*$, the weak convergence by $x_n^* \rightharpoonup x^*$ and the weak-star convergence by $x_n^* \rightharpoonup^* x^*$. The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2,\ \|x\| = \|f\|\}, \qquad \forall x \in E.$$

Now, we define a mapping $\rho : [0, \infty) \to [0, \infty)$, the modulus of smoothness of E, as follows:

$$\rho(t) = \sup\Big\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : x, y \in E, \|x\| = 1, \|y\| = t\Big\}.$$

It is well known that E is uniformly smooth if and only if $\lim_{t\to 0} \frac{\rho(t)}{t} = 0$. Let $q \in \mathbb{R}$ be such that $1 < q \leq 2$. Then a Banach space E is said to be q-uniformly smooth if there exists a constant $c_q > 0$ such that $\rho(t) \leq c_q t^q$ for all $t > 0$. If a Banach space E admits a sequentially continuous duality mapping J from weak topology to weak-star topology, then J is single-valued and also E is smooth; see for more details [14]. In this case, the normalized duality mapping J is said to be weakly sequentially continuous, *i.e.*, if $\{x_n\}_{n\in\mathbb{N}} \subset E$ is a sequence with $x_n \rightharpoonup x \in E$, then $J(x_n) \rightharpoonup^* J(x)$ [14]. If $E = H$ is a Hilbert space, then $J = I$ the identity mapping on H. A Banach space E is said to satisfy the *Opial property* [12] if for any weakly convergent sequence $\{x_n\}_{n\in\mathbb{N}}$ in E with weak limit x,

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in E$ with $y \neq x$. It is well known that all Hilbert spaces, all finite dimensional Banach spaces and the Banach spaces l^p $(1 \leq p < \infty)$ satisfy the Opial property; see, for example [14, 12, 13]. It is also known that if E admits a weakly sequentially continuous duality mapping, then E is smooth and enjoys the Opial property; see for more details [11, 14].

Let C be a nonempty, closed and convex subset of a Banach space E and $\{x_n\}_{n\in\mathbb{N}}$ be a bounded sequence in E. For any $x \in E$, we set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} \|x - x_n\|.$$

The *asymptotic radius* of $\{x_n\}_{n\in\mathbb{N}}$ relative to C is defined by

$$r(C, \{x_n\}) = \inf\{r(x, \{x_n\}) : x \in C\}.$$

The *asymptotic center* of $\{x_n\}_{n\in\mathbb{N}}$ relative to C is the set

$$A(C, \{x_n\}) = \{x \in C : r(x, \{x_n\}) = r(C, \{x_n\})\}.$$

It is well known that, in a uniformly convex Banach space E, $A(C, \{x_n\})$ consists of exactly one point; see [12, 13].

Let l^∞ denote the Banach space of bounded real sequences with the supremum norm. It is well known that there exists a bounded linear functional μ on l^∞ such that the following three conditions hold:
(1) If $\{t_n\}_{n\in\mathbb{N}} \in l^\infty$ and $t_n \geq 0$ for every $n \in \mathbb{N}$, then $\mu(t_n) \geq 0$;
(2) If $t_n = 1$ for every $n \in \mathbb{N}$, then $\mu(t_n) = 1$;
(3) $\mu(\{t_{n+1}\}) = \mu(\{t_n\})$ for all $\{t_n\}_{n\in\mathbb{N}} \in l^\infty$.
Such a functional μ is called a Banach limit and the value of μ at $\{t_n\} \in l^\infty$ is denoted by $\mu_n t_n$ (see, for example [29]).

Lemma 2.1. *([1]) Let C be a nonempty, closed and convex subset of a uniformly convex Banach space E and let $T : C \to E$ be a mapping. Suppose that there exist $x \in C$ and a Banach limit μ such that $\{T^n x\}_{n\in\mathbb{N}}$ is bounded and*

$$\mu_n \|T^n x - Ty\|^2 \leq \mu_n \|T^n x - y\|^2$$

for all $y \in C$. Then T has a fixed point.

Lemma 2.2. *Let E be a Banach space and C be a subset of E. Let $T : C \to E$ be an $(\alpha_1, \alpha_2, \beta_1, \beta_2)$-generalized hybrid mapping for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$. Suppose that $\{T^n x\}_{n\in\mathbb{N}}$ is bounded for some $x \in C$. Then $\mu_n \|T^n x - Ty\|^2 \leq \mu_n \|T^n x - y\|^2$ for all Banach limit μ and for all $y \in C$.*

Proof. Let μ be a Banach limit and take $y \in C$ arbitrarily chosen. Since T is an $(\alpha_1, \alpha_2, \beta_1, \beta_2)$-generalized hybrid mapping, we conclude that

$$\begin{aligned} &\alpha_1 \|T^{n+2}x - Ty\|^2 + \alpha_2 \|T^{n+1}x - Ty\|^2 + (1 - \alpha_1 - \alpha_2)\|T^n x - Ty\|^2 \\ &\leq \beta_1 \|T^{n+2}x - y\|^2 + \beta_2 \|T^{n+1}x - y\|^2 + (1 - \beta_1 - \beta_2)\|T^n x - y\|^2 \end{aligned}$$

for all $n \in \mathbb{N}$. Since μ is a Banach limit, we obtain

$$\begin{aligned} &\alpha_1 \mu_n \|T^n x - Ty\|^2 + \alpha_2 \mu_n \|T^n x - Ty\|^2 + (1 - \alpha_1 - \alpha_2)\mu_n \|T^n x - Ty\|^2 \\ &\leq \beta_1 \mu_n \|T^n x - y\|^2 + \beta_2 \mu_n \|T^n x - y\|^2 + (1 - \beta_1 - \beta_2)\mu_n \|T^n x - y\|^2 \end{aligned}$$

for all $n \in \mathbb{N}$. This implies that

$$\mu_n \|T^n x - Ty\|^2 \leq \mu_n \|T^n x - y\|^2$$

for all $y \in C$, which completes the proof. □

The following result is an immediate consequence of Lemmas 2.1 and 2.2.

Corollary 2.1. *Let E be a uniformly convex Banach space and C be a subset of E. Let $T : C \to E$ be an $(\alpha_1, \alpha_2, \beta_1, \beta_2)$-generalized hybrid mapping for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$. Then $F(T) \neq \emptyset$ if and only if there exists $x \in C$ such that $\{T^n x\}_{n\in\mathbb{N}}$ is bounded.*

Lemma 2.3. *Let E be a Banach space and let C be a nonempty, closed and convex subset of E. Let $T : C \to E$ be an $(\alpha_1, \alpha_2, \beta_1, \beta_2)$-generalized hybrid mapping for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that $F(T) \neq \emptyset$. Then T is quasi-nonexpansive.*

Proof. Let $x \in C$ and $z \in F(T)$. Then we have

$$\begin{aligned} \|Tx - z\|^2 &= \alpha_1 \|T^2 z - Tx\|^2 + \alpha_2 \|Tz - Tx\|^2 + (1 - \alpha_1 - \alpha_2)\|z - Tx\|^2 \\ &\leq \beta_1 \|T^2 z - x\|^2 + \beta_2 \|Tz - x\|^2 + (1 - \beta_1 - \beta_2)\|z - x\|^2 \\ &= \|x - z\|^2. \end{aligned}$$

This implies that

$$\|Tx - z\| \le \|x - z\|,$$

which completes that proof. □

Lemma 2.4. *Let C be a nonempty subset of a Banach space E. Let $T : C \to E$ be an $(\alpha_1, \alpha_2, \beta_1, \beta_2)$-generalized hybrid mapping for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$.* Then for any $x, y \in C$,

$$\|x - Ty\|^2 \le \|x - y\|^2 + |\alpha_1|\theta_{x,y} + |\alpha_2|\eta_{x,y} + |\beta_1|\delta_{x,y} + |\beta_2|\sigma_{x,y},$$

where

$$\begin{aligned}
\theta_{x,y} &= \|x - T^2x\|^2 + 2\|x - T^2x\|\|T^2x - Ty\| + 2\|x - T^2x\|\|x - Ty\|,\\
\eta_{x,y} &= \|x - Tx\|^2 + 2\|x - Tx\|\|Tx - Ty\| + 2\|x - Tx\|\|x - Ty\|,\\
\delta_{x,y} &= \|T^2x - x\|^2 + 2\|T^2x - x\|\|x - y\| + 2\|T^2x - x\|\|T^2x - y\|,\\
\sigma_{x,y} &= \|Tx - x\|^2 + 2\|Tx - x\|\|x - y\| + 2\|Tx - x\|\|Tx - y\|.
\end{aligned}$$

Proof. We divide the proof into several cases.
Case 1. If $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}^+$, then

$$\begin{aligned}
&\|x - Ty\|^2\\
&= \alpha_1\|x - Ty\|^2 + \alpha_2\|x - Ty\|^2 + (1 - \alpha_1 - \alpha_2)\|x - Ty\|^2\\
&\le \alpha_1[\|x - T^2x\| + \|T^2x - Ty\|]^2\\
&+ \alpha_2[\|x - Tx\| + \|Tx - Ty\|]^2 + (1 - \alpha_1 - \alpha_2)\|x - Ty\|^2\\
&\le \alpha_1[\|x - T^2x\|^2 + \|T^2x - Ty\|^2 + 2\|x - T^2x\|\|T^2x - Ty\| + 2\|x - T^2x\|\|x - Ty\|]\\
&+ \alpha_2[\|x - Tx\|^2 + \|Tx - Ty\|^2 + 2\|x - Tx\|\|Tx - Ty\| + 2\|x - Tx\|\|x - Ty\|]\\
&+ (1 - \alpha_1 - \alpha_2)\|x - Ty\|^2\\
&= \alpha_1\|T^2x - Ty\|^2 + \alpha_2\|Tx - Ty\|^2 + (1 - \alpha_1 - \alpha_2)\|x - Ty\|^2 + \alpha_1\theta_{x,y} + \alpha_2\eta_{x,y}\\
&\le \beta_1\|T^2x - y\|^2 + \beta_2\|Tx - y\|^2 + (1 - \beta_1 - \beta_2)\|x - y\|^2 + \alpha_1\theta_{x,y} + \alpha_2\eta_{x,y}\\
&\le \beta_1[\|T^2x - x\| + \|x - y\|]^2 + \beta_2[\|Tx - x\| + \|x - y\|]^2\\
&+ (1 - \beta_1 - \beta_2)\|x - y\|^2 + \alpha_1\theta_{x,y} + \alpha_2\eta_{x,y}\\
&\le \beta_1[\|T^2x - x\|^2 + \|x - y\|^2 + 2\|T^2x - x\|\|x - y\| + 2\|T^2x - x\|\|T^2x - y\|]\\
&+ \beta_2[\|Tx - x\|^2 + \|x - y\|^2 + \|Tx - x\|\|x - y\| + \|Tx - x\|\|Tx - y\|]\\
&+ (1 - \beta_1 - \beta_2)\|x - y\|^2 + \alpha_1\theta_{x,y} + \alpha_2\eta_{x,y}\\
&= \beta_1\|x - y\|^2 + \beta_2\|x - y\|^2 + (1 - \beta_1 - \beta_2)\|x - y\|^2\\
&+ \alpha_1\theta_{x,y} + \alpha_2\eta_{x,y} + \beta_1\delta_{x,y} + \beta_2\sigma_{x,y}\\
&= \|x - y\|^2 + \alpha_1\theta_{x,y} + \alpha_2\eta_{x,y} + \beta_1\delta_{x,y} + \beta_2\sigma_{x,y}.
\end{aligned}$$

Thus we have

$$\|x - Ty\|^2 \le \|x - y\|^2 + \alpha_1\theta_{x,y} + \alpha_2\eta_{x,y} + \beta_1\delta_{x,y} + \beta_2\sigma_{x,y}.$$

Case 2. If $\alpha_1, \alpha_2, \beta_1 \in \mathbb{R}^+$ and $\beta_2 \in \mathbb{R}^-$, then

$$
\begin{aligned}
&\|x-y\|^2 + \alpha_1\theta_{x,y} + \alpha_2\eta_{x,y} + \beta_1\delta_{x,y} \\
&= \beta_1\|x-y\|^2 + \beta_2\|x-y\|^2 + (1-\beta_1-\beta_2)\|x-y\|^2 + \alpha_1\theta_{x,y} + \alpha_2\eta_{x,y} + \beta_1\delta_{x,y} \\
&\geq \beta_1[\|x-y\|^2 + \|T^2x-x\|^2 + 2\|T^2x-x\|\|x-y\| + 2\|T^2x-x\|\|T^2x-y\|] \\
&+ \beta_2[\|Tx-y\| + \|Tx-x\|]^2 + (1-\beta_1-\beta_2)\|x-y\|^2 + \alpha_1\theta_{x,y} + \alpha_2\eta_{x,y} \\
&\geq \beta_1[\|x-y\|^2 + \|T^2x-x\|^2 + 2\|T^2x-x\|\|x-y\| + 2\|T^2x-x\|\|T^2x-y\|] \\
&+ \beta_2[\|Tx-y\|^2 + \|Tx-x\|^2 + 2\|Tx-x\|\|x-y\| + 2\|Tx-x\|\|Tx-y\|] \\
&+ (1-\beta_1-\beta_2)\|x-y\|^2 + \alpha_1\theta_{x,y} + \alpha_2\eta_{x,y} \\
&\geq \beta_1\|T^2x-y\|^2\beta_2\|Tx-y\|^2 \\
&+ \beta_2[\|x-Tx\|^2 + 2\|x-Tx\|\|Tx-y\| + 2\|x-Tx\|\|x-Ty\|] \\
&+ (1-\beta_1-\beta_2)\|x-y\|^2 + \alpha_1\theta_{x,y} + \alpha_2\eta_{x,y} \\
&= \beta_1\|T^2x-y\|^2 + \beta_2\|Tx-y\|^2 + (1-\beta_1-\beta_2)\|x-y\|^2 + \alpha_1\theta_{x,y} + \alpha_2\eta_{x,y} + \beta_2\sigma_{x,y} \\
&\geq \alpha_1\|T^2x-Ty\|^2 + \alpha_2\|Tx-Ty\|^2 + (1-\alpha_1-\alpha_2)\|x-Ty\|^2 \\
&+ \alpha_1\theta_{x,y} + \alpha_2\eta_{x,y} + \beta_2\sigma_{x,y} \\
&\geq \alpha_1\|x-Ty\|^2 + \alpha_2\|x-Ty\|^2 + (1-\alpha_1-\alpha_2)\|x-Ty\|^2 + \beta_2\sigma_{x,y} \\
&= \|x-Ty\|^2 + \beta_2\sigma_{x,y}.
\end{aligned}
$$

Thus we have

$$\|x-Ty\|^2 \leq \|x-y\|^2 + \alpha_1\theta_{x,y} + \alpha_2\eta_{x,y} + \beta_1\delta_{x,y} + (-\beta_2)\sigma_{x,y}.$$

Similarly, we can prove the following possible cases and we omit the details.

Case 3. If $\alpha_1, \alpha_2, \beta_2 \in \mathbb{R}^+$ and $\beta_1 \in \mathbb{R}^-$, then

$$\|x-Ty\|^2 \leq \|x-y\|^2 + \alpha_1\theta_{x,y} + \alpha_2\eta_{x,y} + (-\beta_1)\delta_{x,y} + \beta_2\sigma_{x,y}.$$

Case 4. If $\alpha_1, \beta_1, \beta_2 \in \mathbb{R}^+$ and $\alpha_2 \in \mathbb{R}^-$, then

$$\|x-Ty\|^2 \leq \|x-y\|^2 + \alpha_1\theta_{x,y} + (-\alpha_2)\eta_{x,y} + \beta_1\delta_{x,y} + \beta_2\sigma_{x,y}.$$

Case 5. If $\alpha_1, \alpha_2 \in \mathbb{R}^+$ and $\beta_1, \beta_2 \in \mathbb{R}^-$, then

$$\|x-Ty\|^2 \leq \|x-y\|^2 + \alpha_1\theta_{x,y} + \alpha_2\eta_{x,y} + (-\beta_1)\delta_{x,y} + (-\beta_2)\sigma_{x,y}.$$

Case 6. If $\alpha_1, \beta_2 \in \mathbb{R}^+$ and $\alpha_2, \beta_1 \in \mathbb{R}^-$, then

$$\|x-Ty\|^2 \leq \|x-y\|^2 + \alpha_1\theta_{x,y} + (-\alpha_2)\eta_{x,y} + (-\beta_1)\delta_{x,y} + \beta_2\sigma_{x,y}.$$

Case 7. If $\alpha_1, \beta_1 \in \mathbb{R}^+$ and $\alpha_2, \beta_2 \in \mathbb{R}^-$, then

$$\|x-Ty\|^2 \leq \|x-y\|^2 + \alpha_1\theta_{x,y} + (-\alpha_2)\eta_{x,y} + \beta_1\delta_{x,y} + (-\beta_2)\sigma_{x,y}.$$

Case 8. If $\alpha_1 \in \mathbb{R}^+$ and $\beta_1, \alpha_2, \beta_2 \in \mathbb{R}^-$, then

$$\|x-Ty\|^2 \leq \|x-y\|^2 + \alpha_1\theta_{x,y} + (-\alpha_2)\eta_{x,y} + (-\beta_1)\delta_{x,y} + (-\beta_2)\sigma_{x,y}.$$

Case 9. If $\alpha_1 \in \mathbb{R}^-$ and $\alpha_2, , \beta_1, \beta_2 \in \mathbb{R}^+$, then

$$\|x-Ty\|^2 \leq \|x-y\|^2 + (-\alpha_1)\theta_{x,y} + \alpha_2\eta_{x,y} + \beta_1\delta_{x,y} + \beta_2\sigma_{x,y}.$$

Case 10. If $\alpha_2, \beta_1 \in \mathbb{R}^+$ and $\alpha_1, \beta_2 \in \mathbb{R}^-$, then
$$\|x - Ty\|^2 \leq \|x - y\|^2 + (-\alpha_1)\theta_{x,y} + \alpha_2\eta_{x,y} + \beta_1\delta_{x,y} + (-\beta_2)\sigma_{x,y}.$$
Case 11. If $\alpha_2, \beta_2 \in \mathbb{R}^+$ and $\alpha_1, \beta_1 \in \mathbb{R}^-$, then
$$\|x - Ty\|^2 \leq \|x - y\|^2 + (-\alpha_1)\theta_{x,y} + \alpha_2\eta_{x,y} + (-\beta_1)\delta_{x,y} + \beta_2\sigma_{x,y}.$$
Case 12. If $\beta_1, \beta_2 \in \mathbb{R}^+$ and $\alpha_1, \alpha_2 \in \mathbb{R}^-$, then
$$\|x - Ty\|^2 \leq \|x - y\|^2 + (-\alpha_1)\theta_{x,y} + (-\alpha_2)\eta_{x,y} + \beta_1\delta_{x,y} + \beta_2\sigma_{x,y}.$$
Case 13. If $\alpha_2 \in \mathbb{R}^+$ and $\alpha_1, \beta_1, \beta_2 \in \mathbb{R}^-$, then
$$\|x - Ty\|^2 \leq \|x - y\|^2 + (-\alpha_1)\theta_{x,y} + \alpha_2\eta_{x,y} + \beta_1\delta_{x,y} + (-\beta_2)\sigma_{x,y}.$$
Case 14. If $\beta_2 \in \mathbb{R}^+$ and $\alpha_1, \alpha_2, \beta_1 \in \mathbb{R}^-$, then
$$\|x - Ty\|^2 \leq \|x - y\|^2 + (-\alpha_1)\theta_{x,y} + (-\alpha_2)\eta_{x,y} + (-\beta_1)\delta_{x,y} + \beta_2\sigma_{x,y}.$$
Case 15. If $\beta_1 \in \mathbb{R}^+$ and $\alpha_1, \alpha_2, \beta_2 \in \mathbb{R}^-$, then
$$\|x - Ty\|^2 \leq \|x - y\|^2 + (-\alpha_1)\theta_{x,y} + (-\alpha_2)\eta_{x,y} + \beta_1\delta_{x,y} + (-\beta_2)\sigma_{x,y}.$$
Case 16. If $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}^-$, then
$$\|x - Ty\|^2 \leq \|x - y\|^2 + (-\alpha_1)\theta_{x,y} + (-\alpha_2)\eta_{x,y} + (-\beta_1)\delta_{x,y} + (-\beta_2)\sigma_{x,y}.$$
This completes that proof. □

Proposition 2.1. *(Demiclosedness Principle) Let E be a Banach space with the Opial property and C be a subset of E. Let $T : C \to E$ be an $(\alpha_1, \alpha_2, \beta_1, \beta_2)$-generalized hybrid mapping for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$. If $\{x_n\}_{n\in\mathbb{N}}$ converges weakly to z, $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$ and $\lim_{n\to\infty} \|T^2x_n - x_n\| = 0$, then $Tz = z$. That is, $I - T$ is demiclosed at zero, where I is the identity mapping on E.*

Proof. Since $\{x_n\}_{n\in\mathbb{N}}$ converges weakly to z, $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$ and $\lim_{n\to\infty} \|T^2x_n - x_n\| = 0$, we have that $\{x_n\}_{n\in\mathbb{N}}$, $\{Tx_n\}_{n\in\mathbb{N}}$ and $\{T^2x_n\}_{n\in\mathbb{N}}$ are bounded. Let $M_1 = \sup\{\|x_n\|, \|Tx_n\|, \|T^2x_n\|, \|z\|, \|Tz\| : n \in \mathbb{N}\} < \infty$. It is obvious that
$$\lim_{n\to\infty} \|T^2x_n - Tx_n\| = 0. \tag{2.2}$$
By the definition of 2-generalized hybrid mapping and in view of Lemma 2.2, for all $n \in \mathbb{N}$, we get that
$$\|x_n - Tz\|^2 \leq \|x_n - z\|^2 + |\alpha_1|\theta_n + |\alpha_2|\eta_n + |\beta_1|\delta_n + |\beta_2|\sigma_n,$$
where
$$\begin{aligned}
\theta_n &= \|x_n - T^2x_n\|^2 + 2\|x_n - T^2x_n\|\|T^2x_n - Tz\| + 2\|x_n - T^2x_n\|\|x_n - Tz\|,\\
\eta_n &= \|x_n - Tx_n\|^2 + 2\|x_n - Tx_n\|\|Tx_n - Tz\| + 2\|x_n - Tx_n\|\|x_n - Tz\|,\\
\delta_n &= \|T^2x_n - x_n\|^2 + 2\|T^2x_n - x_n\|\|x_n - z\| + 2\|T^2x_n - x_n\|\|T^2x_n - z\|,\\
\sigma_n &= \|Tx_n - x_n\|^2 + 2\|Tx_n - x_n\|\|x_n - z\| + 2\|Tx_n - x_n\|\|Tx_n - z\|.
\end{aligned}$$
In view of (2.2), we conclude that
$$\lim_{n\to\infty} \theta_n = \lim_{n\to\infty} \eta_n = \lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \sigma_n = 0.$$

This implies

$$\limsup_{n\to\infty} \|x_n - Tz\| \le \limsup_{n\to\infty} \|x_n - z\|.$$

From the Opial property, we obtain $Tz = z$. □

Corollary 2.2. *Let E be a Banach space with the Opial property and C be a subset of E. Let $T : C \to E$ be an (α, β)-generalized hybrid mapping for some $\alpha, \beta \in \mathbb{R}$. If $\{x_n\}_{n\in\mathbb{N}}$ converges weakly to z and $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$, then $Tz = z$. That is, $I - T$ is demiclosed at zero, where I is the identity mapping on E.*

Proof. Since $\{x_n\}_{n\in\mathbb{N}}$ converges weakly to z and $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$, we have that $\{x_n\}_{n\in\mathbb{N}}$ and $\{Tx_n\}_{n\in\mathbb{N}}$ are bounded. Let $M_2 = \sup\{\|x_n\|, \|Tx_n\|, \|z\|, \|Tz\| : n \in \mathbb{N}\} < \infty$. A similar argument as in the proof of Proposition 2.1 shows that for all $n \in \mathbb{N}$

$$\|x_n - Tz\|^2 \le \|x_n - z\|^2 + \alpha\eta_n + \beta\sigma_n,$$

where

$$\eta_n = \|x_n - Tx_n\|^2 + 2\|x_n - Tx_n\|\|Tx_n - Tz\| + 2\|x_n - Tx_n\|\|x_n - Tz\|,$$
$$\sigma_n = \|Tx_n - x_n\|^2 + 2\|Tx_n - x_n\|\|x_n - z\| + 2\|Tx_n - x_n\|\|Tx_n - z\|.$$

Since $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$, we see that

$$\lim_{n\to\infty} \eta_n = \lim_{n\to\infty} \sigma_n = 0.$$

This implies

$$\limsup_{n\to\infty} \|x_n - Tz\| \le \limsup_{n\to\infty} \|x_n - z\|.$$

From the Opial property, we obtain $Tz = z$.

Lemma 2.5. *([31]) Let $r > 0$ be a fixed real number. If E is a uniformly convex Banach space, then there exists a continuous strictly increasing convex function $g : [0, +\infty) \to [0, +\infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + (1-\lambda)y\|^2 \le \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g(\|x - y\|),$$

for all $x, y \in B_r(0) = \{u \in E : \|u\| \le r\}$ and $\lambda \in [0, 1]$.

Lemma 2.6. *([27]) Let E be a uniformly convex Banach space, let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that $0 < a \le t_n \le b < 1$ for all $n \in \mathbb{N}$, and let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be sequences of E such that $\limsup_{n\to\infty} \|x_n\| \le r$, $\limsup_{n\to\infty} \|y_n\| \le r$ and*

$$\lim_{n\to\infty} \|t_n x_n + (1 - t_n)y_n\| = r \quad \text{for some } r \ge 0.$$

Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Let C and D be nonempty subsets of real Banach space E with $D \subset C$. A mapping $Q_D : C \to D$ is said to be *sunny* if

$$Q_D(Q_D x + t(x - Q_D x)) = Q_D x$$

for each $x \in E$ and $t \ge 0$. A mapping $Q_D : C \to D$ is said to be a *retraction* if $Q_D x = x$ for each $x \in C$. If $E = H$ is a real Hilbert space, then $Q_D = P_D$ the metric projection of C onto D.

Lemma 2.7. *([26, 29]) Let C and D be nonempty subsets of a real Banach space E with $D \subset C$ and let $Q_D : C \to D$ be a retraction from C into D. Then Q_D is sunny and nonexpansive if and only if*

$$\langle z - Q_D(z), J(y - Q_D(z))\rangle \leq 0$$

for all $z \in C$ and $y \in D$, where J is the normalized duality mapping of E.

Lemma 2.8. *([26]) Let E be a real Banach space and J be the normalized duality mapping of E. Then,*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x + y)\rangle,$$

for all $x, y \in E$.

Lemma 2.9. *([32]) Let $\{s_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative real numbers satisfying the inequality:*

$$s_{n+1} \leq (1 - \gamma_n)s_n + \gamma_n\delta_n, \qquad \forall n \geq 1,$$

where $\{\gamma_n\}_{n\in\mathbb{N}}$ and $\{\delta_n\}_{n\in\mathbb{N}}$ satisfy the conditions:
(i) $\{\gamma_n\}_{n\in\mathbb{N}} \subset [0, 1]$ and $\sum_{n=1}^{\infty} \gamma_n = \infty$, or equivalently, $\Pi_{n=1}^{\infty}(1 - \gamma_n) = 0$;
(ii) $\limsup_{n\to\infty} \delta_n \leq 0$, or
$(ii)'$ $\sum_{n=1}^{\infty} \gamma_n\delta_n < \infty$.
Then, $\lim_{n\to\infty} s_n = 0$.

Lemma 2.10. *([21]) Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}_{i\in\mathbb{N}}$ of $\{n\}_{n\in\mathbb{N}}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a subsequence $\{m_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \text{ and } a_k \leq a_{m_k+1}.$$

In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.

The following result which is a generalization of Lemma 2.5 has been proved in [7].

Lemma 2.11. *Let E be a uniformly convex Banach space and $B_r := \{x \in E : \|x\| \leq r\}$, $r > 0$. Then there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + \beta y + \gamma z\|^2 \leq \lambda\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \lambda\beta g(\|x - y\|)$$

for all $x, y, z \in B_r$ and all $\lambda, \beta, \gamma \in [0, 1]$ with $\lambda + \beta + \gamma = 1$.

3. Fixed Point and Convergence Theorems in Banach Spaces

In this section, we prove weak and strong convergence theorems for Ishikawa and Halpern iterations of 2-generalized hybrid mappings in a Banach space.

Lemma 3.1. *Let E be a Banach space and let C be a nonempty, closed and convex subset of E. Let $T : C \to E$ be an $(\alpha_1, \alpha_2, \beta_1, \beta_2)$-generalized hybrid mapping for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that the fixed point set $F(T)$ is nonempty. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be two sequences defined by (1.2) such that $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{\gamma_n\}_{n\in\mathbb{N}}$ are arbitrary sequences in $[0, 1]$. Then the following assertions hold:*
(1) $\max\{\|x_{n+1} - z\|, \|y_n - z\|\} \leq \|x_n - z\|$ for any $z \in F(T)$ and for all $n = 1, 2, \ldots$.
(2) $\lim_{n\to\infty} \|x_n - z\|$ exists for any $z \in F(T)$.

(3) $\lim_{n\to\infty} d(x_n, F(T))$ *exists, where* $d(x, F(T))$ *denotes the distance of* x *to the fixed-point set* $F(T)$*.*

Proof. Let $z \in F(T)$. In view of Lemma 2.3, we conclude that

$$\begin{aligned}\|y_n - z\| &= \|\lambda_n Tx_n + (1-\lambda_n)x_n - z\| \\ &\le \lambda_n\|Tx_n - z\| + (1-\lambda_n)\|x_n - z\| \\ &\le \lambda_n\|x_n - z\| + (1-\lambda_n)\|x_n - z\| \\ &= \|x_n - z\|.\end{aligned}$$

Consequently,

$$\begin{aligned}\|x_{n+1} - z\| &= \|\gamma_n Ty_n + (1-\gamma_n)x_n - z\| \\ &\le \gamma_n\|Ty_n - z\| + (1-\gamma_n)\|x_n - z\| \\ &\le \gamma_n\|y_n - z\| + (1-\gamma_n)\|x_n - z\| \\ &\le \gamma_n\|x_n - z\| + (1-\gamma_n)\|x_n - z\| \\ &= \|x_n - z\|.\end{aligned}$$

This implies that $\{\|x_n - z\|\}_{n\in\mathbb{N}}$ is a bounded and nonincreasing sequence for all $z \in F(T)$. Thus we have $\lim_{n\to\infty}\|x_n - z\|$ exists for any $z \in F(T)$. In the same manner, we see that $\{d(x_n, F(T))\}_{n\in\mathbb{N}}$ is also a bounded nonincreasing real sequence, and thus converges. □

The proof of the following corollary is similar to that of Lemma 3.1 and we omit it.

Corollary 3.1. *Let* E *be a Banach space and* C *be a nonempty, closed and convex subset of* E*. Let* $T : C \to E$ *be an* $(\alpha_1, \alpha_2, \beta_1, \beta_2)$*-generalized hybrid mapping for some* $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ *such that the fixed point set* $F(T)$ *is nonempty. Let a sequence* $\{x_n\}_{n\in\mathbb{N}}$ *with* x_1 *in* C *be defined by (1.1) such that* $\{\gamma_n\}_{n\in mathbbN}$ *is an arbitrary sequence in* $[0, 1]$*. Then the following assertions hold:*
(1) $\|x_{n+1} - z\| \le \|x_n - z\|$ *for any* $z \in F(T)$ *and for all* $n = 1, 2, \ldots$*.*
(2) $\lim_{n\to\infty}\|x_n - z\|$ *exists for any* $z \in F(T)$*.*
(3) $\lim_{n\to\infty} d(x_n, F(T))$ *exists, where* $d(x, F(T))$ *denotes the distance of* x *to the fixed-point set* $F(T)$*.*

Theorem 3.1. *Let* C *be a nonempty, closed and convex subset of a uniformly convex Banach space* E *with the Opial property. Let* $T : C \to E$ *be an* $(\alpha_1, \alpha_2, \beta_1, \beta_2)$*-generalized hybrid mapping for some* $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$*. Let a sequence* $\{x_n\}_{n\in\mathbb{N}}$ *with* $x_1 \in C$ *be defined by (1.1) and the sequence* $\{\gamma_n\}_{n\in\mathbb{N}}$ *be chosen so that* $\gamma_n \in [0, 1]$*.*

(i) If $\{x_n\}_{n\in\mathbb{N}}$ *is bounded,* $\liminf_{n\to\infty}\|Tx_n - x_n\| = 0$ *and* $\liminf_{n\to\infty}\|T^2x_n - x_n\| = 0$*, then the fixed point set* $F(T) \ne \emptyset$*.*

(ii) Suppose that $F(T) \ne \emptyset$ *and* $\liminf_{n\to\infty}\gamma_n(1-\gamma_n) > 0$*. Then* $\{x_n\}_{n\in\mathbb{N}}$ *is bounded and* $\liminf_{n\to\infty}\|Tx_n - x_n\| = 0$*.*

Proof. (i) Assume that $\{x_n\}_{n\in\mathbb{N}}$ is bounded, $\liminf_{n\to\infty}\|Tx_n - x_n\| = 0$ and $\liminf_{n\to\infty}\|T^2x_n - x_n\| = 0$. Consequently, there is a bounded subsequence $\{Tx_{n_k}\}_{k\in\mathbb{N}}$

of $\{Tx_n\}_{n\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \|Tx_{n_k} - x_{n_k}\| = \lim_{k\to\infty} \|T^2x_{n_k} - x_{n_k}\| = 0.$$

Suppose $A(C, \{x_{n_k}\}) = \{z\}$. Let $M_3 = \sup\{\|x_{n_k}\|, \|Tx_{n_k}\|, \|z\|, \|Tz\| : k \in \mathbb{N}\} < \infty$. In view of Lemma 2.4, for all $k \in \mathbb{N}$,

$$\|x_{n_k} - Tz\|^2 \le \|x_{n_k} - z\|^2 + \alpha_1\theta_{n_k} + \alpha_2\eta_{n_k} + \beta_1\delta_{n_k} + \beta_2\sigma_{n_k},$$

where

$$\begin{aligned}
&\theta_{n_k} = \|x_{n_k} - T^2x_{n_k}\|^2 + 2\|x_{n_k} - T^2x_{n_k}\|\|T^2x_{n_k} - Tz\| \\
&\ +2\|x_{n_k} - T^2x_{n_k}\|\|x_{n_k} - Tz\|, \\
&\eta_{n_k} = \|x_{n_k} - Tx_{n_k}\|^2 + 2\|x_{n_k} - Tx_{n_k}\|\|Tx_{n_k} - Tz\| \\
&\ +2\|x_{n_k} - Tx_{n_k}\|\|x_{n_k} - Tz\|, \\
&\delta_{n_k} = \|T^2x_{n_k} - x_{n_k}\|^2 + 2\|T^2x_{n_k} - x_{n_k}\|\|x_{n_k} - z\| \\
&\ +2\|T^2x_{n_k} - x_{n_k}\|\|T^2x_{n_k} - z\|, \\
&\sigma_{n_k} = \|Tx_{n_k} - x_{n_k}\|^2 + 2\|Tx_{n_k} - x_{n_k}\|\|x_{n_k} - z\| + 2\|Tx_{n_k} - x_{n_k}\|\|Tx_{n_k} - z\|.
\end{aligned}$$

It is obvious that

$$\lim_{k\to\infty} \theta_{n_k} = \lim_{k\to\infty} \eta_{n_k} = \lim_{k\to\infty} \delta_{n_k} = \lim_{k\to\infty} \sigma_{n_k} = 0.$$

This implies

$$\limsup_{k\to\infty} \|x_{n_k} - Tz\| \le \limsup_{n\to\infty} \|x_{n_k} - z\|.$$

From the Opial property, we obtain $Tz = z$.

(ii) Let $F(T) \ne \emptyset$ and let $z \in F(T)$. It follows from Corollary 3.1 that $\lim_{n\to\infty} \|x_n - z\|$ exists and hence $\{x_n\}_{n\in\mathbb{N}}$ is bounded. In view of Lemma 2.3 and Lemma 2.5, we obtain

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|\gamma_nTx_n + (1-\gamma_n)x_n - z\|^2 \\
&\le \gamma_n\|Tx_n - z\|^2 + (1-\gamma_n)\|x_n - z\|^2 - \gamma_n(1-\gamma_n)g(\|Tx_n - x_n\|) \\
&\le \gamma_n\|x_n - z\|^2 + (1-\gamma_n)\|x_n - z\|^2 - \gamma_n(1-\gamma_n)g(\|Tx_n - x_n\|) \\
&= \|x_n - z\|^2 - \gamma_n(1-\gamma_n)g(\|Tx_n - x_n\|).
\end{aligned} \tag{3.1}$$

In view of (3.1), we conclude that

$$\begin{aligned}
\gamma_n(1-\gamma_n)g(\|Tx_n - x_n\|) &\le \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \\
&\to 0
\end{aligned}$$

as $n \to \infty$. From the assumption $\liminf_{n\to\infty} \gamma_n(1-\gamma_n) > 0$, we have

$$\liminf_{n\to\infty} g(\|Tx_n - x_n\|) = 0.$$

Therefore, from the property of g we deduce that

$$\liminf_{n\to\infty} \|Tx_n - x_n\| = 0.$$

This completes the proof. □

Theorem 3.2. *Let C be a nonempty, closed and convex subset of a uniformly convex Banach space E with the Opial property. Let $T : C \to E$ be an $(\alpha_1, \alpha_2, \beta_1, \beta_2)$-generalized hybrid mapping for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$. Let $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{\gamma_n\}_{n\in\mathbb{N}}$ be sequences such that $\lambda_n \in [a, b]$ and $\gamma_n \in [a, 1]$ for some a, b with $0 < a \le b < 1$ and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence with x_1 in C defined by (1.2).*

(i) If $\{x_n\}_{n\in\mathbb{N}}$ *is bounded,* $\liminf_{n\to\infty}\|Tx_n - x_n\| = 0$ *and* $\liminf_{n\to\infty}\|T^2x_n - x_n\| = 0$*, then the fixed point set* $F(T) \neq \emptyset$*.*

(ii) Assume $F(T) \neq \emptyset$*. Then* $\{x_n\}_{n\in\mathbb{N}}$ *is bounded and* $\liminf_{n\to\infty}\|Tx_n - x_n\| = 0$.

Proof. (i) Assume that $\{x_n\}_{n\in\mathbb{N}}$ is bounded, $\liminf_{n\to\infty}\|Tx_n - x_n\| = 0$ and $\liminf_{n\to\infty}\|T^2x_n - x_n\| = 0$. Consequently, there is a bounded subsequence $\{Tx_{n_k}\}_{k\in\mathbb{N}}$ of $\{Tx_n\}_{n\in\mathbb{N}}$ such that

$$\lim_{k\to\infty}\|Tx_{n_k} - x_{n_k}\| = \lim_{k\to\infty}\|T^2x_{n_k} - x_{n_k}\| = 0.$$

Suppose $A(C, \{x_{n_k}\}) = \{z\}$. Let $M_4 = \sup\{\|x_{n_k}\|, \|Tx_{n_k}\|, \|z\|, \|Tz\| : n \in \mathbb{N}\} < \infty$. In view of Lemma 2.4, for all $k \in \mathbb{N}$,

$$\|x_{n_k} - Tz\|^2 \leq \|x_{n_k} - z\|^2 + \alpha_1\theta_{n_k} + \alpha_2\eta_{n_k} + \beta_1\delta_{n_k} + \beta_2\sigma_{n_k},$$

where

$$\begin{aligned}
&\theta_{n_k} = \|x_{n_k} - T^2x_{n_k}\|^2 + 2\|x_{n_k} - T^2x_{n_k}\|\|T^2x_{n_k} - Tz\| \\
&\quad +2\|x_{n_k} - T^2x_{n_k}\|\|x_{n_k} - Tz\|, \\
&\eta_{n_k} = \|x_{n_k} - Tx_{n_k}\|^2 + 2\|x_{n_k} - Tx_{n_k}\|\|Tx_{n_k} - Tz\| \\
&\quad +2\|x_{n_k} - Tx_{n_k}\|\|x_{n_k} - Tz\|, \\
&\delta_{n_k} = \|T^2x_{n_k} - x_{n_k}\|^2 + 2\|T^2x_{n_k} - x_{n_k}\|\|x_{n_k} - z\| \\
&\quad +2\|T^2x_{n_k} - x_{n_k}\|\|T^2x_{n_k} - z\|, \\
&\sigma_{n_k} = \|Tx_{n_k} - x_{n_k}\|^2 + 2\|Tx_{n_k} - x_{n_k}\|\|x_{n_k} - z\| + 2\|Tx_{n_k} - x_{n_k}\|\|Tx_{n_k} - z\|.
\end{aligned}$$

It is easy to see that

$$\lim_{k\to\infty}\theta_{n_k} = \lim_{k\to\infty}\eta_{n_k} = \lim_{k\to\infty}\delta_{n_k} = \lim_{k\to\infty}\sigma_{n_k} = 0.$$

This implies

$$\limsup_{k\to\infty}\|x_{n_k} - Tz\| \leq \limsup_{n\to\infty}\|x_{n_k} - z\|.$$

From the Opial property, we obtain $Tz = z$.

(ii) Let $F(T) \neq \emptyset$ and take $z \in F(T)$ arbitrarily chosen. Then, in view of Lemma 3.1, $\lim_{n\to\infty}\|x_n - z\|$ exists and hence $\{x_n\}_{n\in\mathbb{N}}$ is bounded. By Lemma 2.3, we have that

$$\|Tx_n - z\| \leq \|x_n - z\|, \ \forall n \in \mathbb{N}.$$

Set

$$\lim_{n\to\infty}\|x_n - z\| = d.$$

This implies that

$$\limsup_{n\to\infty}\|Tx_n - z\| \leq d. \tag{3.2}$$

On the other hand, we have

$$\begin{aligned}
\|x_{n+1} - z\| &= \|\gamma_nTy_n + (1-\gamma_n)x_n - z\| \\
&\leq \gamma_n\|Ty_n - z\| + (1-\gamma_n)\|x_n - z\| \\
&\leq \gamma_n\|y_n - z\| + (1-\gamma_n)\|x_n - z\| \\
&= \gamma_n\|y_n - z\| + \|x_n - z\| - \gamma_n\|x_n - z\|.
\end{aligned}$$

This implies that

$$\|x_{n+1}-z\|-\|x_n-z\| \le \frac{\|x_{n+1}-z\|-\|x_n-z\|}{\gamma_n} \le \|y_n-z\|-\|x_n-z\|.$$

Thus, we have

$$d \le \liminf_{n\to\infty} \|y_n-z\|. \tag{3.3}$$

In view of (3.3), we conclude that

$$d \le \liminf_{n\to\infty} \|y_n-z\| \le \limsup_{n\to\infty} \|y_n-z\| \le d.$$

This means that

$$\lim_{n\to\infty} \|y_n-z\| = d.$$

Therefore

$$\lim_{n\to\infty} \|\lambda_n(Tx_n-z)+(1-\lambda_n)(x_n-z)\| = \lim_{n\to\infty} \|y_n-z\| = d. \tag{3.4}$$

In view of (3.4) and Lemma 2.6, we obtain

$$\lim_{n\to\infty} \|Tx_n-x_n\| = 0.$$

This completes the proof. □

Corollary 3.2. *Let C be a nonempty, closed and convex subset of a uniformly convex Banach space E with the Opial property. Let $T : C \to E$ be an an (α,β)-generalized hybrid mapping for some $\alpha,\beta \in \mathbb{R}$. Let $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{\gamma_n\}_{n\in\mathbb{N}}$ be sequences such that $\lambda_n \in [a,b]$ and $\gamma_n \in [a,1]$ for some a,b with $0 < a \le b < 1$, and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence with x_1 in C defined by (1.2).*
(i) If $\{x_n\}_{n\in\mathbb{N}}$ is bounded and $\liminf_{n\to\infty} \|Tx_n-x_n\| = 0$, then the fixed point set $F(T) \neq \emptyset$.
(ii) Conversely, assume $F(T) \neq \emptyset$. Then $\{x_n\}_{n\in\mathbb{N}}$ is bounded and $\liminf_{n\to\infty} \|Tx_n-x_n\| = 0$.

Proof. (i) Assume that $\{x_n\}_{n\in\mathbb{N}}$ is bounded and $\liminf_{n\to\infty} \|Tx_n-x_n\| = 0$. Consequently, there is a bounded subsequence $\{Tx_{n_k}\}_{k\in\mathbb{N}}$ of $\{Tx_n\}_{n\in\mathbb{N}}$ such that $\lim_{k\to\infty} \|Tx_{n_k} - x_{n_k}\| = 0$. Suppose $A(C,\{x_{n_k}\}) = \{z\}$. Let $M_5 = \sup\{\|x_{n_k}\|, \|Tx_{n_k}\|, \|z\|, \|Tz\| : k \in \mathbb{N}\} < \infty$. In view of Lemma 2.4, for all $k \in \mathbb{N}$,

$$\|x_{n_k}-Tz\|^2 \le \|x_{n_k}-z\|^2 + \alpha\eta_{n_k} + \beta\sigma_{n_k},$$

where

$$\begin{aligned} \eta_{n_k} &= \|x_{n_k}-Tx_{n_k}\|^2 + 2\|x_{n_k}-Tx_{n_k}\|\|Tx_{n_k}-Tz\| \\ &+2\|x_{n_k}-Tx_{n_k}\|\|x_{n_k}-Tz\|, \\ \sigma_{n_k} &= \|Tx_{n_k}-x_{n_k}\|^2 + 2\|Tx_{n_k}-x_{n_k}\|\|x_{n_k}-z\| \\ &+2\|Tx_{n_k}-x_{n_k}\|\|Tx_{n_k}-z\|. \end{aligned}$$

Since $\lim_{k\to\infty} \|Tx_{n_k}-x_{n_k}\| = 0$, we conclude that

$$\lim_{k\to\infty} \eta_{n_k} = \lim_{k\to\infty} \sigma_{n_k} = 0.$$

This implies

$$\limsup_{k\to\infty} \|x_{n_k}-Tz\| \le \limsup_{n\to\infty} \|x_{n_k}-z\|.$$

From the Opial property, we obtain $Tz = z$.
A similar argument as in the proof of Theorem 3.2 proves (ii), which completes the proof. □

Theorem 3.3. *Let C be a nonempty, closed and convex subset of a uniformly convex Banach space E with the Opial property. Let $T : C \to E$ be an (α, β)-generalized hybrid mapping for some $\alpha, \beta \in \mathbb{R}$ such that the fixed point set $F(T)$ is nonempty. Let $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{\gamma_n\}_{n\in\mathbb{N}}$ be sequences in $[0,1]$, and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence with x_1 in C defined by (1.2). Assume that $\liminf_{n\to\infty} \gamma_n(1-\gamma_n) > 0$, and assume, in addition, $\limsup_{n\to\infty} \lambda_n < 1$. Then $\{x_n\}_{n\in\mathbb{N}}$ converges weakly to a fixed point of T.*

Proof. It follows from Theorem 3.2 that $\{x_n\}_{n\in\mathbb{N}}$ is bounded and $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. The uniform convexity of E implies that E is reflexive; see, for example, [29]. Then, there exists a subsequence $\{x_{n_i}\}_{i\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $x_{n_i} \rightharpoonup p \in C$ as $i \to \infty$. In view of Corollary 2.2, we conclude that $p \in F(T)$. We claim that $x_n \rightharpoonup p$ as $n \to \infty$. If not, then there exists a subsequence $\{x_{n_j}\}_{j\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $\{x_{n_j}\}_{j\in\mathbb{N}}$ converges weakly to some $q \in C$ with $p \neq q$. In view of Corollary 2.2, we conclude that $q \in F(T)$. By Lemma 3.1 we conclude that $\lim_{n\to\infty} \|x_n - z\|$ exists for all z in $F(T)$. Thus we obtain by the Opial property that

$$\begin{aligned} \lim_{n\to\infty} \|x_n - p\| &= \lim_{i\to\infty} \|x_{n_i} - p\| & &< \lim_{i\to\infty} \|x_{n_i} - q\| \\ &= \lim_{n\to\infty} \|x_n - q\| & &= \lim_{j\to\infty} \|x_{n_j} - q\| \\ &< \lim_{j\to\infty} \|x_{n_j} - p\| & &= \lim_{n\to\infty} \|x_n - p\|. \end{aligned}$$

This is a contradiction. Thus we have $p = q$, and the desired assertion follows. □

Theorem 3.4. *Let C be a nonempty, compact and convex subset of a uniformly convex Banach space E. Let $T : C \to E$ be an (α, β)-generalized hybrid mapping for some $\alpha, \beta \in \mathbb{R}$. Let $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{\gamma_n\}_{n\in\mathbb{N}}$ be sequences in $[0,1]$. We assume either*

$$\begin{cases} \liminf_{n\to\infty} \gamma_n(1-\gamma_n) > 0, \\ \liminf_{n\to\infty} \lambda_n < 1, \end{cases} \quad or \quad \begin{cases} \limsup_{n\to\infty} \gamma_n(1-\gamma_n) > 0, \\ \limsup_{n\to\infty} \lambda_n < 1. \end{cases}$$

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence with x_1 in C defined by (1.2). Then $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to a fixed point z of T.

Proof. By Corollary 2.1, we see that the fixed point set $F(T)$ of T is nonempty. In view of Theorem 3.2, we obtain that $\{x_n\}_{n\in\mathbb{N}}$ is bounded and $\liminf_{n\to\infty} \|Tx_n - x_n\| = 0$. By the compactness of C, there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $\{x_{n_k}\}_{k\in\mathbb{N}}$ converges strongly to some z in C. We can even assume that $\lim_{k\to\infty} \|Tx_{n_k} - x_{n_k}\| = 0$, and in particular, $\{Tx_{n_k}\}_{k\in\mathbb{N}}$ is bounded. Let $M_6 = \sup\{\|x_{n_k}\|, \|Tx_{n_k}\|, \|z\|, \|Tz\| : k \in \mathbb{N}\} < \infty$. In view of Lemma 2.4, we obtain

$$\|x_{n_k} - Tz\|^2 \leq \|x_{n_k} - z\|^2 + \alpha\eta_{n_k} + \beta\sigma_{n_k},$$

where

$$\eta_{n_k} = \|x_{n_k} - Tx_{n_k}\|^2 + 2\|x_{n_k} - Tx_{n_k}\|\|Tx_{n_k} - Tz\| \\ +2\|x_{n_k} - Tx_{n_k}\|\|x_{n_k} - Tz\|,$$

$$\sigma_{n_k} = \|Tx_{n_k} - x_{n_k}\|^2 + 2\|Tx_{n_k} - x_{n_k}\|\|x_{n_k} - z\| \\ +2\|Tx_{n_k} - x_{n_k}\|\|Tx_{n_k} - z\|.$$

Since $\lim_{k\to\infty} \|Tx_{n_k} - x_{n_k}\| = 0$, we conclude that

$$\lim_{k\to\infty} \eta_{n_k} = \lim_{k\to\infty} \sigma_{n_k} = 0.$$

This implies

$$\limsup_{k\to\infty} \|x_{n_k} - Tz\| \leq \limsup_{n\to\infty} \|x_{n_k} - z\|.$$

From the Opial property, we obtain $Tz = z$. □

Let C be a nonempty, closed and convex subset of a Banach space E. A mapping $T : C \to C$ is said to satisfy *condition* (I) [27] if

> there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that $d(x, Tx) \geq f(d(x, F(T)))$, where $d(x, F(T)) = \inf_{z\in F(T)} d(x, z)$.

Theorem 3.5. *Let C be a nonempty, closed and convex subset of a uniformly convex Banach space E. Let $T : C \to E$ be an $(\alpha_1, \alpha_2, \beta_1, \beta_2)$-generalized hybrid mapping for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that the fixed point set $F(T)$ is nonempty. Let $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{\gamma_n\}_{n\in\mathbb{N}}$ be sequences in $[0, 1]$. We assume either*

$$\begin{cases} \liminf\limits_{n\to\infty} \gamma_n(1-\gamma_n) > 0, \\ \liminf\limits_{n\to\infty} \lambda_n < 1, \end{cases} \quad or \quad \begin{cases} \limsup\limits_{n\to\infty} \gamma_n(1-\gamma_n) > 0, \\ \limsup\limits_{n\to\infty} \lambda_n < 1. \end{cases}$$

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence with x_1 in C defined by (1.2). If T satisfies condition (I), then $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to a fixed point z of T.

Proof. It follows from Theorem 3.2 that

$$\liminf_{n\to\infty} \|Tx_n - x_n\| = 0.$$

Therefore, there is a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \|Tx_{n_k} - x_{n_k}\| = 0.$$

Since T satisfies condition (I), with respect to the sequence $\{x_{n_k}\}_{k\in\mathbb{N}}$, we obtain

$$\lim_{k\to\infty} d(x_{n_k}, F(T)) = 0.$$

This implies that, there exists a subsequence of $\{x_n\}_{n\in\mathbb{N}}$, denoted also by $\{x_{n_k}\}_{k\in\mathbb{N}}$, and a sequence $\{z_k\}_{k\in\mathbb{N}}$ in $F(T)$ such that

$$d(x_{n_k}, z_k) < \frac{1}{2^k}, \quad \forall k \in \mathbb{N}. \tag{3.5}$$

In view of Lemma 3.1, we have that

$$\|x_{n_{k+1}} - z_k\| \leq \|x_{n_k} - z_k\| < \frac{1}{2^k}, \quad \forall k \in \mathbb{N}.$$

This implies that

$$\begin{aligned}\|z_{k+1}-z_k\| &\leq \|z_{k+1}-x_{n_{k+1}}\|+\|x_{n_{k+1}}-z_k\| \\ &\leq \tfrac{1}{2^{(k+1)}}+\tfrac{1}{2^k} \\ &< \tfrac{1}{2^{(k-1)}}, \quad \forall k=1,2,\ldots.\end{aligned}$$

Therefore, $\{z_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $F(T)$. By Proposition 2.1, we know that $F(T)$ is closed in E. This implies that $\lim_{k\to\infty} z_k = z$ for some z in $F(T)$. It follows from (3.5) that $\lim_{k\to\infty} x_{n_k} = z$. By Lemma 3.1, we have that $\lim_{n\to\infty}\|x_n - z\|$ exists. This forces $\lim_{n\to\infty}\|x_n - z\| = 0$. □

Theorem 3.6. *Let C be a nonempty, closed and convex subset of a uniformly convex Banach space E. Let $T : C \to E$ be an $(\alpha_1, \alpha_2, \beta_1, \beta_2)$-generalized hybrid mapping for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that the fixed point set $F(T)$ is nonempty. Let $\{\gamma_n\}_{n\in\mathbb{N}}$ be a sequence in $[0,1]$ such that $\liminf_{n\to\infty} \gamma_n(1-\gamma_n) > 0$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence with x_1 in C defined by (1.1). If T satisfies condition (I), then $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to a fixed point z of T.*

Example 3.1. (i) Let $T : [-1,1] \to [-1,1]$ be defined by $Tx = -x$. Then T is a $(0,1,0,0)$-generalized hybrid mapping. Setting all $\lambda_n = 1$, the Ishikawa iteration (1.2) provides a sequence

$$x_{n+1} = \gamma_n T^2 x_n + (1-\gamma_n)x_n = x_n, \quad \forall n = 1,2,\ldots,$$

no matter how we choose $\{\gamma_n\}_{n\in\mathbb{N}}$. Unless $x_1 = 0$, we can never reach the unique fixed point 0 of T via x_n.

(ii) Let $T : [0,2] \to [0,2]$ be defined by

$$Tx = \begin{cases} 0 \text{ if } x \neq 2, \\ 1 \text{ if } x = 2. \end{cases}$$

Then T is $(\frac{1}{2}, \frac{3}{4}, \frac{1}{3}, \frac{2}{3})$-generalized hybrid mapping with $F(T) = \{0\}$. Indeed, for any $x \in [0,2)$ and $y = 2$, we have that $Tx = 0$, $T^2x = 0$ and $Ty = 1$. Thus we have

$$\begin{aligned}&\tfrac{1}{2}|T^2x-Ty|^2+\tfrac{3}{4}|T^2x-Ty|^2+(1-\tfrac{1}{2}-\tfrac{3}{4})|x-Ty|^2 \\ &= \tfrac{1}{2}|0-1|^2+\tfrac{3}{4}|0-1|^2+(-\tfrac{1}{4})|x-1|^2 \\ &= \tfrac{5}{4}-\tfrac{1}{4}|x-1|^2 \\ &\leq \tfrac{5}{4} \\ &\leq \tfrac{1}{3}|0-2|^2+\tfrac{2}{3}|0-2|^2+(1-\tfrac{1}{3}-\tfrac{2}{3})|x-2|^2 \\ &= 4.\end{aligned}$$

The other cases can be verified similarly. It is worth mentioning that T is neither nonexpansive nor continuous. Now, we define the function $f : [0,\infty) \to [0,\infty)$ by

$$f(x) = \frac{x}{2}, \quad x \in [0,\infty).$$

It is easy to see that T satisfies the condition (I) with respect to f.

Setting all $\lambda_n = 1$, the Ishikawa iteration (1.2) provides a sequence

$$x_{n+1} = \gamma_n T^2 x_n + (1-\gamma_n)x_n, \ \forall n = 1,2,....$$

If $\gamma_n \neq 0,\ \forall n = 1, 2, ...$, then for any starting point $x_1 \in [0, 4]$, we have that $T^2 x_n = 0$ and

$$\begin{aligned} x_{n+1} &= \gamma_n T^2 x_n + (1-\gamma_n) x_n \\ &= (1-\gamma_n) x_n \\ &= (1-\gamma_1)(1-\gamma_2)...(1-\gamma_n) x_1 \\ &= \prod_{k=1}^{n} (1-\gamma_k) x_1, \qquad \forall n = 1, 2, ... \end{aligned}$$

Consider two possible choices of the values of γ_n:
Case 1. If we set $\gamma_n = \frac{1}{2},\ \forall n = 1, 2, ...$, then $\liminf_{n\to\infty} \gamma_n(1-\gamma_n) = \frac{1}{4} > 0$, and $x_n \to 0$, the unique fixed point of T.
Case 2. If we set $\gamma_n = \frac{1}{(n+1)^2},\ \forall n = 1, 2, ...$, then $\liminf_{n\to\infty} \gamma_n(1-\gamma_n) = 0$ and $x_n = \frac{n+2}{2n+2} x_1 \to \frac{x_1}{2}$. Unless $x_1 = 0$, we can never reach the unique fixed point 0 of T via x_n.

This explains why we need to impose some conditions on the parameters in previous theorems.

Theorem 3.7. *Let E be a real uniformly convex Banach space which admits the weakly sequentially continuous duality mapping J and C be a nonempty, closed and convex subset of E. Let $T : C \to C$ be a 2-generalized hybrid mapping such that $F := F(T) \neq \emptyset$. Let $\{\alpha_n\}_{n\in\mathbb{N}}$, $\{\beta_{n,1}\}_{n\in\mathbb{N}}$, $\{\beta_{n,2}\}_{n\in\mathbb{N}}$, $\{\beta_{n,3}\}_{n\in\mathbb{N}}$ be sequences in $[0, 1]$ satisfying the following control conditions:*
(a) $\lim_{n\to\infty} \alpha_n = 0$;
(b) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(c) $\beta_{n,1} + \beta_{n,2} + \beta_{n,3} = 1,\ \forall n \in \mathbb{N}$;
(d) $\liminf_{n\to\infty} \beta_{n,j}\beta_{n,3} > 0,\ j = 1, 2$.
Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by

$$\begin{cases} u \in C,\ x_1 \in C \quad \textit{chosen arbitrarily}, \\ y_n = \beta_{n,1} T x_n + \beta_{n,2} T^2 x_n + \beta_{n,3} x_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n) y_n. \end{cases} \tag{3.6}$$

Then, the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined in (3.6) converges strongly to $Q_F u$, where Q_F is a sunny nonexpansive retraction from E onto F.

Proof. We divide the proof into several steps.
Since T is a quasi-nonexpansive mapping, we know that F is closed and convex. Set

$$z = Q_F u.$$

Step 1. We prove that the sequences $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$, $\{Tx_n\}_{n\in\mathbb{N}}$ and $\{T^2 x_n\}_{n\in\mathbb{N}}$ are bounded.
We first show that $\{x_n\}_{n\in\mathbb{N}}$ is bounded.
Let $p \in F$ be fixed. In view of Lemma 2.11, there exists a continuous strictly increasing

convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that

$$\begin{aligned}\|y_n - p\|^2 &= \|\beta_{n,1}Tx_n + \beta_{n,2}T^2x_n + \beta_{n,3}x_n - p\|^2 \\ &\le \beta_{n,1}\|Tx_n - p\|^2 + \beta_{n,2}\|T^2x_n - p\|^2 + \beta_{n,3}\|x_n - p\|^2 \\ &\quad -\beta_{n,j}\beta_{n,3}g(\|x_n - T^jx_n\|) \\ &\le \beta_{n,1}\|x_n - p\|^2 + \beta_{n,2}\|x_n - p\|^2 + \beta_{n,3}\|x_n - p\|^2 \\ &\quad -\beta_{n,j}\beta_{n,3}\|x_n - T^jx_n\|^2 \\ &= \|x_n - p\|^2 - \beta_{n,j}\beta_{n,3}g(\|x_n - T^jx_n\|) \\ &\le \|x_n - p\|^2,\ j = 1,2.\end{aligned} \tag{3.7}$$

This implies that

$$\begin{aligned}\|x_{n+1} - p\| &= \|\alpha_n u + (1-\alpha_n)y_n - p\| \le \alpha_n\|u - p\| + (1-\alpha_n)\|y_n - p\| \\ &\le \alpha_n\|u - p\| + (1-\alpha_n)\|x_n - p\| \le \max\{\|u - p\|, \|x_n - p\|\}.\end{aligned}$$

By induction, we obtain

$$\|x_{n+1} - p\| \le \max\{\|u - p\|, \|x_1 - p\|\}$$

for all $n \in \mathbb{N}$. This implies that the sequence $\{\|x_n - p\|\}_{n\in\mathbb{N}}$ is bounded and hence the sequence $\{x_n\}_{n\in\mathbb{N}}$ is bounded. This, together with (3.6), implies that the sequences $\{y_n\}_{n\in\mathbb{N}}$, $\{Tx_n\}_{n\in\mathbb{N}}$ and $\{T^2x_n\}_{n\in\mathbb{N}}$ are bounded too.

Step 2. We prove that for any $n \in \mathbb{N}$

$$\|x_{n+1} - z\|^2 \le (1-\alpha_n)\|x_n - z\|^2 + 2\alpha_n\langle u - z, J(x_{n+1} - z)\rangle. \tag{3.8}$$

Let us show (3.8). For each $n \in \mathbb{N}$ and $j = 1, 2$, in view of (3.7), we obtain

$$\|y_n - z\|^2 \le \|x_n - z\|^2 - \beta_{n,j}\beta_{n,3}g(\|x_n - T^jx_n\|).$$

This implies that

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \|\alpha_n u + (1-\alpha_n)y_n - z\|^2 \\ &\le \alpha_n\|u - z\|^2 + (1-\alpha_n)\|y_n - z\|^2 \\ &\le \alpha_n\|u - z\|^2 + (1-\alpha_n)[\|x_n - z\|^2 - \beta_{n,j}\beta_{n,3}g(\|x_n - T^jx_n\|)].\end{aligned} \tag{3.9}$$

Let $M_7 := \sup\{|\|u - z\|^2 - \|x_n - z\|^2| + \beta_{n,j}\beta_{n,3}g(\|x_n - T^jx_n\|) : n \in \mathbb{N},\ j = 1,2\}$. It follows from (3.9) that

$$\beta_{n,j}\beta_{n,3}g(\|x_n - T^jx_n\|) \le \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + \alpha_n M_7. \tag{3.10}$$

In view of Lemma 2.8 and (3.6), we obtain

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \|\alpha_n u + (1-\alpha_n)y_n - z\|^2 \\ &\le \|\alpha_n u + (1-\alpha_n)y_n - z - \alpha_n(u - z)\|^2 \\ &\quad +2\langle\alpha_n(u - z), J(x_{n+1} - z)\rangle \\ &= \|(1-\alpha_n)(y_n - z)\|^2 + 2\alpha_n\langle u - z, J(x_{n+1} - z)\rangle \\ &= (1-\alpha_n)\|y_n - z\|^2 + 2\alpha_n\langle u - z, J(x_{n+1} - z)\rangle \\ &\le (1-\alpha_n)\|x_n - z\|^2 + 2\alpha_n\langle u - z, J(x_{n+1} - z)\rangle.\end{aligned}$$

Step 3. We prove that $x_n \to z$ as $n \to \infty$.

We discuss the following two possible cases:

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\|x_n - z\|\}_{n=n_0}^{\infty}$ is nonincreasing.

Then, the sequence $\{\|x_n - z\|\}_{n\in\mathbb{N}}$ is convergent. Thus we have $\|x_n - z\|^2 - \|x_{n+1} - z\|^2 \to 0$ as $n \to \infty$. This, together with condition (c) and (3.10), implies that

$$\lim_{n\to\infty} g(\|x_n - T^j x_n\|) = 0,\ j = 1, 2.$$

From the properties of g, it follows that

$$\lim_{n\to\infty} \|x_n - T^j x_n\| = 0,\ = 1, 2. \tag{3.11}$$

On the other hand, we have

$$y_n - x_n = \beta_{n,1}(x_n - Tx_n) + \beta_{n,2}(x_n - T^2 x_n),\ \text{and}\ x_{n+1} - y_n = \alpha_n(u - y_n).$$

This implies that

$$\lim_{n\to\infty} \|y_n - x_n\| = 0,\ \text{and}\ \lim_{n\to\infty} \|x_{n+1} - y_n\| = 0. \tag{3.12}$$

By the triangle inequality, we conclude that

$$\|x_{n+1} - x_n\| \le \|x_{n+1} - y_n\| + \|y_n - x_n\|.$$

It follows from (3.12) that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.13}$$

Since $\{x_n\}_{n\in\mathbb{N}}$ is bounded, there exists a subsequence $\{x_{n_i}\}_{i\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $x_{n_i+1} \rightharpoonup y \in F(T)$. This, together with Lemma 2.7, implies that

$$\begin{aligned} \limsup_{n\to\infty}\langle u - z, J(x_{n+1} - z)\rangle &= \lim_{i\to\infty}\langle u - z, J(x_{n_i+1} - z)\rangle \\ &= \langle u - z, J(y - z)\rangle \\ &\le 0. \end{aligned} \tag{3.14}$$

Thus we have the desired result by Lemma 2.9.

Case 2. Suppose that there exists a subsequence $\{n_i\}_{i\in\mathbb{N}}$ of $\{n\}_{n\in\mathbb{N}}$ such that

$$\|x_{n_i} - z\| < \|x_{n_i+1} - z\|$$

for all $i \in \mathbb{N}$. Then, by Lemma 2.10, there exists a nondecreasing sequence $\{m_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ such that $m_k \to \infty$,

$$\|z - x_{m_k}\| < \|z - x_{m_k+1}\| \text{ and } \|z - x_k\| \le \|x_{m_k+1} - z\|$$

for all $k \in \mathbb{N}$. This, together with (3.9), implies that

$$\beta_{m_k}(1 - \beta_{m_k}) g(\|x_{m_k} - Tx_{m_k}\|) \le \|x_{m_k} - z\|^2 - \|x_{m_k+1} - z\|^2 + \alpha_{m_k} M_7 \le \alpha_{m_k} M_7$$

for all $k \in \mathbb{N}$. Then, by conditions (a) and (c), we get

$$\lim_{k\to\infty} \|x_{m_k} - Tx_{m_k}\| = 0.$$

By the same argument as Case 1, we arrive at

$$\limsup_{k\to\infty}\langle u - z, J(x_{m_k} - z)\rangle = \limsup_{k\to\infty}\langle u - z, J(x_{m_k+1} - z)\rangle \le 0.$$

It follows from (3.8) that

$$\|x_{m_k+1} - z\|^2 \quad \le (1 - \alpha_{m_k})\|x_{m_k} - z\|^2 + \alpha_{m_k}\langle u - z, J(x_{m_k} - z)\rangle. \tag{3.15}$$

Since $\|x_{m_k} - z\| \leq \|x_{m_k+1} - z\|$, we have that

$$\begin{aligned}\alpha_{m_k}\|x_{m_k} - z\|^2 &\leq \|x_{m_k} - z\|^2 - \|x_{m_k+1} - z\|^2 + \alpha_{m_k}\langle u - z, J(x_{m_k} - z)\rangle \\ &\leq 2\alpha_{m_k}\langle u - z, J(x_{m_k} - z)\rangle.\end{aligned} \tag{3.16}$$

In particular, since $\alpha_{m_k} > 0$, we obtain

$$\|x_{m_k} - z\|^2 \leq \langle u - z, J(x_{m_k} - z)\rangle.$$

In view of (3.16), we deduce that

$$\lim_{k\to\infty} \|x_{m_k} - z\| = 0.$$

This, together with (3.14), implies that

$$\lim_{k\to\infty} \|x_{m_k+1} - z\| = 0.$$

On the other hand, we have $\|x_k - z\| \leq \|x_{m_k+1} - z\|$ for all $k \in \mathbb{N}$ which implies that $x_{m_k} \to z$ as $k \to \infty$. Thus, we have $x_n \to z$ as $n \to \infty$, which completes the proof. □

Let C be a nonempty, closed and convex subset of a Banach space E and $T : C \to C$ be a 2-generalized hybrid mapping such that $F(T) \neq \emptyset$. For any real numbers $\beta, \gamma, \delta \in (0, 1)$ with $\beta + \gamma + \delta = 1$, we define a mapping $T_{\beta,\gamma,\delta} : C \to C$ by

$$T_{\beta,\gamma,\delta}x = \beta Ix + \gamma Tx + \delta T^2x, \ (x \in C), \tag{3.17}$$

where I is the identity mapping on E. It is easy to see that $F(T_{\beta,\gamma,\delta}) = F(T)$. The following strong convergence result provides an affirmative answer to open question 1.1 in the case where the mapping T is a 2-generalized hybrid mapping.

Theorem 3.8. *Let E be a real uniformly convex Banach which admits the weakly sequentially continuous duality mapping J and C be a nonempty, closed and convex subset of E. Let $T : C \to C$ be a 2-generalized hybrid mapping such that $F := F(T) \neq \emptyset$. Let $\{\alpha_n\}_{n\in\mathbb{N}}$ be a sequence in $[0, 1]$ and let $\beta, \gamma, \delta \in (0, 1)$ be real numbers satisfying the following control conditions:*
(a) $\lim_{n\to\infty} \alpha_n = 0$;
(b) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(c) $\beta + \gamma + \delta = 1$.
Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by

$$\begin{cases} u \in C, \ x_1 \in C \quad \textit{chosen arbitrarily}, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)T_{\beta,\gamma,\delta}x_n, \end{cases}$$

where $T_{\beta,\gamma,\delta}$ is defined by (3.17). Then, the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $Q_F u$, where Q_F is a sunny nonexpansive retraction from E onto F.

Remark 3.1. The introduction of condition (I) for the mapping T in theorem 3.5 and the auxiliary mapping $T_{\beta,\gamma,\delta}$ in theorem 3.8 yields strong convergence theorems of Ishikawa's and Halpern's type iterations for 2-generalized hybrid mappings and hence resolves in the affirmative the open problem raised by Hojo, Takahashi and Termwuttipong in [17].

4. Preliminaries on CAT(0) Spaces

A metric space X is a CAT(0) space if it is geodesically connected, and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane. The precise definition is given below. It is well known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a CAT(0) space. Other examples include pre-Hilbert spaces, CAT(0)-trees (see, for example [2]), Euclidean building (see, for example [3]), and the complex Hilbert ball with a hyperbolic metric (see, for example [13]). For a thorough discussion of other spaces and of the fundamental role they play in geometry, see, for example, [3]. Burago et al. [5] contains a somewhat more elementary treatment, and Gromov [15] is a deeper study.

Let (X,d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from x to y) is a map c from a closed interval $[0,l] \subset \mathbb{R}$ to X such that $c(0)=x,\ c(l)=y$, and $d(c(t),c(t'))=|t-t'|$ for all $t,t' \in [0,l]$. In particular, c is an isometry and $d(x,y)=l$. The image α of c is called a *geodesic* (or *metric*) *segment* joining x and y. When it is unique, this geodesic is denoted by $[x,y]$. The space (X,d) is said to be a *geodesic space* if every two points of X are joined by a geodesic, and X is said to be a *uniquely geodesic* if there exists exactly one geodesic joining x and y for each $x,y \in X$. A subset Y of X is said to be *convex* if Y includes every geodesic segment joining any two of its points.

A *geodesic triangle* $\Delta(x_1,x_2,x_3)$ in a geodesic space (X,d) consists of three points x_1,x_2,x_3 in X (the vertices of Δ), together with a geodesic segment between each pair of vertices (the edges of Δ). A *comparison triangle* for a geodesic triangle $\Delta(x_1,x_2,x_3)$ in a geodesic space (X,d) is a triangle $\bar{\Delta}(x_1,x_2,x_3) := \Delta(\bar{x}_1,\bar{x}_2,\bar{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\bar{x}_i,\bar{x}_j)=d(x_i,x_j)$ for $i,j \in \{1,2,3\}$.

A geodesic space is said to be a *CAT(0) space* if all geodesic triangles satisfy the following comparison axiom CAT(0). Let Δ be a geodesic triangle in X, and let $\bar{\Delta}$ be a comparison triangle for Δ. Then Δ is said to satisfy the CAT(0) *inequality* if, for all $x,y \in \Delta$ and all comparison points $\bar{x},\bar{y} \in \Delta,\ d(x,y) \le d_{\mathbb{E}^2}(\bar{x},\bar{y})$. It is easy to see that a CAT(0) space is uniquely geodesic.

If x,y_1,y_2 are points in a CAT(0) space, and if y_0 is the midpoint of the segment $[y_1,y_2]$, then the CAT(0) inequality implies that

$$d(x,y_0) \le \frac{1}{2}d(x,y_1)^2 + \frac{1}{2}d(x,y_2)^2 - \frac{1}{4}d(y_1,y_2)^2. \tag{CN}$$

This is the (CN) inequality of Bruhat and Tits [4]. By using the (CN) inequality, it is easy to see that the CAT(0) Banach spaces are uniformly convex. In fact, a geodesic space is a CAT(0) space if and only if it satisfies the (CN) inequality [2].

We now collect some properties in CAT(0) spaces. For more details on CAT(0) spaces, we refer the readers to [6, 10, 23].

Lemma 4.1. *(*[10]*) Let (X,d) be a CAT(0) space. Then the following assertions hold:*
(i) For $x,y \in X$ and $t \in [0,1]$, there exists a unique point $z \in [x,y]$ such that

$$d(x,z) = td(x,y) \text{ and } d(y,z) = (1-t)d(x,y). \tag{4.1}$$

We use the notation $(1-t)x \oplus ty$ *for the unique point* z *satisfying (4.1).*
(ii) For $x, y \in X$ *and* $t \in [0,1]$*, we have*

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x,z) + td(y,z).$$

The notion of asymptotic centers in a Banach space can be extended to a CAT(0) space as well, by simply replacing the distance defined by $\|\cdot - \cdot\|$ with the one by the metric $d(\cdot,\cdot)$. In particular, in a CAT(0) space, $A(C,\{x_n\})$ consists of exactly one point where C is a closed and convex set and $\{x_n\}_{n\in\mathbb{N}}$ is a bounded sequence; see [9, Proposition 7].

Definition 4.2. *(*[18, 20]*) A sequence* $\{x_n\}_{n\in\mathbb{N}}$ *in a CAT(0) space* X *is said to* Δ*-converge to* $x \in X$ *if* x *is the unique asymptotic center of* $\{u_n\}_{n\in\mathbb{N}}$ *for every subsequence* $\{u_n\}_{n\in\mathbb{N}}$ *of* $\{x_n\}_{n\in\mathbb{N}}$*. In this case, we write* $\Delta - \lim_{n\to\infty} x_n = x$*, and we call* x *the* $\Delta - \lim$ *of* $\{x_n\}_{n\in\mathbb{N}}$*.*

Lemma 4.3. *(*[18]*) Every bounded sequence in a complete CAT(0) space* X *always has a* Δ*-convergent subsequence.*

Lemma 4.4. *(*[8]*) If* C *is a closed and convex subset of a complete CAT(0) space* X*, and if* $\{x_n\}_{n\in\mathbb{N}}$ *is a bounded sequence in* C*, then the asymptotic center of* $\{x_n\}_{n\in\mathbb{N}}$ *is in* C*.*

Lemma 4.5. *(*[19]*) Let* X *be a complete CAT(0) space and let* $x \in X$*. Suppose that* $0 < b \leq t_n \leq c < 1$*, and* x_n, y_n *in* X *such that* $\limsup_{n\to\infty} d(x_n, x) \leq r$*,* $\limsup_{n\to\infty} d(y_n, x) \leq r$ *and* $\lim_{n\to\infty} d(t_n x_n \oplus (1-t_n)y_n, x) = r$ *for some* $r \geq 0$ *and* $n = 1, 2, \ldots$*. Then* $\lim_{n\to\infty} d(x_n, y_n) = 0$*.*

Recall that the *Ishikawa iteration* in CAT(0) spaces is described as follows: for any initial point x_1 in C, we define the iterates $\{x_n\}_{n\in\mathbb{N}}$ by

$$\begin{cases} y_n = \lambda_n T x_n \oplus (1-\lambda_n)x_n, \\ x_{n+1} = \gamma_n T y_n \oplus (1-\gamma_n)x_n, \end{cases} \tag{4.2}$$

where the sequences $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{\gamma_n\}_{n\in\mathbb{N}}$ satisfy some appropriate conditions.

We introduce the notion of 2-generalized hybrid mappings in CAT(0) spaces.

Definition 4.6. Let C be a nonempty subset of a CAT(0) space X. A mapping $T : C \to X$ is said to be
(1) *generalized hybrid* if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha d(Tx,Ty)^2 + (1-\alpha)d(x,Ty)^2 \leq \beta(Tx,y)^2 + (1-\beta)d(x,y)^2, \quad \forall x,y \in C.$$

(2) 2-*generalized hybrid* or $(\alpha_1,\alpha_2,\beta_1,\beta_2)$-*generalized hybrid* if there exist α_1, α_2, β_1, $\beta_2 \in \mathbb{R}$ such that

$$\begin{aligned} &\alpha_1 d(T^2x,Ty)^2 + \alpha_2 d(Tx,Ty)^2 + (1-\alpha_1-\alpha_2)d(x,Ty)^2 \\ &\leq \beta_1 d(T^2x,y)^2 + \beta_2 d(Tx,y)^2 + (1-\beta_1-\beta_2)d(x,y)^2, \quad \forall x,y \in C. \end{aligned}$$

Clearly, $(0,1,1,1)$-generalized hybrid maps are exactly nonexpansive maps.

The proofs of the following results are similar to those in Sections 2 and 3.

Lemma 4.7. *Let C be a nonempty subset of a CAT(0) space X, and let $T : C \to X$ be be an $(\alpha_1, \alpha_2, \beta_1, \beta_2)$-generalized hybrid mapping for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that $F(T) \neq \emptyset$. Then T is quasi-nonexpansive.*

Lemma 4.8. *Let C be a nonempty subset of a CAT(0) space X. Let $T : C \to X$ be an $(\alpha_1, \alpha_2, \beta_1, \beta_2)$-generalized hybrid mapping for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$. Then for any $x, y \in C$,*

$$d(x, Ty)^2 \leq d(x, y)^2 + |\alpha_1|\theta_{x,y} + |\alpha_2|\eta_{x,y} + |\beta_1|\delta_{x,y} + |\beta_2|\sigma_{x,y}, \tag{4.3}$$

where

$$\begin{aligned} \theta_{x,y} &= d(x, T^2x)^2 + 2d(x, T^2x)d(T^2x, Ty) + 2d(x, T^2x)d(x, Ty), \\ \eta_{x,y} &= d(x, Tx)^2 + 2d(x, Tx)d(Tx, Ty) + 2d(x, Tx)d(x, Ty), \\ \delta_{x,y} &= d(T^2x, x)^2 + 2d(T^2x, x)d(x, y) + 2d(T^2x, x)d(T^2x, y), \\ \sigma_{x,y} &= d(Tx, x)^2 + 2d(Tx, x)d(x, y) + 2d(Tx, x)d(Tx, y). \end{aligned} \tag{4.4}$$

Lemma 4.9. *Let C be a nonempty, closed and convex subset of a CAT(0) space X. Let $T : C \to C$ be an $(\alpha_1, \alpha_2, \beta_1, \beta_2)$-generalized hybrid mapping for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ with a nonempty fixed point set $F(T)$. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be two sequences defined by (4.2) such that $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{\gamma_n\}_{n\in\mathbb{N}}$ are arbitrary sequences in $[0, 1]$.*
Then the following assertions hold:
(1) $\max\{d(x_{n+1}, z), d(y_n, z)\} \leq d(x_n, z)$ for any $z \in F(T)$ and for $n = 1, 2, \ldots$.
(2) $\lim_{n\to\infty} d(x_n, z)$ exists for any $z \in F(T)$.
(3) $\lim_{n\to\infty} d(x_n, F(T))$ exists.

Lemma 4.10. *([6]) Let C be a nonempty and convex subset of a CAT(0) space X, and let $T : C \to C$ be a quasi-nonexpansive map whose fixed point set is nonempty. Then $F(T)$ is closed, convex and hence contractible.*

The following result is deduced from Lemmas 4.7 and 4.10.

Lemma 4.11. *Let C be a nonempty and convex subset of a CAT(0) space X, and let $T : C \to C$ be an $(\alpha_1, \alpha_2, \beta_1, \beta_2)$-generalized hybrid mapping for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$. Then $F(T)$ is closed, convex, and hence contractible.*

Lemma 4.12. *Let C be a nonempty, closed and convex subset of a complete CAT(0) space X. Let $T : C \to C$ be an $(\alpha_1, \alpha_2, \beta_1, \beta_2)$-generalized hybrid mapping for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$. If $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in C such that $d(Tx_n, x_n) \to 0$ and $\Delta - \lim_{n\to\infty} x_n = z$ for some $z \in X$, then $z \in C$ and $Tz = z$.*

Proof. In view of Lemma 4.4, it follows that $z \in C$. By Lemma 4.8, we deduce that for any $x, y \in C$,

$$d(x, Ty)^2 \leq d(x, y)^2 + |\alpha_1|\theta_{x,y} + |\alpha_2|\eta_{x,y} + |\beta_1|\delta_{x,y} + |\beta_2|\sigma_{x,y}, \tag{4.3}$$

where

$$\begin{aligned} \theta_{x,y} &= d(x, T^2x)^2 + 2d(x, T^2x)d(T^2x, Ty) + 2d(x, T^2x)d(x, Ty), \\ \eta_{x,y} &= d(x, Tx)^2 + 2d(x, Tx)d(Tx, Ty) + 2d(x, Tx)d(x, Ty), \\ \delta_{x,y} &= d(T^2x, x)^2 + 2d(T^2x, x)d(x, y) + 2d(T^2x, x)d(T^2x, y), \\ \sigma_{x,y} &= d(Tx, x)^2 + 2d(Tx, x)d(x, y) + 2d(Tx, x)d(Tx, y). \end{aligned} \tag{4.4}$$

for all $n \in \mathbb{N}$. Thus we have

$$\limsup_{n\to\infty} d(x_n, Tz) \leq \limsup_{n\to\infty} d(x_n, z).$$

By the uniqueness of asymptotic centers, we obtain that $Tz = z$.

5. Fixed point and convergence theorems in CAT(0) spaces

In this section, we extend our results in Section 3 to CAT(0) spaces.

Theorem 5.1. *Let C be a nonempty, closed and convex subset of a complete CAT(0) space X and let $T : C \to C$ be an $(\alpha_1, \alpha_2, \beta_1, \beta_2)$-generalized hybrid mapping for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$. Let $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{\gamma_n\}_{n\in\mathbb{N}}$ be sequences in $[0,1]$ such that $0 < \liminf_{k\to\infty} \gamma_{n_k} \leq \limsup_{k\to\infty} \gamma_{n_k} < 1$ for a subsequence $\{\gamma_{n_k}\}_{k\in\mathbb{N}}$ of $\{\gamma_n\}_{n\in\mathbb{N}}$. We assume also that $\limsup_{k\to\infty} \lambda_{n_k} < 1$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence with x_1 in C defined by (4.2).*
(i) If $\{x_n\}_{n\in\mathbb{N}}$ is bounded, $\liminf_{n\to\infty} d(Tx_n, x_n) = 0$ and $\liminf_{n\to\infty} d(T^2x_n, x_n) = 0$, then the fixed point set $F(T) \neq \emptyset$.
(ii) Assume $F(T) \neq \emptyset$. Then $\{x_n\}_{n\in\mathbb{N}}$ is bounded and $\liminf_{n\to\infty} d(Tx_n, x_n) = 0$.

Proof. (i) By simply replacing $\| \cdot - \cdot \|$ with $d(\cdot, \cdot)$ in the proof of Theorem 3.2, we have the desired result $F(T) \neq \emptyset$.
(ii) Suppose that $F(T) \neq \emptyset$ and $z \in F(T)$ is arbitrarily chosen. By Lemma 4.9, $\lim_{n\to\infty} d(x_n, z)$ exists and $\{x_n\}_{n\in\mathbb{N}}$ is bounded. Let

$$\lim_{n\to\infty} d(x_n, z) = l. \tag{5.1}$$

It follows from Lemmas 4.7 and 4.1(ii) that

$$\begin{aligned} d(Ty_n, z) &\leq d(y_n, z) \\ &= d(\lambda_n Tx_n \oplus (1-\lambda_n)x_n, z) \\ &\leq \lambda_n d(Tx_n, z) + (1-\lambda_n) d(x_n, z) \\ &\leq \lambda_n d(x_n, z) + (1-\lambda_n) d(x_n, z) \\ &= d(x_n, z). \end{aligned}$$

Thus, we have

$$\limsup_{n\to\infty} d(Ty_n, z) \leq \limsup_{n\to\infty} d(y_n, z) \leq \limsup_{n\to\infty} d(x_n, z) = l. \tag{5.2}$$

On the other hand, it follows from (4.2) and (5.1) that

$$\lim_{n\to\infty} d(\gamma_n Ty_n \oplus (1-\gamma_n)x_n, z) = \lim_{n\to\infty} d(x_{n+1}, z) = l. \tag{5.3}$$

In view of (5.1)-(5.3) and Lemma 4.5, we conclude that

$$\lim_{k\to\infty} d(Ty_{n_k}, x_{n_k}) = 0.$$

By simply replacing $\| \cdot - \cdot \|$ with $d(\cdot, \cdot)$ in the proof of Theorem 3.2, we have the desired result $\lim_{k\to\infty} d(Tx_{n_k}, x_{n_k}) = 0$. □

Theorem 5.2. *Let C be a nonempty, closed and convex subset of a complete CAT(0) space X and let $T : C \to C$ be an $(\alpha_1, \alpha_2, \beta_1, \beta_2)$-generalized hybrid mapping for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$. Let $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{\gamma_n\}_{n\in\mathbb{N}}$ be sequences in $[0,1]$ such that*

$0 < \liminf_{k\to\infty} \gamma_{n_k} \leq \limsup_{k\to\infty} \gamma_{n_k} < 1$ *for a subsequence* $\{\gamma_{n_k}\}_{k\in\mathbb{N}}$ *of* $\{\gamma_n\}_{n\in\mathbb{N}}$. *We assume also that* $\limsup_{k\to\infty} \lambda_{n_k} < 1$. *Let* $\{x_n\}_{n\in\mathbb{N}}$ *be a sequence with* x_1 *in* C *defined by (4.2). If* $F(T) \neq \emptyset$*, then* $\{x_{n_k}\}_{k\in\mathbb{N}}$ Δ*-converges to a fixed point of* T.

Proof. It follows from Theorem 5.1 that $\{x_n\}_{n\in\mathbb{N}}$ is bounded and

$$\lim_{k\to\infty} d(Tx_{n_k}, x_{n_k}) = 0.$$

Denote by $\omega_w(x_{n_k}) := \cup A(C, \{u_n\})$, where the union is taken over all subsequences $\{u_n\}_{n\in\mathbb{N}}$ of $\{x_{n_k}\}_{k\in\mathbb{N}}$. We prove that $\omega_w(x_{n_k}) \subset F(T)$. Let $u \in \omega_w(x_{n_k})$. Then there exists a subsequence $\{u_n\}_{n\in\mathbb{N}}$ of $\{x_{n_k}\}_{k\in\mathbb{N}}$ such that $A(C, \{u_n\}) = \{u\}$. In view of Lemmas 4.3 and 4.4, there exists a subsequence $\{v_n\}_{n\in\mathbb{N}}$ of $\{u_n\}_{n\in\mathbb{N}}$ such that $\Delta - \lim_{n\to\infty} v_n = v$ for some v in C. Since $\lim_{n\to\infty} d(Tv_n, v_n) = 0$, Lemma 4.12 implies that $v \in F(T)$. By Lemma 4.9, the limit $\lim_{n\to\infty} d(x_n, v)$ exists. We claim that $u = v$. For else, the uniqueness of asymptotic centers implies that

$$\begin{aligned}\limsup_{n\to\infty} d(v_n, v) &< \limsup_{n\to\infty} d(v_n, u) \leq \limsup_{n\to\infty} d(u_n, u)\\ &< \limsup_{n\to\infty} d(u_n, v) = \limsup_{n\to\infty} d(x_n, v)\\ &= \limsup_{n\to\infty} d(v_n, v),\end{aligned}$$

which is a contradiction. Thus, we have $u = v \in F(T)$ and hence $\omega_w(x_{n_k}) \subset F(T)$.

Now, we prove that $\{x_{n_k}\}_{k\in\mathbb{N}}$ Δ-converges to a fixed point of T. It suffices to show that $\omega_w(x_{n_k})$ consists of exactly one point. Let $\{u_n\}_{n\in\mathbb{N}}$ be a subsequence of $\{x_{n_k}\}_{k\in\mathbb{N}}$. In view of Lemmas 4.3 and 4.4, there exists a subsequence $\{v_n\}_{n\in\mathbb{N}}$ of $\{u_n\}_{n\in\mathbb{N}}$ such that $\Delta - \lim_{n\to\infty} v_n = v$ for some v in C. Let $A(C, \{u_n\}) = \{u\}$ and $A(C, \{x_{n_k}\}) = \{x\}$. By the argument mentioned above we have $u = v$ and $v \in F(T)$. We show that $x = v$. If it is not the case, then the uniqueness of asymptotic centers implies that

$$\begin{aligned}\limsup_{n\to\infty} d(v_n, v) &< \limsup_{n\to\infty} d(v_n, x) \leq \limsup_{n\to\infty} d(x_n, x)\\ &< \limsup_{n\to\infty} d(x_n, v) = \limsup_{n\to\infty} d(v_n, v),\end{aligned}$$

which is a contradiction. Thus we have the desired result. □

Theorem 5.3. *Let* C *be a nonempty, compact and convex subset of a complete CAT(0) space* X *and let* $T : C \to C$ *be an* $(\alpha_1, \alpha_2, \beta_1, \beta_2)$*-generalized hybrid mapping for some* $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ *such that the fixed point set* $F(T)$ *is nonempty. Let* $\{\lambda_n\}_{n\in\mathbb{N}}$ *and* $\{\gamma_n\}_{n\in\mathbb{N}}$ *be sequences in* $[0, 1]$ *such that* $0 < \liminf_{k\to\infty} \gamma_{n_k} \leq \limsup_{k\to\infty} \gamma_{n_k} < 1$ *for a subsequence* $\{\gamma_{n_k}\}_{k\in\mathbb{N}}$ *of* $\{\gamma_n\}_{n\in\mathbb{N}}$. *We assume also that* $\limsup_{k\to\infty} \lambda_{n_k} < 1$. *Let* $\{x_n\}_{n\in\mathbb{N}}$ *be a sequence with* x_1 *in* C *defined by (4.2). Then* $\{x_n\}_{n\in\mathbb{N}}$ *converges in metric to a fixed point of* T.

Proof. Using Lemma 4.8 and simply replacing $\| \cdot - \cdot \|$ with $d(\cdot, \cdot)$ in the proof of Theorem 3.4, we conclude the desired result. □

As in the proof of Theorem 3.5, we can verify the following result.

Theorem 5.4. *Let C be a nonempty, closed and convex subset of a complete CAT(0) space X and let $T : C \to C$ be an $(\alpha_1, \alpha_2, \beta_1, \beta_2)$-generalized hybrid mapping for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that the fixed point set $F(T)$ is nonempty. Let $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{\gamma_n\}_{n\in\mathbb{N}}$ be sequences in $[0,1]$ such that $0 < \liminf_{k\to\infty} \gamma_{n_k} \leq \limsup_{k\to\infty} \gamma_{n_k} < 1$ for a subsequence $\{\gamma_{n_k}\}_{k\in\mathbb{N}}$ of $\{\gamma_n\}_{n\in\mathbb{N}}$. We assume also that $\limsup_{k\to\infty} \lambda_{n_k} < 1$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence with x_1 in C defined by (4.2). If T satisfies condition (I), then $\{x_n\}_{n\in\mathbb{N}}$ converges in metric to a fixed point of T.*

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