Fixed Point Theory, 15(2014), No. 2, 495-500
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

THE FIXED POINT ALTERNATIVE THEOREM AND SET-VALUED FUNCTIONAL EQUATIONS

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Abstract. We use the fixed point alternative theorem to prove the stability of the set-valued function equation

$$c(x)F(h(x)) = F(x).$$

This result enable us to prove the stability of some set-valued functional equations.
Key Words and Phrases: Set-valued mappings, functional inequalities, non-expensive mappings, fixed point.
2010 Mathematics Subject Classification: 39B52, 39B69, 47S09, 47H10.

1. Introduction

One of the main topics in functional equations is Hyers-Ulam stability which was originated from a question of S. M. Ulam [25]. D. H. Hyers [12] gave the first significant partial solution to Ulam's question. The theorem of Hyers was generalized by T. Aoki [1] for additive mappings and by Th.M. Rassias [22] for linear mappings by considering an unbounded Cauchy difference. The paper of Th.M. Rassias has provided a lot of influence in the development of what we now call Hyers-Ulam-Rassias stability of functional equations.

It should be noted that almost all proofs in this topic used Hyers method. In 1991, Baker [3] used the Banach fixed point theorem to prove Hyers-Ulam stability for a non-linear functional equation. V. Radu [21], in 2003, employed the fixed point alternative theorem [9] to establish the stability of Cauchy additive functional equation. Using such an elegant idea, several authors applied the method to investigate the stability of some functional equations, (see e. g. [5, 6, 7, 10, 13, 14, 16, 17, 19, 20]).

The theory of set-valued functions was fairly systematically developed for the first time in Berge's book [4]. It is of interest to investigate the Hyers-Ulam stability of set-valued functional equations and inclusions. Although there are much less results of Hyers-Ulam stability for set-valued ones than those for single-valued ones, some interesting results were obtained by several mathematicians (e.g. [2, 11, 15, 18, 23, 24, 26]).

In this paper, we apply the fixed point alternative theorem to prove the stability of set-valued functional equations. More precisely, we will prove the stability of functional equation

$$c(x)F\Big(h(x)\Big) = F(x)$$

in the space of compact convex subsets of a Banach space. We will show that this result can be applied to prove the stability of set-valued Cauchy functional equation. Our method may be applied to prove the stability of several other set-valued functional equations.

2. Main results

Hereafter, unless otherwise state, we will assume that X is a semigroup and Y is a Banach space. If $A, B \subset Y$ and $\lambda \in \mathbb{R}$, we define

$$A + B = \{a + b : a \in A,\ b \in B\}, \quad \lambda A = \{\lambda a : a \in A\}.$$

One can easily see that for each $A, B \subset Y$ and $\lambda, \mu \geq 0$,

$$\lambda(A + B) = \lambda A + \lambda B, \quad (\lambda + \mu)A \subseteq \lambda A + \mu A.$$

Moreover, if A is convex, then $(\lambda + \mu)A = \lambda A + \mu A$. We denote by $CC(Y)$ the collection of all non-empty compact convex subsets of Y. Let

$$\mathcal{H}(A, B) = \inf\{s > 0 : A \subset B + sK,\ B \subset A + sK\},$$

where K is the closed unit ball in Y and $A, B \subset Y$ are non-empty closed bounded sets. The function $\mathcal{H}$ is a metric called the *Hausdorff metric* induced by the space Y. It is known that if Y is a Banach space, then $\mathcal{H}$ defines a complete metric on $CC(Y)$ [8].

The following result reveals some basic properties of Hausdorff distance.

Theorem 2.1. [8, Page 188] *Let Y be a real normed space. If $A, B, X \in CC(Y)$ and m is a positive number, then*

$$\mathcal{H}(A + X, B + X) = \mathcal{H}(A, B),$$
$$\mathcal{H}(mA, mB) = m\mathcal{H}(A, B).$$

Definition 2.2. Let Ω be a nonempty set and $d : \Omega \times \Omega \to [0, \infty]$ satisfy the following properties:

(i) $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$(symmetry),
(iii) $d(x, z) \leq d(x, y) + d(y, z)$(triangle inequality),

for all $x, y, z \in \Omega$. Then (Ω, d) is called a generalized metric space. (Ω, d) is called complete if every d-Cauchy sequence in Ω is d-convergent.

We recall the following result by Diaz and Margolis.

Theorem 2.3. (The fixed point alternative theorem [9]) *Suppose that a complete generalized metric space (Ω, d) and a strictly contractive mapping $\mathcal{J} : \Omega \to \Omega$ with the Lipschitz constant $0 < L < 1$ are given. Then, for a given element $x \in \Omega$, exactly one of the following assertions is true:*
either
(a) $d(\mathcal{J}^n x, \mathcal{J}^{n+1} x) = \infty$ for all $n \geq 0$ or

(b); there exists a natural number k such that $d(\mathcal{J}^n x, \mathcal{J}^{n+1} x) < \infty$ for all $n \geq k$. Actually, if (b) holds, then the sequence $\{\mathcal{J}^n x\}$ is convergent to a fixed point x^* of $\mathcal{J}$ and
(b1) x^* is the unique fixed point of $\mathcal{J}$ in $\mathcal{F} := \{y \in \Omega, d(\mathcal{J}^k x, y) < \infty\}$;
(b2) $d(y, x^*) \leq \frac{d(y,\mathcal{J}y)}{1-L}$ for all $y \in \mathcal{F}$.

Definition 2.4. Let Ω denote the set of all functions $F : X \to CC(Y)$ and $\varphi : X \to [0, \infty)$ be a mapping. We define a function $d_\varphi : \Omega \times \Omega \to [0, \infty]$ by

$$d_\varphi(F, G) = \inf\{a > 0 \mid \forall x \in X, \mathcal{H}(F(x), G(x)) \leq a\varphi(x)\} \quad (F, G \in \Omega).$$

Lemma 2.5. *(Ω, d_φ) is a complete generalized metric space for each $\varphi : X \to [0, \infty)$.*

Proof. We first prove that (Ω, d_φ) is a generalized metric space. Fix $F, G, H \in \Omega$. Let $d_\varphi(F, G) = 0$. Then for each $x \in X$ and $a > 0$, we have $\mathcal{H}(F(x), G(x)) < a\varphi(x)$. This means that for each $x \in X$, $F(x) = G(x)$.
Conversely, if $F = G$, then it follows from the definition that $d_\varphi(F, G) = 0$. Clearly, d_φ is symmetric. To prove the triangle inequality note that if either $d(F, H) = \infty$ or $d(H, G) = \infty$, then $d_\varphi(F, G) \leq d_\varphi(F, H)+d_\varphi(H, G)$. Suppose that $d_\varphi(F, H) = \alpha < \infty$ and $d_\varphi(H, G) = \beta < \infty$. Then for each $\epsilon > 0$, we can find real numbers a_1, a_2 such that

$$\alpha < a_1 < \alpha + \epsilon \text{ and } \beta < a_2 < \beta + \epsilon.$$

Then for each $x \in X$, we have $\mathcal{H}(F(x), H(x)) \leq a_1\varphi(x)$ and $\mathcal{H}(H(x), G(x)) \leq a_2\varphi(x)$. It follows that

$$\mathcal{H}(F(x), G(x)) \leq (a_1 + a_2)\varphi(x) \quad (x \in X).$$

This means that for each $\epsilon > 0$, $\quad d_\varphi(F, G) \leq \alpha + \beta + 2\epsilon$. This proves the triangle inequality. Next, we will show that (Ω, d_φ) is a complete generalized metric space. Let $\{F_n\}$ be a Cauchy sequence in generalized metric space (Ω, d_φ). For each $\epsilon > 0$ there exists natural number N_ϵ such that for all $n, m > N_\epsilon$ we have $d_\varphi(F_n, F_m) < \epsilon$. It follows that

$$\mathcal{H}(F_n(x), F_m(x)) < \varphi(x)\varepsilon \quad (x \in X, n, m > N_\varepsilon).$$

Hence $\{F_n(x)\}$ is a Cauchy sequence in complete metric space $(CC(Y), \mathcal{H})$. Let

$$F(x) = \lim_{n\to\infty} F_n(x) \quad (x \in X).$$

We have to show that $F_n \to F$ in (Ω, d_φ). The above argument shows that for each $\varepsilon > 0$, there is some N_ε such that

$$\mathcal{H}(F_n(x), F_m(x)) < \varphi(x)\varepsilon \quad (n, m > N_\varepsilon, x \in X).$$

By taking limit of the above inequality as $m \to \infty$, we see that

$$\mathcal{H}(F_n(x), F(x)) < \varphi(x)\varepsilon \quad (n > N_\varepsilon, x \in X).$$

Hence

$$d_\varphi(F_n, F) \leq \varepsilon \quad (n > N_\varepsilon).$$

This shows that (Ω, d_φ) is a generalized complete metric space.

Let $h : X \to X$ and $\varphi, c : X \to [0, \infty)$ are given functions. We inductively define $c_0(x) = 1$, $c_1(x) = c(x)$ and for each $n > 1$, $c_n(x) = c(x)c_{n-1}(h(x))$ for each $x \in X$. Moreover, let $h^0(x) = x$, $h^1(x) = h(x)$ and for each $n > 1$, define

$$h^n(x) = \underbrace{ho\ldots oh}_{n-terms}(x) \text{ for each } x \in X.$$

The following Theorem is the main result of this paper.

Theorem 2.6. *Let $F : X \to CC(Y)$ satisfy the inequality*

$$\mathcal{H}(c(x)F(h(x)), F(x)) < \varphi(x) \quad (x \in X). \tag{2.1}$$

If for some $0 < L < 1$,

$$c(x)\varphi(h(x)) \leq L\varphi(x) \quad (x \in X),$$

then there is a unique set-valued function $G : X \to CC(Y)$ such that

$$c(x)G(h(x)) = G(x)$$

and

$$\mathcal{H}(F(x), G(x)) < \frac{\varphi(x)}{1 - L} \tag{2.2}$$

for all $x \in X$.

Proof. Let Ω denote the set of all functions $H : X \to CC(Y)$. In view of Lemma 2.5, (Ω, d_φ) is a generalized complete metric space. Define $\mathcal{J} : \Omega \to \Omega$ by

$$\mathcal{J}(H)(x) = c(x)H(h(x)) \quad (H \in \Omega, x \in X).$$

Let $H_1, H_2 \in \Omega$ and for some $\alpha > 0$, $d_\varphi(H_1, H_2) < \alpha$, then

$$\begin{aligned} \mathcal{H}\Big(\mathcal{J}(H_1)(x), \mathcal{J}(H_2)(x)\Big) &= c(x)\mathcal{H}\Big(H_1(h(x)), H_2(h(x))\Big) \\ &\leq \alpha c(x)\varphi(h(x)) \\ &\leq \alpha L\varphi(x) \quad (x \in X). \end{aligned}$$

Thus $d_\varphi\Big(\mathcal{J}(H_1), \mathcal{J}(H_2)\Big) \leq L\alpha$. It follows that

$$d_\varphi\Big(\mathcal{J}(H_1), \mathcal{J}(H_2)\Big) \leq Ld_\varphi(H_1, H_2) \quad (H_1, H_2 \in \Omega).$$

Hence $\mathcal{J}$ is strictly contractive mapping with Lipschitz constant L on Ω. Let $F_n = \mathcal{J}^n(F)$ for each $n \in \mathbb{N}$. By induction on n, we will show that for each $n \geq 1$,

$$\mathcal{H}\Big(F_n(x), F_{n-1}(x)\Big) \leq L^n\varphi(x) \quad (x \in X). \tag{2.3}$$

For $n = 1$, (2.3) is (2.1). Let for some $n \geq 1$, (2.3) holds. Then

$$\begin{aligned} \mathcal{H}\Big(F_{n+1}(x), F_n(x)\Big) &= \mathcal{H}\Big(c(x)F_n(h(x)), c(x)F_{n-1}(h(x))\Big) \\ &\leq c(x)\mathcal{H}\Big(F_n(h(x)), F_{n-1}(h(x))\Big) \\ &\leq L^n c(x)\varphi(h(x)) \\ &\leq L^{n+1}\varphi(x) \quad (x \in X). \end{aligned}$$

Hence (2.3) holds for each $n \geq 1$. It follows that

$$d_\varphi\Big(\mathcal{J}^n(F), \mathcal{J}^{n-1}(F)\Big) \leq L^n \quad (n \in \mathbb{N}).$$

In view of Theorem 2.3, the sequence $\{\mathcal{J}^n(F)\}$ is convergent to a fixed point G of $\mathcal{J}$, G is the unique fixed point of $\mathcal{J}$ in $\mathcal{F} := \{H \in \Omega, d(\mathcal{J}^k(F), H) < \infty\}$ and $d_\varphi(H, G) \leq \frac{d_\varphi(H,\mathcal{J}(F))}{1-L}$ for all $H \in \mathcal{F}$. It follows that

$$G(x) = \mathcal{J}(G)(x) = c(x)G(h(x)) \quad (x \in X)$$

and

$$d_\varphi(F, G) \leq \frac{d_\varphi(F, \mathcal{J}(F))}{1-L} \leq \frac{1}{1-L}.$$

Hence (2.2) holds.

The next result gives an application of Theorem 2.6.

Theorem 2.7. Let $F : X \to CC(Y)$ satisfies the following inequality

$$\mathcal{H}\Big(F(x+y), F(x) + F(y)\Big) \leq \psi(x, y) \quad (x, y \in X), \tag{2.4}$$

where $\psi : X \times X \to [0, \infty)$ is a function with the following properties:
(i) $\psi(2x, 2x) \leq L\psi(x, x)$ for each $x \in X$, where $0 < L < 1$.
(ii) $\lim_{n\to\infty} 2^{-n}\psi(2^n x, 2^n y) = 0$ for each $x, y \in X$.
Then there exists a unique additive function $A : X \to CC(Y)$ such that

$$\mathcal{H}\Big(F(x), A(x)\Big) \leq \frac{\psi(x, x)}{2(1-L)} \quad (x \in X). \tag{2.5}$$

Proof. Put $y = x$ in (2.4) to obtain $\mathcal{H}\Big(F(2x), 2F(x)\Big) \leq \psi(x, x) \quad (x, y \in X)$.
It follows from the above inequality that for

$$c(x) = \tfrac{1}{2},\ h(x) = 2x \text{ and } \varphi(x) = \tfrac{1}{2}\psi(x, x),\ (x \in X),$$

the conditions of Theorem 2.6 hold. Hence there is a unique function $A : X \to CC(Y)$ which is defined by $A(x) = \lim\limits_{n\to\infty} 2^{-n}F(2^n x)$ $(x \in X)$ and satisfies (2.5) and $A(2x) = 2A(x)$ for each $x \in X$. Since for each $x, y \in X$ and $n \geq 1$,

$$\begin{aligned}\mathcal{H}\Big(A(x+y), A(x) + A(y)\Big) &\leq \mathcal{H}\Big(A(x+y), 2^{-n}F(2^n(x+y))\Big)\\ &+ \mathcal{H}\Big(2^{-n}F(2^n(x+y)), 2^{-n}F(2^x) + 2^{-n}F(2^n y)\Big)\\ &+ \mathcal{H}\Big(2^{-n}F(2^x) + 2^{-n}F(2^n y), A(x) + A(y)\Big)\end{aligned}$$

and the right hand side of the above inequality tends to zero as $n \to \infty$, A is additive.

Remark. A similar argument as it was used in Theorem 2.7 may be applied to prove the stability of other set-valued functional equations.

References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces,* J. Math. Soc. Japan, **2**(1950), 64–66.

[2] J. Brzdęk, D. Popa, B. Xu, *Selection of set-valued maps satisfying a linear inclusion in single variable,* Nonlinear Anal., **74**(2011), 324–330.

[3] J.A. Baker, *The stability of certain functional equations,* Proc. Amer. Math. Soc, **112**(1991), 729–732.
[4] C. Berge, *Topological Spaces,* Dover Publications, 1963.
[5] L. Cădariu, V. Radu, *On the stability of the Cauchy functional equation: a fixed point approach,* Grazer Math. Ber., **346**(2004), 43–52.
[6] L. Cădariu, V. Radu, *Remarks on the stability of monomial functional equations,* Fixed Point Theory, **8**(2007), 201–218.
[7] L. Cădariu, V. Radu, *Fixed point methods for the generalized stability of functional equations in a single variable,* Fixed Point Theory Appl., **2008**(2008), Art. ID 749392.
[8] S. Czerwik, *Functional equations and inequalities in several variables*, World Scientific Publ., 2002.
[9] J.B. Diaz, B. Margolis, *A fixed point theorem of the alternative for the contractions on a generalized complete metric space*, Bull. Amer. Math. Soc., **74**(1968), 305-309.
[10] M. Eshaghi Gordji, H. Khodaei, J.M. Rassias, *Fixed point methods for the stability of general quadratic functional equation,* Fixed Point Theory, **12**(2011), 71–82.
[11] Z. Gajda, R. Ger, *Subadditive mulifunctions and Hyers-Ulam stability,* Numer. Math., **80**, (1987), 281–291.
[12] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A., **27**(1941), 222–224.
[13] S.-M. Jung, T.-S. Kim, K.-S. Lee, *A fixed point approach to the stability of quadratic functional equation,* Bull. Korean Math. Soc., **43**(2006), No. 3, 531–541.
[14] D. Mihet, *The fixed point method for fuzzy stability of the Jensen functional equation,* Fuzzy Sets and Systems, **160**(2009), 1663–1667.
[15] K. Nikodem, W. Zhang, *On a multivalued iterative equation*, Publ. Math. Debrecen, **64**(2004), 427-435.
[16] A.K. Mirmostafaee, *Non-Archimedean stability of quadratic equations*, Fixed Point Theory, **11**(2010), 67–75.
[17] A.K. Mirmostafaee, *Non-Archimedean stability of the monomial functional equations*, Tamsui Oxford J. Math. Sci., **26**(2010), 221–235.
[18] C. Park, D. O'Regan, R. Saadati, *Stability of some set-valued functional equations*, Appl. Math. Lett., **24**(2011), 1910–1914.
[19] C. Park, *Generalized Hyers-Ulam Stability of Quadratic Functional Equations: A Fixed Point Approach*, Fixed Point Theory Appl., Volume 2008 (2008), Article ID 493751, 9 pages-doi:10.1155/2008/493751.
[20] J.M. Rassias, *Alternative contraction principle and Ulam stability problem,* Math. Sci. Res. J., **9**(7)(2005), 190–199.
[21] V. Radu, *The fixed point alternative and stability of functional equations,* Fixed Point Theory, **4**(2003), no. 1. 91–96.
[22] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., **72**(1978), 297–300.
[23] A. Smajdor, *Additive selections of superadditive set-valued functions,* Aequations Math., **39**(1990), 121–128.
[24] A. Smajdor, *Subadditive and subquadratic set-valed functions,* Prac Naukowe Uniwersytetu Śląskiego W Katowicach, Katowice, **889**(1987).
[25] S.M. Ulam, *Problems in Modern Mathematics (Chapter VI, Some Questions in Analysis: §1, Stability)*, Science Editions, John Wiley & Sons, New York, 1964.
[26] Z. Wanxiong, X. Bing, *Hyers-Ulam stability for a multivalued equation,* Acta Math. Scientia, **28**(2008), 54–62.

Received: May 02, 2012; Accepted: May 05, 2013