Fixed Point Theory, 15(2014), No. 2, 487-494
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

PERIODIC SOLUTIONS OF SECOND ORDER NON-AUTONOMOUS DIFFERENTIAL SYSTEMS

SHENGJUN LI*, FANG-FANG LIAO** AND HAILONG ZHU***

*College of Information Sciences and Technology
Hainan University, Haikou 570228, China
E-mail: shjli626@126.com

** Nanjing College of Information Technology
Nanjing 210046, China
E-mail: liaofangfang8178@sina.com

***School of Statistics and Applied Mathematics
Anhui University of Finance and Economics
Bengbu 233030, China
E-mail: hai-long-zhu@163.com

Abstract. We study the existence of nonnegative solutions for second order nonlinear differential systems with periodic boundary conditions. In this class of problems, where the associated Green's function may take on negative values, and the nonlinear term is allowed to be singular. Our method is based on the Guo-Krasnosel'skii fixed point theorem of cone expansion and compression type, involving a new type of cone. Recent results in the literature, even in the scalar case, are complemented, generalized and improved.
Key Words and Phrases: Nonnegative solutions, existence, Guo-Krasnosel'skii fixed point theorem, differential systems.
2010 Mathematics Subject Classification: 34B15, 47H10.

1. Introduction

In this paper, we study the existence of nonnegative solutions for the n-dimensional nonlinear system:

$$x'' + a(t)x = h(t)g(x), \tag{1.1}$$

and boundary conditions

$$x(0) = x(T), \quad x'(0) = x'(T), \tag{1.2}$$

where $h(t) = \operatorname{diag}(h_1(t), \cdots, h_n(t))$, $a(t) = \operatorname{diag}(a_1(t), \cdots, a_n(t))$, $(n > 1)$ are continuous T-periodic functions, and $g(x) = (g_1(x), \cdots, g_n(x))^T \in \mathbb{C}(\mathbb{R}^n, \mathbb{R}^n)$. As usual, by a T-periodic nonnegative solution, we mean a function $x(t) = (x_1(t), \cdots, x_n(t))^T \in \mathbb{C}^2(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^n)$ solving (1.1) and such that $x_i(t) \geq 0$ for all t, $i = 1, 2, \cdots, n$.

During the last few decades, the study of the existence of periodic solutions for the second order differential equations and systems with the nonlinearity has attracted the attention of many researchers [5, 10, 12, 11, 14, 17, 20, 21, 22]. Usually, in the literature, the proof is based on variational methods [1, 15], or topological methods, which were started with the pioneering paper of Lazer and Solimini [10]. In particular, the method of upper and lower solutions [2, 12], degree theory [20, 21], Schauder's fixed point theorem [5, 8] , a nonlinear Leray-Schauder alternative principle [4, 6, 11] and some fixed point theorem in cones for completely continuous operators [3, 7, 16] are the most relevant tools.

Guo-Krasnosel'skii fixed point theorem on compression and expansion of cones has been used to study positive solutions for systems of ordinary, functional differential equations [9, 16, 18, 19]. The proof of the main results in this paper is based on it. In the process of the paper, we define a new cone and a new norm by the scalar product, which is different to other papers. By the use of the new cone, we do not need the positivity of the Green function, however, the positivity of the Green function plays a very important role in [6, 7], and therefore they cannot cover the critical case, such as $k = \pi/T$ when $a_i(t) \equiv k^2$, whereas the result in [12] covers such a case.

As mentioned above, this paper is mainly motivated by the recent paper [3, 9, 18]. And the remaining part of this paper is organized as follows. In Section 2, some preliminary results are given. In Section 3, by employing the Guo-Krasnosel'skii fixed point theorem, we establish the main result. To illustrate the new results, some applications are also given.

2. Preliminaries

We say that the linear system

$$x'' + a(t)x = 0 \tag{2.1}$$

is nonresonant if its unique T-periodic solution is the trivial one. When (2.1) is nonresonant, as a consequence of Fredholm's alternative, the nonhomogeneous system

$$x'' + a(t)x = l(t)$$

admits a unique T-periodic solution which can be written as

$$x(t) = \int_0^T G(t,s)l(s)ds,$$

where $G(t,s) = \mathrm{diag}(G_1(t,s), \cdots, G_n(t,s))$ is the Green function of (2.1), associated with (1.2), and $l(t) = (l_1(t), \cdots, l_n(t)) \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^n)$.

Throughout this paper, we always assume that the following standing hypothesis (A) is satisfied:

(A) The Hill equation $x'' + a_i(t)x = 0$ is nonresonant, and the Green function associated with (1.2) verifies $\int_0^T G_i(t,s)ds > 0$, for all t, $i = 1, 2, \cdots, n$.

In other words, the (strict) anti-maximum principle holds for $(2.1) - (1.2)$.

Remark 2.1. If $a_i(t) \equiv k^2$, condition (A) is equivalent to $0 < k^2 \leq \lambda_1 = (\pi/T)^2$, where λ_1 is the first eigenvalue of the homogeneous equation $x'' + k^2x = 0$ with Dirichlet boundary conditions $x(0) = x(T) = 0$. In this case, we have

$$G_i(t,s) = \begin{cases} \frac{\sin k(t-s) + \sin k(T-t+s)}{2k(1-\cos kT)}, & 0 \leq s \leq t \leq T, \\ \frac{\sin k(s-t) + \sin k(T-s+t)}{2k(1-\cos kT)}, & 0 \leq t \leq s \leq T, \end{cases}$$

and

$$0 \le G_i(t,s) \le \frac{1}{2k\sin\frac{kT}{2}}, \quad \int_0^T G_i(t,s)ds = \frac{1}{k^2}.$$

For a nonconstant function $a_i(t)$, there is not an explicit expression of the Green function, but there is an L^p-criterion proved in [16], which is given in the following lemma for the sake of completeness. To describe these, given an exponent $q \in [1,\infty]$, the best constant in the Sobolev inequality

$$C\|u\|_q^2 \le \|u'\|_2^2 \qquad \text{for all } u \in H_0^1(0,T),$$

is denoted by $\mathbf{M}(q)$. The explicit formula for $\mathbf{M}(q)$ is known, that is,

$$\mathbf{M}(q) = \begin{cases} \frac{2\pi}{qT^{1+2/q}}\left(\frac{2}{q+2}\right)^{1-2/q}\left(\frac{\Gamma(1/q)}{\Gamma(1/2+1/q)}\right)^2, & \text{for } 1 \le q < \infty, \\ \frac{4}{T}, & \text{for } q = \infty, \end{cases}$$

where $\Gamma(\cdot)$ is the Gamma function of Euler.

The $a_i(t) \succ 0$ means that $a_i(t) \ge 0$ for all $t \in [0,T]$, and it is positive for t in a subset of positive measure. The usual L^p-norm is denoted by $\|\cdot\|_p$, and the conjugate exponent of p is denoted by $q : 1/p + 1/q = 1$.

Lemma 2.2. [16] For each $i = 1,2,\cdots,n$, assume that $a_i(t) \succ 0$, and $a_i \in L^p[0,T]$ for some $1 \le p \le \infty$. If

$$\|a_i\|_p \le \mathbf{M}(2q).$$

Then the standing hypothesis (A) holds.

Under hypothesis (A), we always denote

$$M_i = \max_{0\le s,t\le T} G_i(t,s), \qquad \tau_i = \min_{0\le t\le T}\int_0^T G_i(t,s)ds,$$

and

$$M = \max\{M_1,\cdots,M_n\}, \qquad \tau = \min\{\tau_1,\cdots,\tau_n\}, \qquad \sigma = \frac{\tau}{M}.$$

One may readily see that $0 < \sigma \le 1$. When $a_i(t) \equiv k^2$ and $0 < k \le \pi/T$, we have

$$\tau = \frac{1}{k^2}, \quad M = \frac{1}{2k\sin\frac{kT}{2}}, \quad \sigma = \frac{2}{k}\sin\frac{kT}{2}.$$

Let us fix some notation, we will use $\mathbb{R}_+ = [0,\infty)$, $\mathbb{R}_+^n = \prod_{i=1}^n \mathbb{R}_+$. Given $x = (x_1,\cdots,x_n), y = (y_1,\cdots,y_n) \in \mathbb{R}^n$, the usual scalar product is denoted by (x,y), that is $(x,y) = \sum_{i=1}^n x_iy_i$. The usual Euclidean norm is denoted by $|x|$, whereas $|x|_1 = \sum_{i=1}^n |x_i|$ is the l_1-norm. More generally, for a fixed vector $v \in \mathbb{R}_+^n$, we have a well-defined norm

$$|x|_v = \sum_{i=1}^n v_i|x_i|.$$

Obviously, $|x|_v = |x|_1$ if $v = (1,\cdots,1)$. Let $\|\cdot\|$ denote the supremum norm of $\mathbb{C}[0,T]$, and take $X = \mathbb{C}[0,T]\times\cdots\times\mathbb{C}[0,T]$ (n copies). For $x(t) = (x_1(t),\cdots,x_n(t)) \in X$, the natural norm becomes

$$\|x\|_v = \sum_{i=1}^n v_i \|x_i\| = \sum_{i=1}^n v_i \cdot \max_t |x_i(t)|.$$

Obviously X is a Banach space.

Definition 2.3. Let X be a Banach space and let K be a closed, nonempty subset of X. K is a cone if

(i) $\alpha u + \beta v \in K$ for all $u, v \in K$ and all $\alpha, \beta > 0$,

(ii) $u, -u \in K$ implies $u = 0$.

Let $X = \mathbb{C}[0,T] \times \cdots \times \mathbb{C}[0,T]$ (n copies), we write $x(t) \geq 0$, if $(x_1(t), \ldots, x_n(t)) \in \mathbb{R}^n_+$ and define

$$K = \{x \in X : x(t) \geq 0 \quad \text{for all } t \in [0,T] \text{ and } \int_0^T (v, x(t))dt \geq \sigma \|x\|_v, v \in \mathbb{R}^n_+\}.$$

One may verify that K is a cone in X. In fact, clearly K is closed and nonempty. Moreover, for $x, y \in K$ and $a, b > 0$, we have

$$\begin{aligned}\int_0^T (v, ax(t) + by(t))dt &= a \int_0^T (v, x(t))dt + b \int_0^T (v, y(t))dt \\ &\geq a\sigma \|x\|_v + b\sigma \|y\|_v \geq \sigma \|ax + by\|_v.\end{aligned}$$

Suppose now that $h_i : [0,T] \to \mathbb{R}$, $g_i : \mathbb{R}^n \to \mathbb{R}$ are continuous functions. Define the integral operator $T : X \to X$ by

$$(\mathrm{T}x) = (\mathrm{T}_1 x, \cdots, \mathrm{T}_n x)^T$$

where

$$(\mathrm{T}_i x)(t) = \int_0^T G_i(t,s) h_i(s) g_i(x(s))ds, \quad i = 1, 2, \cdots, n,$$

for $x \in X, t \in [0,T]$.

Lemma 2.4. $T : X \to K$ is well defined.

Proof. Let $x \in X$ and we have

$$\begin{aligned}\big(v, (Tx)(t)\big) = \sum_{i=1}^n v_i \, |(T_i x)(t)| &= \sum_{i=1}^n v_i \left| \int_0^T G_i(t,s) h_i(s) g_i(x(s))ds \right| \\ &\leq \sum_{i=1}^n v_i M_i \left| \int_0^T h_i(s) g_i(x(s))ds \right|.\end{aligned}$$

which implies $\|Tx\|_v \leq M \left| \int_0^T (v, h(s)g(x(s))ds \right|$.

On the other hand,

$$\begin{aligned}\int_0^T (v, (Tx)(t))dt &= \int_0^T \sum_{i=1}^n v_i \left| \int_0^T G_i(t,s) h_i(s) g_i(x(s))ds \right| dt \\ &\geq \tau \int_0^T (v, h(s)g(x(s))ds.\end{aligned}$$

Thus $\int_0^T (v, Tx(t))dt \geq \sigma \|Tx\|_v$, i.e.,$Tx \in K$, and the proof is completed.

Since $h_i : [0,T] \to \mathbb{R}$, $g_i : \mathbb{R}^n \to \mathbb{R}$ are continuous functions, it is easy to see

Lemma 2.5. $T : X \to K$ is continuous and completely continuous.

3. Main results

In this section, we state and prove the new existence results for (1.1). In order to prove our main results, the following well-known Guo-Krasnosel'skii fixed point theorem of cone and expansion and compression type is need, which can be found in [13].

Theorem 3.1. Let X be a Banach space and K $(\subset X)$ be a cone. Assume that $\Omega_1,\ \Omega_2$ are open subsets of X with $0 \in \Omega_1, \bar{\Omega}_1 \subset \Omega_2$, and let

$$\mathcal{A} : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$$

be a continuous and completely continuous operator such that either

(i) $\|\mathcal{A}u\| \geq \|u\|, u \in K \cap \partial\Omega_1$ and $\|\mathcal{A}u\| \leq \|u\|, u \in K \cap \partial\Omega_2;$ *or*

(ii) $\|\mathcal{A}u\| \leq \|u\|, u \in K \cap \partial\Omega_1$ and $\|\mathcal{A}u\| \geq \|u\|, u \in K \cap \partial\Omega_2$.

Then $\mathcal{A}$ has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

Now we present our main existence result of nonnegative solution to problem (1.1). For convenience, we introduce the notation

$$f_0 = \lim_{|x|_v \to 0^+} \frac{(v, g(x))}{|x|_v}, \qquad f_\infty = \lim_{|x|_v \to \infty} \frac{(v, g(x))}{|x|_v}.$$

Theorem 3.2. Suppose that $a(t)$ satisfies conditions (A). Furthermore, we assume that

(H_1) $h_i(t) : [0,T] \to \mathbb{R}_+$ are continuous with $h_i(t) > 0$. $i = 1, 2, \cdots, n$.

(H_2) $g_i(x) : \mathbb{R}_+^n \to \mathbb{R}_+$ are continuous. $i = 1, 2, \cdots, n$.

Then problem (1.1) has a nontrivial solution x with $x(t) \geq 0$ for $t \in [0,T]$, if one of the following conditions hold.

(i) $f_0 = \infty$ and $f_\infty = 0$.

(ii) $f_0 = 0$ and $f_\infty = \infty$.

To prove Theorem 3.2, we define new functions

$$g_i^*(x) = \max_t g_i(x)(i = 1, 2, \cdots, n), \qquad g^*(x) = (g_1^*(x), \cdots, g_n^*(x))$$

and let

$$f_0^* = \lim_{|x|_v \to 0^+} \frac{(v, g^*(x))}{|x|_v}, \qquad f_\infty^* = \lim_{|x|_v \to \infty} \frac{(v, g^*(x))}{|x|_v}.$$

The following Lemma is needed in the proof of Theorem 3.2.

Lemma 3.3.[18] Assume (H_2) holds. Then $f_0 = f_0^*$ and $f_\infty = f_\infty^*$.

Proof of Theorem 3.2. Part (i). By (H_2) and $f_0 = \infty$, we set

$$\gamma = \min\{h_1(t), \cdots, h_n(t)\}, \quad \lambda = \frac{TM}{\tau^2 \gamma},$$

there exist $R_1 > 0$, such that

$$|x|_v \leq R_1 \Rightarrow (v, g(x)) \geq \lambda |x|_v.$$

For any $r > 0$, let

$$\Omega_r = \{x \in K : \|x\|_v < r\}.$$

First we show

$$\|Tx\|_v \geq \|x\|_v \quad \text{for} \quad x \in K \cap \partial\Omega_{R_1}. \tag{3.1}$$

In fact, $x \in K \cap \partial\Omega_{R_1}$, then $R_1 = \|x\|_v \geq |x|_v$.

$$\begin{aligned}
\|Tx\|_v &= \frac{1}{T}\int_0^T \|Tx\|_v dt \geq \frac{1}{T}\int_0^T (v, (Tx)(t))dt \\
&= \frac{1}{T}\int_0^T \sum_{i=1}^n v_i \left| \int_0^T G_i(t,s)h_i(s)g_i(x(s))ds \right| dt \\
&\geq \frac{\lambda\gamma\tau}{T}\int_0^T (v, x(s))ds \\
&\geq \frac{\lambda\gamma\tau^2}{TM}\|x\|_v = \|x\|_v.
\end{aligned}$$

Since $f_\infty = 0$, Lemma 3.3 implies $f_\infty^* = 0$. Thus, there exists $R_2 > |x|_v$, $|x|_v \in (R_1, +\infty)$ such that

$$(v, g^*(x)) \leq \frac{|x|_v}{TM\|h\|}.$$

where $\|h\| = \max_{i\in\{1,\cdots,n\}} \sup_{t\in[0,T]} h_i(t)$.

Next, we show

$$\|Tx\|_v \leq \|x\|_v \quad \text{for} \quad x \in K \cap \partial\Omega_{R_2}. \tag{3.2}$$

To see this, let $x \in K \cap \partial\Omega_{R_2}$, then $\|x\|_v = R_2$.

$$\begin{aligned}
\|Tx\|_v &= \sum_{i=1}^n v_i \max_t \left| \int_0^T G_i(t,s)h_i(s)g_i(x(s))ds \right| \\
&\leq M\|h\| \int_0^T \sum_{i=1}^n v_i |g_i^*(x(s))| ds \\
&\leq M\|h\| T\big(v, g^*(x)\big) \\
&\leq \|x\|_v.
\end{aligned}$$

Now (3.1),(3.2) and Theorem 3.1(i) guarantee that T has a fixed point $x \in K \cap (\bar{\Omega}_{R_2} \backslash \Omega_{R_1})$.

Part (ii). Since $f_\infty = \infty$, we can choose $r_2 > |x|_v$, and $|x|_v$ sufficiently large such that

$$(v, g(x)) \geq \lambda |x|_v.$$

Let $x \in K \cap \partial\Omega_{r_2}$, then $\|x\|_v = r_2$. Similar to part(i) $f_0 = \infty$, we have

$$\|Tx\|_v \geq \|x\|_v \quad \text{for} \quad x \in K \cap \partial\Omega_{r_2}. \tag{3.3}$$

If $f_0 = 0$, Lemma 3.3 implies $f_0^* = 0$. Thus, there exists $r_1 \in (0, r_2)$ such that

$$|x|_v \leq r_1 \Rightarrow (v, g^*(x)) \leq \frac{|x|_v}{TM\|h\|}.$$

Let $x \in K \cap \partial\Omega_{r_1}$, then $\|x\|_v = r_1$. Similar to part (i) $f_\infty = 0$, we have

$$\|Tx\|_v \leq \|x\|_v \quad \text{for} \quad x \in K \cap \partial\Omega_{r_1}. \tag{3.4}$$

Now (3.3),(3.4) and Theorem 3.1(ii) guarantee that T has a fixed point $x \in K \cap (\bar{\Omega}_{r_2} \backslash \Omega_{r_1})$. Clearly, $x(t) \geq 0$ is a nontrivial solution of (1.1). This completes the proof of the theorem.

Corollary 3.4. Assume that $g \in \mathbb{C}(\mathbb{R}^n, \mathbb{R}^n), h \in \mathbb{C}(\mathbb{R}, \mathbb{R}^n)$ and there exists a vector $v \in \mathbb{R}^n_+$, and continuous positive functions $a(t), b(t)$ such that

(F) $\qquad \dfrac{a(t)}{|x|_v^\alpha} \leq (v, g(x)) \leq \dfrac{b(t)}{|x|_v^\alpha}$, for all t and $x \in \mathbb{R}^n_+$.

Then problem (1.1) has a solution x with $x(t) \geq 0$.

Proof. We will apply Theorem 3.2, with the above functions g and h, we see that (H_1) and (H_2) hold. Moreover, it is easy to see that

$$f_0 = \infty \quad \text{and} \quad f_\infty = 0, \quad \text{if} \quad \alpha \in (-1, \infty),$$

and

$$f_0 = 0 \quad \text{and} \quad f_\infty = \infty, \quad \text{if} \quad \alpha \in (-\infty, -1).$$

Then the conclusion follows from Theorem 3.2(i) if $\alpha \in (-1, \infty)$ and Theorem 3.2(ii) if $\alpha \in (-\infty, -1)$.

Acknowledgment. This work is supported by the National Natural Science Foundation of China (Grant No.11161017, 11301001), Hainan Natural Science Foundation (Grant No.113001), Excellent Youth Scholars Foundation and the Natural Science Foundation of Anhui Province of PR China (NO. 2013SQRL030ZD)

References

[1] A. Ambrosetti, V. Coti Zelati, *Periodic Solutions of Singular Lagrangian Systems*, Birkhäuser, Boston, MA, 1993.

[2] D. Bonheure, C. De Coster, *Forced singular oscillators and the method of lower and upper solutions*, Topol. Meth. Nonlinear Anal., **22**(2003), 297-317.

[3] J. Chu, D. Franco, *Non-collision periodic solutions of second order singular dynamical systems*, J. Math. Anal. Appl., **344**(2008), 898-905.

[4] J. Chu, M. Li, *Positive periodic solutions of Hill's equations with singular nonlinear perturbations*, Nonlinear Anal., **69**(2008), 276-286.

[5] J. Chu, P. J. Torres, *Applications of Schauder's fixed point theorem to singular differential equations*, Bull. Lond. Math. Soc., **39**(2007), 653-660.

[6] J. Chu, P.J. Torres, M. Zhang, *Periodic solutions of second order non-autonomous singular dynamical systems*, J. Diff. Eq., **239**(2007), 196-212.

[7] D. Franco, J.R.L. Webb, *Collisionless orbits of singular and nonsingular dynamical systems*, Discrete Contin. Dyn. Syst., **15**(2006), 747-757.

[8] D. Franco, P.J. Torres, *Periodic solutions of singular systems without the strong force condition*, Proc. Amer. Math. Soc., **136**(2008), 1229-1236.
[9] J.R. Graef, L. Kong, H. Wang, *A periodic boundary value problem with vanishing Green's function*, Appl. Math. Lett., **21**(2008), 176-180.
[10] A.C. Lazer, S. Solimini, *On periodic solutions of nonlinear differential equations with singularities*, Proc. Amer. Math. Soc., **99**(1987), 109-114.
[11] S. Li, L. Liang, Z. Xiu, *Positive solutions for nonlinear differential equations with periodic boundary condition*, J. Appl. Math. doi:10.1155/2012/528719.
[12] I. Rachunková, M. Tvrdý, I. Vrkoč, *Existence of nonnegative and nonpositive solutions for second order periodic boundary value problems*, J. Diff. Eq., **176**(2001), 445-469.
[13] D. O'Regan, *Existence Theory for Nonlinear Ordinary Differential Equations*, Kluwer Academic Publ., Dordrecht, 1997.
[14] M. Schechter, *Periodic non-autonomous second-order dynamical systems*, J. Diff. Eq., **223**(2006), 290-302.
[15] K. Tanaka, *A note on generalized solutions of singular Hamiltonian systems*, Proc. Amer. Math. Soc., **122**(1994), 275-284.
[16] P.J. Torres, *Existence of one-signed periodic solutions of some second-order differential equations via a Krasnoselskii fixed point theorem*, J. Diff. Eq., **190**(2003), 643-662.
[17] P.J. Torres, *Existence and stability of periodic solutions for second order semilinear differential equations with a singular nonlinearity*, Proc. Roy. Soc. Edinburgh Sect. A., **137**(2007), 195-201.
[18] H. Wang, *On the number of positive solutions of nonlinear systems*, J. Math. Anal., **281**(2003), 287-306.
[19] F. Wang, F. Zhang, Y. Ya, *Existence of positive solutions of Neumann boundary value problem via a convex functional compression-expansion fixed point theorem*, Fixed Point Theory, **11**(2010), 395-400.
[20] P. Yan, M. Zhang, *Higher order nonresonance for differential equations with singularities*, Math. Meth. Appl. Sci., **26**(2003), 1067-1074.
[21] M. Zhang, *Periodic solutions of equations of Ermakov-Pinney type,* Adv. Nonlinear Stud., **6**(2006), 57-67.
[22] S. Zhang, Q. Zhou, *Nonplanar and noncollision periodic solutions for $N-$body problems*, Discrete Contin. Dyn. Syst., **10**(2004), 679-685.

Received: November 30, 2012; Accepted: March 28, 2013