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MULTIPLE POSITIVE SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR P-LAPLACIAN IMPULSIVE DYNAMIC EQUATIONS ON TIME SCALES

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Abstract. In this paper, by using the classical fixed-point index theorem for compact maps and Leggett-Williams fixed point theorem, some sufficient conditions for the existence of multiple positive solutions for a class of second-order p-Laplacian boundary value problem with impulse on time scales are established. We also give an example to illustrate our results
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1. Introduction

It is known that the theory of impulsive differential equations have become more important in recent years in some mathematical models of real processes and phenomena. For the introduction of the basic theory of impulsive equations, see [1, 12] and the references therein.

The theory of dynamic equations on time scales has been developing rapidly and have received much attention in recent years. The study unifies existing results from the theory of differential and finite difference equations and provides powerful new tools for exploring connections between the traditionally separated fields. We refer to the books by Bohner and Peterson [3, 4].

Recently, the boundary value problems for impulsive differential equations have been studied extensively. To identify a few, we refer to the reader to see [7, 8, 9, 11, 18, 20]. However, the corresponding theory of such equations is still in the beginning stages of its development, especially the impulsive dynamic equations on time scales, see [2, 15, 19]. There is not so much work on impulsive boundary value problems with p-Laplacian on time scales except that in [5, 6, 10, 16, 17]. To our knowledge, no paper has considered the second-order BVP with integral boundary conditions for p-Laplacian impulsive dynamic equations on time scales. This paper fills this gap in the literature.

In this paper, we are concerned with the existence of many positive solutions of the following boundary value problem for p-Laplacian impulsive dynamic equations on time scales

$$-(\phi_p(u^{\triangle}))^{\nabla}(t) = f(t, u(t)), \quad t \in [0,1]_{\mathbb{T}}, \quad t \neq t_k, \quad k = 1, 2, ..., m \tag{1.1}$$

$$u(t_k^+) - u(t_k^-) = I_k(u(t_k)), \quad k = 1, 2, ..., m \tag{1.2}$$

$$\alpha u(0) - \beta u^{\triangle}(0) = \int_0^1 u(s) \triangle s, \quad u^{\triangle}(1) = 0 \tag{1.3}$$

where $\mathbb{T}$ is a time scale, $0, 1 \in \mathbb{T}, [0,1]_{\mathbb{T}} = [0,1] \cap \mathbb{T}, t_k \in (0,1) \cap \mathbb{T}, k = 1, 2, ..., m$ with $0 < t_1 < t_2 < ... < t_m < 1, \alpha > 1, \beta > 0$, $\phi_p(s)$ is a p-Laplacian function, i.e., $\phi_p(s) = |s|^{p-2}s$ for $p > 1, (\phi_p)^{-1}(s) = \phi_q(s)$ where $\frac{1}{p} + \frac{1}{q} = 1$.

We assume that the following conditions are satisfied:

$(H1)$ $f \in \mathcal{C}([0,1] \times [0,\infty), [0,\infty))$;

$(H2)$ $I_k \in \mathcal{C}([0,\infty), [0,\infty))$, $t_k \in [0,1]_{\mathbb{T}}$ and $u(t_k^+) = \lim_{h \to 0} u(t_k + h)$, $u(t_k^-) = \lim_{h \to 0} u(t_k - h)$ represent the right and left limits of $u(t)$ at $t = t_k, k = 1, 2, ..., m$.

We remark that by a solution u of (1.1)-(1.3) we mean $u : \mathbb{T} \to \mathbb{R}$ is delta differentiable, $u^{\triangle} : \mathbb{T}^k \to \mathbb{R}$ is nabla differentiable on $\mathbb{T}^k \cap \mathbb{T}_k$ and $u^{\triangle\nabla} : \mathbb{T}^k \cap \mathbb{T}_k \to \mathbb{R}$ is continuous, and satisfies the impulsive and boundary value conditions (1.2)-(1.3).

By using the fixed point index theory in the cone [13], we get the existence of at least two or more positive solutions for the BVP (1.1)-(1.3). By using Leggett-Williams fixed point theorem [14], we also establish the existence of triple positive solutions for BVP (1.1)-(1.3).

The organization of this paper is as follows. In section 2, we provide some necessary background. In section 3, the main results for problem (1.1)-(1.3) are given. In section 4, we give an example.

2. Preliminaries

In this section, we list some background material from the theory of cones in Banach spaces.

Definition 2.1. Let $(E, \|.\|)$ be a real Banach space. A nonempty, closed set $K \subset E$ is said to be a cone provided the following are satisfied:
(i) if $x, y \in K$ and $a, b \geq 0$, then $ax + by \in K$;
(ii)if $y \in K$ and $-y \in K$, then $y = 0$.
If $K \subset E$ is a cone, then K can induce a partially order relation $\leq$ on E by $x \leq y$, if and only if $y - x \in K$, for all $x, y \in E$.

Definition 2.2. A function $f : \mathbb{T} \to \mathbb{R}$ is called concave on $I_{\mathbb{T}} = I \cap \mathbb{T}$, if $f(\lambda t + (1 - \lambda)s) \geq \lambda f(t) + (1 - \lambda) f(s)$, for all $t, s \in I_{\mathbb{T}}$ and all $\lambda \in [0,1]$ such that $\lambda t + (1 - \lambda)s \in I_{\mathbb{T}}$.

Remark 2.3. If $u^{\triangle\nabla} \leq 0$ on $[0,1]_{\mathbb{T}^k \cap \mathbb{T}_k}$, then we say that u is concave on $[0,1]_{\mathbb{T}}$.

Theorem 2.4. [13] Let K be a cone in a real Banach space E. Let D be an open bounded subset of E with $D_K = D \cap K \neq \emptyset$ and $\bar{D}_K \neq K$. Assume that $T : \bar{D}_K \to K$ is completely continuous such that $x \neq Tx$ for $x \in \partial D_K$. Then the following results hold:

(*i*) If $\|Tx\| \leq \|x\|, x \in \partial D_K$, then $i_K(T, D_K) = 1$.

(*ii*) If there exists $e \in K \setminus \{0\}$ such that $x \neq Tx + \lambda e$ for all $x \in \partial D_K$ and all $\lambda > 0$, then $i_K(T, D_K) = 0$.

(*iii*) Let U be open in K such that $\bar{U} \subset D_K$. If $i_K(T, D_K) = 1$ and $i_K(T, U_K) = 0$, then T has a fixed point in $D_K \setminus \bar{U}_K$. The same result holds if $i_K(T, D_K) = 0$ and $i_K(T, U_K) = 1$.

Let E be a real Banach space with cone K, a map $\theta : K \to [0, \infty)$ is said to be nonnegative continuous concave functional on K if θ is continuous and

$$\theta(tx + (1-t)y) \geq t\theta(x) + (1-t)\theta(y)$$

for all $x, y \in K$ and $t \in [0,1]_{\mathbb{T}}$. Let a, b be two numbers such that $0 < a < b$ and θ a nonnegative continuous concave functional on K. We define the following convex sets:

$$K_a = \{x \in K : \|x\| < a\},$$

$$K(\theta, a, b) = \{x \in K : a \leq \theta(x), \|x\| < b\}.$$

Theorem 2.5. [14] (Leggett-Williams fixed point theorem) Let $T : \bar{K}_c \to \bar{K}_c$ be completely continuous and θ be a nonnegative continuous concave functional on P such that $\theta(x) \leq \|x\|$ for all $x \in \bar{K}_c$. Suppose there exists $0 < d < a < b \leq c$ such that

(*i*) $\{x \in K(\theta, a, b) : \theta(x) > a\} \neq \emptyset$ and $\theta(Tx) > a$ for $x \in K(\theta, a, b)$,

(*ii*) $\|Tx\| < d$ for $\|x\| \leq d$,

(*iii*) $\theta(Tx) > a$ for $x \in K(\theta, a, c)$ with $\|Tx\| > b$.

Then T has at least three fixed points x_1, x_2, x_3 in $\bar{K}_c$ such that $\|x_1\| < d, a < \theta(x_2)$ and $\|x_3\| > d$ with $\theta(x_3) < a$.

3. Main results

In this section, by defining an appropriate Banach space and cone, we impose the growth conditions on f, I_k which allow us to apply the theorems in section 2 to establish the existence results of the positive solutions for the BVP (1.1)-(1.3).

Let $J' = [0,1]_{\mathbb{T}} \setminus \{t_1, t_2, ..., t_m\}$. We define

$$E = \Big\{u : [0,1]_{\mathbb{T}} \to \mathbb{R} \;\; \textit{is continuous at} \;\; t \neq t_k, \;\; \textit{there exist} \;\; u(t_k^-) \;\; \textit{and} \\ u(t_k^+) \;\; \textit{with} \;\; u(t_k^-) = u(t_k) \;\; \textit{for} \;\; k = 1, 2, ..., m\Big\},$$

which is a Banach space with the norm $\|u\| = \sup_{t \in [0,1]_{\mathbb{T}}} |u(t)|$.

By a solution of (1.1)-(1.3), we mean a function $u \in E \cap \mathcal{C}^2(J')$ which satisfies (1.1)-(1.3). We define a cone $K \subset E$ as

$$K = \Big\{ u \in E : u \text{ is a concave, nonnegative and nondecreasing function,} \\ \alpha u(0) - \beta u^{\triangle}(0) = \int_0^1 u(s) \triangle s \Big\}.$$

Lemma 3.1. Suppose that $(H1)$ and $(H2)$ are satisfied. Then $u \in E \cap \mathcal{C}^2(J')$ is a positive solution of the impulsive boundary value problem (1.1)-(1.3) if and only if $u(t)$ is a solution of the following integral equation

$$u(t) = \frac{\beta}{\alpha - 1}\phi_q\left(\int_0^1 f(s, u(s))\nabla s\right) + \frac{1}{\alpha - 1}\int_0^1 (1 - \sigma(\tau))\phi_q\left(\int_\tau^1 f(r, u(r))\nabla r\right)\Delta\tau \\ + \int_0^t \phi_q\left(\int_s^1 f(r, u(r))\nabla r\right)\Delta s + \frac{1}{\alpha - 1}\sum_{k=1}^m I_k(u_{t_k})(1 - t_k) + \sum_{t_k < t} I_k(u_{t_k}). \tag{3.1}$$

Proof. Integrating of (1.1) from t to 1, one has

$$-\phi_p(u^{\triangle}(1)) + \phi_p(u^{\triangle}(t)) = \int_t^1 f(s, u(s))\nabla s.$$

By the boundary condition (1.3), we have

$$u^{\triangle}(t) = \phi_q\left(\int_t^1 f(s, u(s))\nabla s\right).$$

Integrating the differential equation above from 0 to t, we get

$$u(t) = u(0) + \int_0^t \phi_q\left(\int_s^1 f(r, u(r))\nabla r\right)\Delta s + \sum_{t_k < t} I_k(u_{t_k}). \tag{3.2}$$

Applying the boundary condition (1.3), one has

$$u(0) = \frac{\beta}{\alpha - 1}\phi_q\left(\int_0^1 f(s, u(s))\nabla s\right) + \frac{1}{\alpha - 1}\int_0^1 (1 - \sigma(\tau))\phi_q\left(\int_\tau^1 f(r, u(r))\nabla r\right)\Delta\tau \\ + \frac{1}{\alpha - 1}\sum_{k=1}^m I_k(u_{t_k})(1 - t_k). \tag{3.3}$$

Therefore, by (3.2) and (3.3), we have

$$u(t) = \frac{\beta}{\alpha - 1}\phi_q\left(\int_0^1 f(s, u(s))\nabla s\right) + \frac{1}{\alpha - 1}\int_0^1 (1 - \sigma(\tau))\phi_q\left(\int_\tau^1 f(r, u(r))\nabla r\right)\Delta\tau \\ + \int_0^t \phi_q\left(\int_s^1 f(r, u(r))\nabla r\right)\Delta s + \frac{1}{\alpha - 1}\sum_{k=1}^m I_k(u_{t_k})(1 - t_k) + \sum_{t_k < t} I_k(u_{t_k}).$$

Then, sufficient proof is complete.

Conversely, let u be as in (3.1). If we take the delta derivative of both sides of (3.1), then

$$u^{\triangle}(t) = \phi_q\left(\int_t^1 f(s,u(s))\nabla s\right),$$

$$\text{i.e.,}\quad \phi_p(u^{\triangle}(t)) = \int_t^1 f(s,u(s))\nabla s.$$

So $u^{\triangle}(1) = 0$. It is easy to see that $u(t)$ satisfy (1.2) and (1.3). Furthermore, from $(H1), (H2)$ and (3.1), it is clear that $u(t) \geq 0$. The proof is complete. □

Lemma 3.2. If $u \in K$, then $\min\limits_{t\in[0,1]_{\mathbb{T}}} u(t) \geq \gamma\|u\|$, where $\gamma = \dfrac{\int_0^1 s\Delta s}{\alpha - 1 + \int_0^1 s\Delta s}$.

Proof. Since $u \in K$, nonnegative and nondecreasing

$$\|u\| = u(1), \quad \min_{t\in[0,1]_{\mathbb{T}}} u(t) = u(0).$$

On the other hand, $u(t)$ is concave on $[0,1]_{\mathbb{T}} \setminus \{t_1, ..., t_m\}$. So, for every $t \in [0,1]_{\mathbb{T}}$, we have

$$u(1) - u(0) \leq \frac{u(t) - u(0)}{t},$$

i.e.,

$$tu(1) + (1-t)u(0) \leq u(t).$$

Therefore,

$$\int_0^1 su(1)\triangle s + \int_0^1 (1-s)u(0)\triangle s \leq \int_0^1 u(s)\triangle s.$$

This together with $\alpha u(0) - \beta u^{\triangle}(0) = \int_0^1 u(s)\triangle s$, implies that

$$u(0) \geq \frac{\int_0^1 s\Delta s}{\alpha - 1 + \int_0^1 s\Delta s} u(1).$$

The Lemma is proved. □

Define $T : K \to E$ by

$$(Tu)(t) = \frac{\beta}{\alpha-1}\phi_q\left(\int_0^1 f(s,u(s))\nabla s\right) + \frac{1}{\alpha-1}\int_0^1 (1-\sigma(\tau))\phi_q\left(\int_\tau^1 f(r,u(r))\nabla r\right)\Delta\tau$$

$$+ \int_0^t \phi_q\left(\int_s^1 f(r,u(r))\nabla r\right)\Delta s + \frac{1}{\alpha-1}\sum_{k=1}^m I_k(u_{t_k})(1-t_k) + \sum_{t_k<t} I_k(u_{t_k}). \quad (3.4)$$

From (3.4) and Lemma 3.1, it is easy to obtain the following result.

Lemma 3.3. Assume that $(H1)$ and $(H2)$ hold. Then $T : K \to K$ is completely continuous.

We define

$$\begin{aligned}\Omega_\rho &= \{u \in K : \min_{t\in[0,1]_{\mathbb{T}}} u(t) < \gamma\rho\} \\ &= \{u \in K : \gamma\|u\| \le \min_{t\in[0,1]_{\mathbb{T}}} u(t) < \gamma\rho\}\end{aligned}$$

and

$$K_\rho = \{u \in K : \|u\| < \rho\}.$$

Lemma 3.4. [13] Ω_ρ has the following properties:
(a) Ω_ρ is open relative to K.
(b) $K_{\gamma\rho} \subset \Omega_\rho \subset K_\rho$.
(c) $u \in \partial\Omega_\rho$ if and only if $\min_{t\in[0,1]_{\mathbb{T}}} u(t) = \gamma\rho$.
(d) If $u \in \partial\Omega_\rho$, then $\gamma\rho \le u(t) \le \rho$ for $t \in [0,1]_{\mathbb{T}}$.

Now for convenience we introduce the following notations. Let

$$f_a^b = \min\left\{\min_{t\in[0,1]_{\mathbb{T}}} \frac{f(t,u)}{\phi_p(a)} : u \in [a,b]\right\},$$

$$f_{\gamma\rho}^\rho = \min\left\{\min_{t\in[0,1]_{\mathbb{T}}} \frac{f(t,u)}{\phi_p(\rho)} : u \in [\gamma\rho,\rho]\right\},$$

$$f_0^\rho = \max\left\{\max_{t\in[0,1]_{\mathbb{T}}} \frac{f(t,u)}{\phi_p(\rho)} : u \in [0,\rho]\right\},$$

$$I_0^\rho(k) = \max\left\{I_k(u) : u \in [0,\rho]\right\},$$

$$\frac{1}{l} = \frac{\beta + (1+m)\alpha}{\alpha - 1}, \quad \frac{1}{L} = \frac{\beta}{\alpha - 1}.$$

Theorem 3.5. Suppose $(H1)$ and $(H2)$ hold.

$(H3)$ If there exist $\rho_1, \rho_2, \rho_3 \in (0,\infty)$ with $\rho_1 < \gamma\rho_2$ and $\rho_2 < \rho_3$ such that

$$f_0^{\rho_1} < \phi_p(l),\ I_0^{\rho_1}(k) < l\rho_1,\ f_{\gamma\rho_2}^{\rho_2} > \phi_p(L),\ f_0^{\rho_3} < \phi_p(l),\ I_0^{\rho_3}(k) < l\rho_3,$$

then problem (1.1)-(1.3) has at least two positive solutions u_1, u_2 with $u_1 \in \Omega_{\rho_2} \setminus \bar{K}_{\rho_1},\ u_2 \in K_{\rho_3} \setminus \bar{\Omega}_{\rho_2}$.

$(H4)$ If there exist $\rho_1, \rho_2, \rho_3 \in (0,\infty)$ with $\rho_1 < \rho_2 < \gamma\rho_3 < \rho_3$ such that

$$f_{\gamma\rho_1}^{\rho_1} > \phi_p(L),\ f_0^{\rho_2} < \phi_p(l),\ I_0^{\rho_2}(k) < l\rho_2,\ f_{\gamma\rho_3}^{\rho_3} > \phi_p(L),$$

then problem (1.1)-(1.3) has at least two positive solutions u_1, u_2 with $u_1 \in K_{\rho_2} \setminus \bar{\Omega}_{\rho_1},\ u_2 \in \Omega_{\rho_3} \setminus \bar{K}_{\rho_2}$.

Proof. We only consider the condition $(H3)$. If $(H4)$ holds, then the proof is similar to that of the case when $(H3)$ holds. By Lemma 3.3, we know that the operator $T : K \to K$ is completely continuous.

First, we show that $i_K(T, K_{\rho_1}) = 1$. In fact, by (3.4), $f_0^{\rho_1} < \phi_p(l)$ and $I_0^{\rho_1}(k) < l\rho_1$, we have for $u \in \partial K_{\rho_1}$,

$$
\begin{aligned}
(Tu)(t) &= \frac{\beta}{\alpha-1}\phi_q\left(\int_0^1 f(s,u(s))\nabla s\right) + \int_0^t \phi_q\left(\int_s^1 f(r,u(r))\nabla r\right)\triangle s \\
&+ \frac{1}{\alpha-1}\int_0^1 (1-\sigma(s))\phi_q\left(\int_s^1 f(r,u(r))\nabla r\right)\Delta s \\
&+ \frac{1}{\alpha-1}\sum_{k=1}^m I_k(u(t_k))(1-t_k) + \sum_{t_k<t} I_k(u(t_k)) \\
&\le \frac{\beta}{\alpha-1}\phi_q\left(\int_0^1 f(s,u(s))\nabla s\right) + \int_0^1 \phi_q\left(\int_0^1 f(r,u(r))\nabla r\right)\triangle s \\
&+ \frac{1}{\alpha-1}\int_0^1 \phi_q\left(\int_0^1 f(r,u(r))\nabla r\right)\Delta s + \frac{1}{\alpha-1}\sum_{k=1}^m I_k(u(t_k)) \\
&+ \sum_{k=1}^m I_k(u(t_k)) \\
&= \frac{\beta}{\alpha-1}\phi_q\left(\int_0^1 f(s,u(s))\nabla s\right) + \phi_q\left(\int_0^1 f(r,u(r))\nabla r\right) \\
&+ \frac{1}{\alpha-1}\phi_q\left(\int_0^1 f(r,u(r))\nabla r\right) + \frac{\alpha}{\alpha-1}\sum_{k=1}^m I_k(u(t_k)) \\
&= \frac{\beta+\alpha}{\alpha-1}\phi_q\left(\int_0^1 f(s,u(s))\nabla s\right) + \frac{\alpha}{\alpha-1}\sum_{k=1}^m I_k(u(t_k)) \\
&< \frac{\beta+\alpha}{\alpha-1}\phi_q\left(\int_0^1 \phi_p(l\rho_1)\nabla s\right) + \frac{\alpha}{\alpha-1}\sum_{k=1}^m l\rho_1 \\
&= \frac{\beta+\alpha}{\alpha-1}l\rho_1 + \frac{\alpha}{\alpha-1}l\rho_1 m \\
&= l\rho_1\left[\frac{\beta+(m+1)\alpha}{\alpha-1}\right] = \rho_1,
\end{aligned}
$$

i.e., $\|Tu\| < \|u\|$ for $u \in \partial K_{\rho_1}$. By (i) of Theorem 2.4, we obtain that $i_K(T, K_{\rho_1}) = 1$.

Secondly, we show that $i_K(T, \Omega_{\rho_2}) = 0$. Let $e(t) \equiv 1$. Then $e \in \partial K_1$. We claim that

$$u \ne Tu + \lambda e,\ u \in \partial\Omega_{\rho_2},\ \lambda > 0.$$

Suppose that there exists $u_0 \in \partial\Omega_{\rho_2}$ and $\lambda_0 > 0$ such that

$$u_0 = Tu_0 + \lambda_0 e. \tag{3.5}$$

Then, Lemma 3.2, Lemma 3.4 (d) and (3.5) imply that for $t \in [0,1]_{\mathbb{T}}$

$$\begin{aligned} u_0 &= Tu_0 + \lambda_0 e \geq \gamma \|Tu_0\| + \lambda_0 \\ &\geq \gamma \frac{\beta}{\alpha - 1} \phi_q \left(\int_0^1 f(s, u(s)) \nabla s \right) + \lambda_0 \\ &> \gamma \frac{\beta}{\alpha - 1} \phi_q \left(\int_0^1 \phi_p(L\rho_2) \nabla s \right) + \lambda_0 \\ &= \gamma \frac{\beta}{\alpha - 1} L\rho_2 + \lambda_0 = \gamma\rho_2 + \lambda_0, \end{aligned}$$

i.e., $\gamma\rho_2 > \gamma\rho_2 + \lambda_0$, which is a contradiction. Hence by (ii) of Theorem 2.4, it follows that $i_K(T, \Omega_{\rho_2}) = 0$.

Finally, similar to the proof of $i_K(T, K_{\rho_1}) = 1$, we can prove that $i_K(T, K_{\rho_3}) = 1$. Since $\rho_1 < \gamma\rho_2$ and Lemma 3.4 (b), we have $\bar{K}_{\rho_1} \subset K_{\gamma\rho_2} \subset \Omega_{\rho_2}$. Similarly with $\rho_2 < \rho_3$ and Lemma 3.4 (b), we have $\bar{\Omega}_{\rho_2} \subset K_{\rho_2} \subset K_{\rho_3}$. Therefore (iii) of Theorem 2.4 implies that BVP (1.1)-(1.3) has at least two positive solutions $u_1,\ u_2$ with $u_1 \in \Omega_{\rho_2} \setminus \bar{K}_{\rho_1},\ u_2 \in K_{\rho_3} \setminus \bar{\Omega}_{\rho_2}$. □

Theorem 3.5 can be generalized to obtain many solutions.

Theorem 3.6. Suppose $(H1)$ and $(H2)$ hold. Then we have the following assertions.

$(H5)$ If there exists $\{\rho_i\}_{i=1}^{2m_0+1} \subset (0,\infty)$ with $\rho_1 < \gamma\rho_2 < \rho_2 < \rho_3 < \gamma\rho_4 < \dots < \gamma\rho_{2m_0} < \rho_{2m_0} < \rho_{2m_0+1}$ such that

$$f_0^{\rho_{2m-1}} < \phi_p(l),\ I_0^{\rho_{2m-1}} < l\rho_{2m-1},\ (m = 1, 2, ..., m_0, m_0 + 1),$$

$$f_{\gamma\rho_{2m}}^{\rho_{2m}} > \phi_p(L),\ (m = 1, 2, ..., m_0),$$

then problem (1.1)-(1.3) has at least $2m_0$ solutions in K.

$(H6)$ If there exists $\{\rho_i\}_{i=1}^{2m_0} \subset (0,\infty)$ with $\rho_1 < \gamma\rho_2 < \rho_2 < \rho_3 < \gamma\rho_4 < \dots < \gamma\rho_{2m_0} < \rho_{2m_0}$ such that

$$f_0^{\rho_{2m-1}} < \phi_p(l),\ I_0^{\rho_{2m-1}} < l\rho_{2m-1},\quad f_{\gamma\rho_{2m}}^{\rho_{2m}} > \phi_p(L),\ (m = 1, 2, ..., m_0),$$

then problem (1.1)-(1.3) has at least $2m_0 - 1$ solutions in K.

Theorem 3.7. Suppose $(H1)$ and $(H2)$ hold. Then we have the following assertions.

$(H7)$ If there exists $\{\rho_i\}_{i=1}^{2m_0+1} \subset (0,\infty)$ with $\rho_1 < \rho_2 < \gamma\rho_3 < \rho_3 < \dots < \rho_{2m_0} < \gamma\rho_{2m_0+1} < \rho_{2m_0+1}$ such that

$$f_{\gamma\rho_{2m-1}}^{\rho_{2m-1}} > \phi_p(L),\ (m = 1, 2, ..., m_0, m_0 + 1),$$

$$f_0^{\rho_{2m}} < \phi_p(l),\ I_0^{\rho_{2m}} < l\rho_{2m},\ (m = 1, 2, ..., m_0),$$

then problem (1.1)-(1.3) has at least $2m_0$ solutions in K.

$(H8)$ If there exists $\{\rho_i\}_{i=1}^{2m_0} \subset (0,\infty)$ with $\rho_1 < \rho_2 < \gamma\rho_3 < \rho_3 < \dots < \gamma\rho_{2m_0-1} < \rho_{2m_0-1} < \rho_{2m_0}$ such that

$$f_{\gamma\rho_{2m-1}}^{\rho_{2m-1}} > \phi_p(L),\ f_0^{\rho_{2m}} < \phi_p(l),\ I_0^{\rho_{2m}} < l\rho_{2m},\ (m = 1, 2, ..., m_0),$$

then problem (1.1)-(1.3) has at least $2m_0 - 1$ solutions in K.

Theorem 3.8. Let $0 < d < a < \gamma b < b \le c$ and assume that conditions $(H1)$, $(H2)$ hold, and the following conditions hold:

$(H9)$ There exists a constant c_k such that $I_k(y) \le c_k$ for $k = 1, 2, ..., m$;
$(H10)$ $f_a^b > \phi_p(L)$;
$(H11)$ $f_0^c \le \phi_p(R),\ f_0^d < \phi_p(R)$ where $0 < R < \dfrac{\alpha - 1}{\beta + \alpha},\ d > \dfrac{\frac{\alpha}{\alpha-1}\sum_{k=1}^{m} c_k}{1 - R\left(\frac{\beta+\alpha}{\alpha-1}\right)}$.

Then the BVP (1.1)-(1.3) has at least three nonnegative solutions $u_1,\ u_2$, and u_3 in $\bar{K}_c$ such that

$$\|u_1\| < d, \quad a < \beta(u_2) \text{ and } \|u_3\| > d \text{ with } \beta(u_3) < a.$$

Proof. By $(H1)$ and $(H2)$, $T : K \to K$ is completely continuous. For $u \in K$, let

$$\theta(u) = \min_{t\in[0,1]_{\mathbb{T}}} u(t),$$

then it is easy to check that θ is a nonnegative continuous concave functional on K with $\theta(u) \le \|u\|$ for $u \in K$.

Let $u \in \bar{K}_c$, by $(H9)$ and $(H11)$ we have

$$\begin{aligned}
\|Tu\| &= \max_{t\in[0,1]_{\mathbb{T}}} Tu(t) \\
&= \max_{t\in[0,1]_{\mathbb{T}}} \left\{ \frac{\beta}{\alpha-1}\phi_q\left(\int_0^1 f(s,u(s))\nabla s\right) + \int_0^t \phi_q\left(\int_s^1 f(r,u(r))\nabla r\right)\triangle s \right. \\
&+ \frac{1}{\alpha-1}\int_0^1 (1-\sigma(s))\phi_q\left(\int_s^1 f(r,u(r))\nabla r\right)\Delta s \\
&+ \left. \frac{1}{\alpha-1}\sum_{k=1}^{m} I_k(u(t_k))(1-t_k) + \sum_{t_k<t} I_k(u(t_k)) \right\} \\
&\le \frac{\beta}{\alpha-1}\phi_q\left(\int_0^1 \phi_p(Rc)\nabla s\right) + \int_0^1 \phi_q\left(\int_0^1 \phi_p(Rc)\nabla r\right)\triangle s \\
&+ \frac{1}{\alpha-1}\int_0^1 \phi_q\left(\int_0^1 \phi_p(Rc)\nabla r\right)\Delta s + \frac{1}{\alpha-1}\sum_{k=1}^{m} c_k + \sum_{k=1}^{m} c_k \\
&= Rc\left(\frac{\beta+\alpha}{\alpha-1}\right) + \frac{\alpha}{\alpha-1}\sum_{k=1}^{m} c_k \\
&< c,
\end{aligned}$$

that is $\|Tu\| < c$.

Now, we show condition (ii) of Theorem 2.5 holds. Let $\|u\| \le d$, it follows from $(H11)$ that

$$\begin{aligned}
\|Tu\| &= \max_{t\in[0,1]_{\mathbb{T}}} Tu(t)\\
&= \max_{t\in[0,1]_{\mathbb{T}}} \left\{ \frac{\beta}{\alpha-1}\phi_q\left(\int_0^1 f(s,u(s))\nabla s\right) + \int_0^t \phi_q\left(\int_s^1 f(r,u(r))\nabla r\right)\triangle s\right.\\
&+ \frac{1}{\alpha-1}\int_0^1 (1-\sigma(s))\phi_q\left(\int_s^1 f(r,u(r))\nabla r\right)\Delta s\\
&+ \left.\frac{1}{\alpha-1}\sum_{k=1}^m I_k(u(t_k))(1-t_k) + \sum_{t_k<t} I_k(u(t_k))\right\}\\
&\le \frac{\beta}{\alpha-1}\phi_q\left(\int_0^1 \phi_p(Rd)\nabla s\right) + \int_0^1 \phi_q\left(\int_0^1 \phi_p(Rd)\nabla r\right)\triangle s\\
&+ \frac{1}{\alpha-1}\int_0^1 \phi_q\left(\int_0^1 \phi_p(Rd)\nabla r\right)\Delta s + \frac{1}{\alpha-1}\sum_{k=1}^m c_k + \sum_{k=1}^m c_k\\
&= Rd\left(\frac{\beta+\alpha}{\alpha-1}\right) + \frac{\alpha}{\alpha-1}\sum_{k=1}^m c_k\\
&< d.
\end{aligned}$$

So, condition (ii) of Theorem 2.5 holds.

Next, we show that $\{u \in K(\theta,a,b) : \theta(u) > a\} \neq \emptyset$ and $\theta(Tu) > a$ for $u \in K(\theta,a,b)$. In fact, take $u(t) = \frac{a+b}{2} > a$, so $u \in \{u \in K(\theta,a,b) : \theta(u) > a\}$. Also, $u \in K(\theta,a,b)$ and $(H10)$ condition imply that

$$\begin{aligned}
\theta(Tu) &= \min_{t\in[0,1]_{\mathbb{T}}} (Tu)(t) = Tu(0)\\
&= \frac{\beta}{\alpha-1}\phi_q\left(\int_0^1 f(s,u(s))\nabla s\right)\\
&+ \frac{1}{\alpha-1}\int_0^1 (1-\sigma(s))\phi_q\left(\int_s^1 f(r,u(r))\nabla r\right)\Delta s\\
&+ \frac{1}{\alpha-1}\sum_{k=1}^m I_k(u(t_k))(1-t_k)\\
&> \frac{\beta}{\alpha-1}\phi_q\left(\int_0^1 f(s,u(s))\nabla s\right)\\
&> \frac{\beta}{\alpha-1}\phi_q\left(\int_0^1 \phi_p\left(\frac{a(\alpha-1)}{\beta}\right)\nabla s\right) = a.
\end{aligned}$$

Finally, we show that $\theta(Tu) > a$ for $u \in K(\theta,a,c)$ with $\|Tu\| > b$. From Lemma 3.2, we have

$$\theta(Tu) = \min_{t\in[0,1]_{\mathbb{T}}} (Tu)(t) \ge \gamma\|Tu\| > \gamma b > a.$$

So $\theta(Tu) > a$ is satisfied. Therefore all of the conditions of Theorem 2.5 hold. Hence BVP (1.1)-(1.3) has at least three positive solutions u_1, u_2 and u_3 such that

$$\|u_1\| < d, \quad a < \theta(u_2) \text{ and } \|u_3\| > d \quad \text{with} \quad \theta(u_3) < a. \qquad \square$$

4. Example

Example 4.1. Consider the following second-order impulsive p-Laplacian boundary value problem,

$$-(\phi_2(u^{\triangle}(t)))^{\nabla} = f(t, u(t)), \quad t \in [0,1]_{\mathbb{T}} \setminus \{\tfrac{1}{2}\}, \tag{4.1}$$

$$u(\tfrac{1}{2}^{+}) - u(\tfrac{1}{2}^{-}) = I_1(u(\tfrac{1}{2})), \tag{4.2}$$

$$u^{\Delta}(1) = 0,\ 2u(0) - u^{\Delta}(0) = \int_0^1 u(s)\triangle s, \tag{4.3}$$

where

$$f(t,u) = \begin{cases} \frac{1}{26}(1-u), & u \in [0,1]; \\ 51(u-1), & u \in (1,4]; \\ \frac{7}{11}(u-4) + 153, & u \in (4,\infty); \end{cases}$$

and

$$I_1(u) = \frac{u}{10}.$$

Taking $\rho_1 = 1,\ \rho_2 = 4,\ \rho_3 = 70,\ \alpha = 2$ and $\beta = 1$, we have $l = \frac{1}{5},\ L = 1,\ \gamma = \frac{1}{3}$. We can obtain that

$$\rho_1 < \gamma\rho_2 \quad \text{and} \quad \rho_2 < \rho_3.$$

Now, we show that $(H3)$ is satisfied:

$$f_0^1 = \frac{1}{26} < \phi_2(\frac{1}{5}) = \frac{1}{25}, \quad f_{\frac{4}{3}}^4 = \frac{51}{48} > \phi_2(1) = 1,$$

$$f_0^{70} < 0.039 < \phi_2(\frac{1}{5}) = 0.04, \quad I_0^1 = (\frac{1}{10}) < l\rho_1 = \frac{1}{5} \quad \text{and} \quad I_0^{70} = 7 < l\rho_3 = 14.$$

Then, all conditions of Theorem 3.5 hold. Hence, we get the BVP (4.1)-(4.3) has at least two positive solutions.

References

[1] M. Benchohra, J. Henderson, S. Ntouyas, *Impulsive Differential Equations and Inclusions*, New York, 2006.

[2] M. Benchohra, S.K. Ntouyas, A. Ouahab, *Extremal solutions of second order impulsive dynamic equations on time scales*, J. Math. Anal. Appl., **324**(2006), 425-434.

[3] M. Bohner, A. Peterson, *Dynamic Equations on Time Scales:An Introduction with Applications*, Birkhäuser, Boston, 2001.

[4] M. Bohner, A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2003.

[5] H. Chen, H. Wang, Q. Zhang, T. Zhou, *Double positive solutions of boundary value problems for p-Laplacian impulsive functional dynamic equations on time scales*, Comput. Math. Appl., **53**(2007), 1473-1480.

[6] H. Chen, H. Wang, *Triple positive solutions of boundary value problems for p-Laplacian impulsive dynamic equations on time scales*, Math. Comput. Modelling, **47**(2008), 917-924.

[7] M. Feng, B. Du, W. Ge, *Impulsive boundary value problems with integral boundary conditions and one-dimensional p-Laplacian*, Nonlinear Anal., **70**(2009), 3119-3126.

[8] D.J. Guo, *Existence of solutions of boundary value problems for nonlinear second order impulsive differential equations in Banach spaces*, J. Math. Anal. Appl., **181**(1994), 407-421.

[9] L. Hu, L. Liu, Y. Wu, *Positive solutions of nonlinear singular two-point boundary value problems for second-order impulsive differential equations*, Appl. Math. Comput., **196**(2008), 550-562.

[10] Y. Jin, Z. Zhang, J. Yang, N. Song, *Positive solutions to boundary value problems for m impulsive points p-Laplacian dynamic equations on time scales*, J. Univ. Sci. Technol. China, **41**(2011), 497-503.

[11] I.Y. Karaca, *On positive solutions for fourth-order boundary value problem with impulse*, J. Comput. Appl. Math., **225**(2009), 356-364.

[12] V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.

[13] K.Q. Lan, *Multiple positive solutions of semilinear differential equations with singularities*, J. London Math. Soc., **63**(2)(2001), 690-704.

[14] R.W. Leggett, L.R. Williams, *Multiple positive fixed points of nonlinear operators on ordered Banach spaces*, Indiana Univ. Math. J., **28**(1979), 673-688.

[15] J. Li, J. Shen, *Existence results for second-order impulsive boundary value problems on time scales*, Nonlinear Anal., **70**(2009), 1648-1655.

[16] P. Li, H. Chen, Y. Wu, *Multiple positive solutions of n-point boundary value problems for p-Laplacian impulsive dynamic equations on time scales*, Comput. Math. Appl., **60**(2010), 2572-2582.

[17] R. Liang, J. Shen, *Triple positive solutions to BVP for p-Laplacian impulsive dynamic equations on time scales*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal., **18**(2011), 719-730.

[18] S. Liang, J. Zhang, *The existence of countably many positive solutions for some nonlinear singular three-point impulsive boundary value problems*, Nonlinear Anal., **71**(10)(2009), 4588-4597.

[19] Y. Xing, Q. Wang, D. Chen, *Antiperiodic boundary value problem for second-order impulsive differential equations on time scales*, Adv. Difference Eq., Art., ID 567329(2009), 1-14.

[20] X. Zhang, W. Ge, *Impulsive boundary value problems involving the one-dimensional p-Laplacian*, Nonlinear Anal., **70**(2009), 1692-1701.

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