Fixed Point Theory, 15(2014), No. 2, 463-474
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

PREŠIĆ-KANNAN-RUS FIXED POINT THEOREM ON PARTIALLY ORDER METRIC SPACES

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Abstract. The purpose of this paper is to present some fixed point theorems for operators $f\colon X^k \to X$ satisfying a general Presić-type contractivity condition on a partially order complete metric space X. Moreover, we give an application of our main theorem to the study of nonlinear differential equations.

Key Words and Phrases: Prešić-type contractive mapping, partially ordered set, fixed point.
2010 Mathematics Subject Classification: 47H10, 54H25, 34B15.

1. Introduction and preliminaries

Banach proved a fixed point theorem for contraction mappings in complete metric spaces. In fact, he supposed that (X, d) is a complete metric space and $T : X \to X$ is a contraction, namely,

$$d(Tx, Ty) \le kd(x, y)$$

for all $x, y \in X$ and $0 < k < 1$. Then he concluded that T has a unique fixed point in X.

Recently, Bhaskar and Lakshmikantham [2] proved the existence of fixed point in partially ordered metric spaces. Let $(X, \le)$ be a partially ordered set and d a metric on X such that (X, d) is a complete metric space. Further, we endow the product space $X \times X$ with the following partial order:

$$(u, v) \le (x, y) \Leftrightarrow x \ge u, y \le v$$

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for $(x, y), (u, v) \in X \times X$. We begin with the following theorem that establishes existence of a fixed point theorem for a function F on the product space $X \times X$.

Theorem 1.1. *Let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property on X, namely, $F(x, y)$ is nondecreasing in x and is nonincreasing in y, that is, for all $x, y \in X$ and $x_1, x_2, y_1, y_2 \in X$,*

$$x_1 \le x_2 \Rightarrow F(x_1, y) \le F(x_2, y)$$

$$y_1 \le y_2 \Rightarrow F(x, y_1) \ge F(x, y_2).$$

Assume that there exists a $k \in [0, 1)$ with

$$d(F(x, y), F(u, v)) \le \frac{k}{2}[d(x, u) + d(y, v)],$$

for $x \ge u, y \le v$. If there exists $x_0, y_0 \in X$ such that

$$x_0 \le F(x_0, y_0), y_0 \ge F(y_0, x_0),$$

then there exist $x, y \in X$ such that

$$x = F(x, y), y = F(y, x),$$

namely, F has a coupled fixed point.

Existence of a fixed point for contraction type mappings in partially ordered metric spaces has been considered recently in [1, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14], where some applications to matrix equations, ordinary differential equations and integral equations are presented.

Prešić [13] proved the existence and uniqueness of fixed points for operators satisfying a special type of contraction condition, and also providing a so-called multi-step iteration method for approximating the fixed points. In the sequel we shall consider (X, d) a metric space. Prešić' condition generalizes Banach's contraction condition, namely,

$$d(f(x), f(y)) \le \alpha d(x, y)$$

for all $x, y \in X$, where $f : X \to X$ is an operator and $\alpha \in [0, 1)$ is a constant, by considering instead

$$d(f(x_0, x_1, \cdots, x_{k-1}), f(x_1, x_2, \cdots, x_k)) \le \sum_{i=1}^{k} \alpha_i c \cdot d(x_{i-1}, x_i)$$

for all $x_0, x_1, \cdots, x_k \in X$, where k is a positive integer, $f : X^k \to X$ is an operator and $\alpha_1, \cdots, \alpha_k \in \mathbb{R}_+$ are constants such that $\sum_{i=1}^{k} \alpha_i < 1$.

Several general Presić type results followed in literature, see for example the papers due to Rus [15, 16], Şerban [17] and Taskovic [18].

Theorem 1.2. (Rus [15]) *Let (X, d) be a complete metric space, k a positive integer, and $\varphi : \mathbb{R}_+^k \to \mathbb{R}_+$ a function with the properties:*

a) $\varphi(r) \leq \varphi(s)$ *for* $r, s \in \mathbb{R}_+^k, r \leq s$*;*
b) $\varphi(r, r, \cdots, r) < r$ *for* $r \in \mathbb{R}_+$ $r > 0$*;*
c) φ *is continuous;*
d) $\sum_{i=0}^{\infty} \varphi^i(r) < \infty$*;*
e) $\varphi(r, 0, \cdots, 0) + \varphi(0, r, 0, \cdots, 0) + \cdots + \varphi(0, \cdots, 0, r) \leq \varphi(r, r, \cdots, r)$ *for any* $r \in \mathbb{R}_+$*;*
and let $f : X^k \to X$ *be an operator such that*

$$d(f(x_0, x_1, \cdots, x_{k-1}), f(x_1, x_2, \cdots, x_k)) \leq \varphi(d(x_0, x_1), d(x_1, x_2), \cdots, d(x_{k-1}, x_k))$$

for all $x_0, x_1, \cdots, x_k \in X$.
Then
i) there exists a unique $x^* \in X$ *solution of the equation*

$$x = f(x, x, \cdots, x);$$

ii) the sequence $\{x_n\}_{n \geq 0}$ *with* $x_0, x_1, \cdots, x_{k-1} \in X$ *and*

$$x_n = f(x_{n-k}, x_{n-k+1}, \cdots, x_{n-1}), \quad for \quad n \geq k,$$

converges to x^*;
iii) the rate of convergence for $\{x_n\}_{n \geq 0}$ *is given by*

$$d(x_n, x^*) \leq k \sum_{i=0}^{\infty} \varphi^{[\frac{n+i}{k}]}(d_0, \cdots, d_0),$$

where $d_0 = \max\{d(x_0, x_1), d(x_1, x_2), \cdots, d(x_{k-1}, x_k)\}$.

Let $(X, \leq)$ be a partially ordered set. We endow the product space X^k with the following partial order: for $(x_1, x_2, \cdots, x_k), (y_1, y_2, \cdots, y_k) \in X^k$

$$(x_1, x_2, \cdots, x_k) \leq (y_1, y_2, \cdots, y_k) \Leftrightarrow x_i \leq y_i \text{ for } i \in \{1, 2, \cdots, k\}.$$

Definition 1.3. *We say that* $x^* \in X$ *is a fixed point of* $f : X^k \to X$ *if*

$$f(x^*, x^*, \cdots, x^*) = x^*.$$

In this paper, we investigate some fixed point theorems for operators $f : X^k \to X$ satisfying a general Presić-type contractivity condition on partially order complete metric space X. Moreover, we give an application of our main theorem in the study of nonlinear difference equations.

2. Main results

We start our work with the following theorem which can be regarded as an extension of Theorem 1.2.

Theorem 2.1. *Let* $(X, \leq)$ *be a partially ordered set and* d *be a metric on* X *such that* (X, d) *is a complete metric space,* k *a positive integer. Suppose that there exists a function* $\varphi : \mathbb{R}_+^k \to \mathbb{R}_+$ *with the properties:*

a) $\varphi(r) \leq \varphi(s)$ *for* $r, s \in \mathbb{R}_+^k, r \leq s$*;*
b) $\varphi(r, r, ..., r) < r$ *for* $r \in \mathbb{R}_+,\ r > 0$*;*
c) φ *is continuous;*
d) $\sum_{i=0}^{\infty} \varphi^i(r) < \infty$*;*
e) $\varphi(r, 0, \cdots, 0) + \varphi(0, r, 0, \cdots, 0) + \cdots + \varphi(0, \cdots, 0, r) \leq \varphi(r, r, \cdots, r)$ *for any* $r \in \mathbb{R}_+$*;*
and let $f : X^k \to X$ *be a continuous mapping having the increasing property on* X *such that*

$$d(f(x_0.x_1, \cdots, x_{k-1}), f(x_1, x_2, \cdots, x_k)) \leq \varphi(d(x_0, x_1), d(x_1, x_2), \cdots, d(x_{k-1}, x_k)) \tag{2.1}$$

for all $x_0, x_1, \cdots, x_{k-1}, x_k \in X$*, where* $x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_{k-1} \leq x_k$*. If there exist* $x_0, x_1, \cdots, x_{k-1} \in X$ *such that* $x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_{k-1} \leq f(x_0, x_1, \cdots, x_{k-1})$*, then there exists* $x^* \in X$ *such that* $x^* = f(x^*, x^*, \cdots, x^*)$*.*

Proof. Let $f(x_0, x_1, \cdots, x_{k-1}) = x_k$. Since

$$x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_{k-1} \leq f(x_0, x_1, \cdots, x_{k-1}),$$

due to the increasing property of f, we have

$$x_k = f(x_0, x_1, \cdots, x_{k-1}) \leq f(x_1, x_2, \cdots, x_k) = x_{k+1}.$$

In this manner, we construct the increasing sequence $\{x_n\}_{n \geq 0}$ such that

$$x_n = f(x_{n-k}, \cdots, x_{n-1}),\ n \geq k.$$

We denote

$$d_0 = max\{d(x_0, x_1), d(x_1, x_2), \cdots, d(x_{k-1}, x_k)\} \tag{2.2}$$

and this is positive, assuming that $x_0, x_1, \cdots, x_k$ are not equal.

The following estimations hold. By hypothesis,

$$\begin{aligned} d(x_k, x_{k+1}) &= d(f(x_0, x_1, \cdots, x_{k-1}), f(x_1, x_2, \cdots, x_k)) \\ &\leq \varphi(d(x_0, x_1), d(x_1, x_2), \cdots, d(x_{k-1}, x_k)) \\ &\leq \varphi(d_0, d_0, \cdots, d_0) < d_0, \end{aligned}$$

$$\begin{aligned} d(x_{k+1}, x_{k+2}) &= d(f(x_1, x_2, \cdots, x_k), f(x_2, x_3, \cdots, x_{k+1})) \\ &\leq \varphi(d(x_1, x_2), d(x_2, x_3), \cdots, d(x_{k-1}, x_k), d(x_k, x_{k+1})) \\ &\leq \varphi(d_0, d_0, \cdots, d_0, \varphi(d_0, \cdots, d_0)) < d_0, \end{aligned}$$

$$\vdots$$

$$\begin{aligned} d(x_{2k-1}, x_{2k}) &= d(f(x_{k-1}, \cdots, x_{2k-2}), f(x_k, \cdots, x_{2k-1})) \\ &\leq \varphi(d(x_{k-1}, x_k), \cdots, d(x_{2k-2}, x_{2k-1})) \\ &\leq \varphi(d_0, \varphi(d_0, \cdots, d_0), \cdots, \varphi(d_0, \cdots, d_0)) \\ &\leq \varphi(d_0, d_0, \cdots, d_0) < d_0, \end{aligned}$$

$$\begin{aligned} d(x_{2k}, x_{2k+1}) &= d(f(x_k, \cdots, x_{2k-1}), f(x_{k+1}, \cdots, x_{2k})) \\ &\leq \varphi(d(x_k, x_{k+1}), \cdots, d(x_{2k-1}, x_{2k})) \\ &\leq \varphi(\varphi(d_0, \cdots, d_0), \varphi(d_0, \cdots, d_0), \cdots, \varphi(d_0, \cdots, d_0)) \\ &= \varphi^2(d_0, \cdots, d_0) \leq \varphi(d_0, \cdots, d_0) < d_0, \end{aligned}$$

$$\begin{aligned} d(x_{2k+1}, x_{2k+2}) &= d(f(x_{k+1}, \cdots, x_{2k}), f(x_{k+2}, \cdots, x_{2k+1})) \\ &\leq \varphi(d(x_{k+1}, x_{k+2}), \cdots, d(x_{2k}, x_{2k+1})) \\ &\leq \varphi^2(d_0, d_0, \cdots, d_0, \varphi(d_0, \cdots, d_0)) \\ &\leq \varphi^2(d_0, \cdots, d_0) < d_0 \end{aligned}$$

and so on

$$d(x_n, x_{n+1}) \leq \varphi^{[\frac{n}{k}]}(d_0, \cdots, d_0), \quad n \geq k. \tag{2.3}$$

Thus, for some integer $p \geq 1$, we obtain that

$$d(x_n, x_{n+p}) \leq \varphi^{[\frac{n}{k}]}(d_0, \cdots, d_0) + \cdots + \varphi^{[\frac{n+p-1}{k}]}(d_0, \cdots, d_0), \quad (n \geq 0). \tag{2.4}$$

Denoting

$$l = [\frac{n}{k}], m = [\frac{n+p-1}{k}], \tag{2.5}$$

we have $m \geq l$. Besides, the above relation (2.4) implies further estimation

$$\begin{aligned} d(x_n, x_{n+p}) \leq & \underbrace{\varphi^l(d_0, \cdots, d_0) + \cdots + \varphi^l(d_0, \cdots, d_0)}_{k-times} + \\ & \underbrace{+\varphi^{l+1}(d_0, \cdots, d_0) + \cdots + \varphi^{l+1}(d_0, \cdots, d_0)}_{k-times} + \\ & + \cdots + \underbrace{\varphi^m(d_0, \cdots, d_0) + \cdots + \varphi^m(d_0, \cdots, d_0)}_{k-times}, \end{aligned}$$

and so

$$d(x_n, x_{n+p}) \leq k \sum_{i=l}^{m} \varphi^i(d_0, \cdots, d_0), \quad (n \geq 0, p \geq 1). \tag{2.6}$$

Denoting $S_m = \sum_{i=0}^{m} \varphi^i(d_0, \cdots, d_0)$, we have

$$\sum_{i=l}^{m} \varphi^i(d_0, \cdots, d_0) = S_m - S_{l-1}, \quad m \geq l.$$

Then from assumption d) upon φ, the limit

$$S = \lim_{m\to\infty} S_m$$

exists. From (2.5), it follows that

$$\lim_{l\to\infty} \sum_{i=l}^{m} \varphi^i(d_0, \cdots, d_0) = S - S = 0,$$

and in view of (2.6), $d(x_n, x_{n+p}) \to 0$, as $n \to \infty$. This means that $\{x_n\}_{n\geq 0}$ is a Cauchy sequence. Since X is a complete metric space, there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$. On the other hand, since $x_n = f(x_{n-k}, \cdots, x_{n-1})$ for all $n \geq k$ and f is a continuous mapping, we have

$$\begin{aligned} x^* = \lim_{n\to\infty} x_n &= \lim_{n\to\infty} f(x_{n-k}, \cdots, x_{n-1}) \\ &= f(\lim_{n\to\infty} x_{n-k}, \cdots, \lim_{n\to\infty} x_{n-1}) \\ &= f(x^*, \cdots, x^*). \end{aligned}$$

This completes the proof.

Theorem 2.2. *Let (X, d) be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Assume that X has the following property:*

$$if\ a\ nondecreasing\ sequence\ \{x_n\} \to x,\ then\ x_n \leq x\ for\ all\ n. \tag{2.7}$$

Let $f : X^k \to X$ be a mapping having the increasing property on X such that

$$d(f(x_0, x_1, \cdots, x_{k-1}), f(x_1, x_2, \cdots, x_k)) \leq \varphi(d(x_0, x_1), d(x_1, x_2), \cdots, d(x_{k-1}, x_k))$$

for all $x_0, x_1, \cdots, x_{k-1}, x_k \in X$, where $x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_{k-1} \leq x_k$ and φ satisfies the conditions of Theorem 2.1. If there exist $x_0, x_1, \cdots, x_{k-1} \in X$ such that $x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_{k-1} \leq f(x_0, x_1, \cdots, x_{k-1})$, then there exists $x^* \in X$ such that $x^* = f(x^*, x^*, \cdots, x^*)$.

Proof. Following the proof of Theorem 2.1, we only have to show that $x^* = f(x^*, \cdots, x^*)$.

Let $\epsilon > 0$. Since $\{x_n\} \to x^*$, there exists $m \in \mathbb{N}$ such that, for all $n \geq m$, we have

$$d(x_n, x^*) < \frac{\epsilon}{2}.$$

Taking $n \in \mathbb{N}$, we get

$$\begin{aligned} &d(f(x^*, x^*, \cdots, x^*), x^*) \\ &\leq d(f(x^*, x^*, \cdots, x^*), f(x_{m+1}, x_{m+2}, \cdots, x_{m+k})) + d(f(x_{m+1}, x_{m+2}, \cdots, x_{m+k}), x^*) \\ &\leq \varphi(d(x^*, x_{m+1}), d(x^*, x_{m+2}), \cdots, d(x^*, x_{m+k})) + d(x_{m+k+1}, x^*) \\ &\leq \varphi(\frac{\epsilon}{2}, \frac{\epsilon}{2}, \cdots, \frac{\epsilon}{2}) + d(x_{m+k+1}, x^*) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

This implies that $f(x^*, x^*, ..., x^*) = x^*$.

One can prove that the fixed point is in fact unique, provided that the product space X^k endowed with the partial order mentioned earlier has the following property:

$$\text{for } (x_1, x_2, \cdots, x_k), (y_1, y_2, \cdots, y_k) \in X^k, \text{ there exists a } (z_1, z_2, \cdots, z_k) \in X^k \text{ that is comparable to } (x_1, x_2, \cdots, x_k) \text{ and } (y_1, y_2, \cdots, y_k). \tag{2.8}$$

Theorem 2.3. *If we add condition (2.8) to the hypothesis of Theorem 2.1, then we obtain the uniqueness of the fixed point of f.*

Proof. If $y^* \in X$ is another fixed point of f, then we show that

$$d(x^*, y^*) = 0.$$

We consider two cases:
Case 1: If x^* is comparable to y^*, then

$$\begin{aligned} d(x^*, y^*) &= d(f(x^*, x^*, \cdots, x^*), f(y^*, y^*, \cdots, y^*)) \\ &\leq \varphi(d(x^*, y^*), \cdots, d(x^*, y^*)) \\ &< d(x^*, y^*). \end{aligned}$$

This contradiction implies that $x^* = y^*$.

Case 2 : If x^* is not comparable to y^*, then there exists $z \in X$ which is comparable to x^* and y^*.
We denote $f^n(z, \cdots, z) = f(f^{n-1}(z, \cdots, z), \cdots, f^{n-1}(z, \cdots, z))$ for $n = 1, 2, \cdots$. Monotonicity implies that $f^n(z, ..., z)$ is comparable to $f^n(x^*, \cdots, x^*) = x^*$ and $f^n(y^*, \cdots, y^*) = y^*$ for $n = 1, 2, \cdots$. Moreover

$$\begin{aligned} & d(x^*, f^n(z, \cdots, z)) = d(f^n(x^*, \cdots, x^*), f^n(z, \cdots, z)) \\ & \leq \varphi(d(f^{n-1}(x^*, \cdots, x^*), f^{n-1}(z, \cdots, z)), \cdots, d(f^{n-1}(x^*, \cdots, x^*), f^{n-1}(z, \cdots, z))) \\ & < d(f^{n-1}(x^*, \cdots, x^*), f^{n-1}(z, \cdots, z)) \\ & = d(x^*, f^{n-1}(z, \cdots, z)). \end{aligned}$$

Consequently, $\{d(x^*, f^n(z, \cdots, z))\}$ is a nonnegative and decreasing sequence and hence there is a limit γ. We claim $\gamma = 0$, otherwise, from the last inequality, we can obtain $\gamma \leq \varphi(\gamma, \gamma, \cdots, \gamma) < \gamma$. This contradiction implies $\gamma = 0$. Analogously, it can be proved that $\lim_{n \to \infty} d(y^*, f^n(z, \cdots, z)) = 0$.

Finally,

$$d(x^*, y^*) \leq d(x^*, f^n(z, \cdots, z)) + d(f^n(z, \cdots, z), y^*),$$

and taking the limit as $n \to \infty$, we obtain that $d(x^*, y^*) = 0$.

3. Application to ordinary differential equations

In this section, by application of Theorem 2.2, we prove the existence of solution of an ordinary differential equation. Hence, we prove the existence of solution for the following first–order periodic problem

$$\begin{cases} u'(t) = f(t, u(t), u(t), \cdots, u(t)), & t \in I = [0, T] \\ u(0) = u(T), \end{cases} \tag{3.1}$$

where $T > 0$ and $f : I \times \mathbb{R}^k \to \mathbb{R}$ is a continuous function.
Previously, we consider the space $C(I)$ $(I = [0, T])$ of continuous functions defined on I. Obviously, this space with the metric given by

$$d(x, y) = \sup\{|x(t) - y(t)| : t \in I\}$$

for $x, y \in C(I)$, is a complete metric space. $C(I)$ can also be equipped with a partial order given by

$$x, y \in C(I), \quad x \le y \Leftrightarrow x(t) \le y(t)$$

for $t \in I$. Obviously, for $x, y \in C(I)$, there exists a lower bound or an upper bound, since, for $x, y \in C(I)$, the functions $\max\{x, y\}$ and $\min\{x, y\}$ are least upper and greatest lower bounds of x and y, respectively. Moreover, in [9], it was proved that $(C(I), \le)$ with the above mentioned metric satisfies the condition (2.7). Let Φ denote the class of those functions $\varphi : \mathbb{R}_+^k \to \mathbb{R}_+$ with the properties:
a) $\varphi(r_1, r_2, \cdots, r_k) \le \varphi(s_1, s_2, \cdots, s_k)$ for $(r_1, \cdots, r_k), (s_1, \cdots, s_k) \in \mathbb{R}_+^k$, $r_i \le s_i$ for all $i \in \{1, 2, ..., k\}$;
b) $\varphi(r, r, \cdots, r) < r$ for $r \in \mathbb{R}_+$, $r > 0$;
c) φ is continuous;
d) $\sum_{i=0}^{\infty} \varphi^i(r) < \infty$;
e) $\varphi(r, 0, \cdots, 0) + \varphi(0, r, 0, \cdots, 0) + \cdots + \varphi(0, \cdots, 0, r) \le \varphi(r, r, \cdots, r)$ for any $r \in \mathbb{R}_+$.

Now we give the following definition.

Definition 3.1. *A lower solution for (3.1) is a function $\alpha \in C^1(I)$ such that*

$$\begin{cases} \alpha'(t) \le f(t, \alpha(t), \alpha(t), \cdots, \alpha(t)), & t \in I = [0, T] \\ \alpha(0) \le \alpha(T) \end{cases}$$

Theorem 3.2. *Consider the problem (3.1) with $f : I \times \mathbb{R}^k \to \mathbb{R}$ continuous and suppose that there exist $\lambda_1, \lambda_2, \cdots, \lambda_k > 0$ such that for $x_1, x_2, \cdots, x_k, y_1, y_2, \cdots, y_k \in$*

$\mathbb{R}$ *with* $x_i \geq y_i$ *for all* $1 \leq i \leq k$

$$\begin{aligned} 0 &\leq f(t, x_1, x_2, \cdots, x_k) + \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k \\ &\quad - [f(t, y_1, y_2, \cdots, v_k) + \lambda_1 y_1 + \lambda_2 y_2 + \cdots + \lambda_k y_k] \\ &\leq \sum_{i=1}^{k} \lambda_i \cdot \ \varphi(x_1 - y_1, x_2 - y_2, \cdots, x_k - y_k), \end{aligned}$$

where $\varphi \in \Phi$. *Then the existence of a lower solution for (3.1) provides the existence of a unique solution of (3.1).*

Proof. Problem (3.1) can be written as

$$\begin{cases} u'(t) + \lambda_1 u(t) + \cdots + \lambda_k u(t) = f(t, u(t), u(t), \cdots, u(t)) + \lambda_1 u(t) + \cdots + \lambda_k u(t), \\ u(0) = u(T) \end{cases}$$

for $t \in I = [0, T]$. This problem is equivalent to the integral equation

$$u(t) = \int_0^T G(t, s)[f(s, u(s), \cdots, u(s)) + \sum_{i=1}^{k} \lambda_i \cdot u(s)]ds,$$

where

$$G(t, s) = \begin{cases} \frac{e^{\sum_{i=1}^{k} \lambda_i \cdot (T+s-t)}}{e^{\sum_{i=1}^{k} \lambda_i \cdot T} - 1}, & 0 \leq s < t \leq T \\ \\ \frac{e^{\sum_{i=1}^{k} \lambda_i \cdot (s-t)}}{e^{\sum_{i=1}^{k} \lambda_i \cdot T} - 1}, & 0 \leq t < s \leq T. \end{cases}$$

Define $F : C(I)^k \to C(I)$ by

$$F(x_1, x_2, \cdots, x_k)(t)$$

$$= \int_0^T G(t, s)[f(s, x_1(s), x_2(s), \cdots, x_k(s)) + \lambda_1 x_1(s) + \lambda_2 x_2(s) + \cdots + \lambda_k x_k(s)]ds.$$

Note that if $x \in C(I)$ is a fixed point of F, then $x \in C^1(I)$ is a solution of (3.1). Now we will check that hypotheses in Theorem 2.2 are satisfied. The mapping F is increasing since, by hypothesis, for $(u_1, u_2, \cdots, u_k), (v_1, v_2, \cdots, v_k)$ in $C(I)^k$ with $u_i \geq v_i$ for $0 \leq i \leq k$,

$$\begin{aligned} &f(t, u_1(t), u_2(t), \cdots, u_k(t)) + \lambda_1 u_1(t) + \lambda_2 u_2(t) + \cdots + \lambda_k u_k(t) \\ &\geq f(t, v_1(t), v_2(t), \cdots, v_k(t)) + \lambda_1 v_1(t) + \lambda_2 v_2(t) + \cdots + \lambda_k v_k(t) \end{aligned}$$

for all $t \in I$, which implies that

$$\begin{aligned} F(u_1, u_2, \cdots, u_k)(t) &= \int_0^T G(t, s)[f(s, u_1(s), \cdots, u_k(s)) + \lambda_1 u_1(s) + \cdots + \lambda_k u_k(s)]ds \\ &\geq \int_0^T G(t, s)[f(s, v_1(s), \cdots, v_k(s)) + \lambda_1 v_1(s) + \cdots + \lambda_k v_k(s)]ds \\ &= F(v_1, v_2, \cdots, v_k)(t) \end{aligned}$$

for all $t \in I$ by using that $G(t,s) > 0$ for $(t,s) \in I \times I$. Besides, for all $u_0, u_1, \cdots, u_{k-1}, u_k \in C(I)$, where $u_0 \leq u_1 \leq \cdots \leq u_{k-1} \leq u_k$ and $\varphi \in \Phi$,

$$\begin{aligned}
&d(F(u_0, u_1, \cdots, u_{k-1}), F(u_1, u_2, \cdots, u_k)) \\
&= \sup_{t\in I} |F(u_0, u_1, \cdots, u_{k-1}) - F(u_1, u_2, \cdots, u_k)(t)| \\
&\leq \sup_{t\in I} \int_0^T G(t,s)|[f(s, u_0(s), \cdots, u_{k-1}(s)) + \lambda_1 u_0(s) + \lambda_2 u_1(s) + \cdots + \\
&\quad \lambda_k u_{k-1}(s)] - [f(s, u_1(s), \cdots, u_k(s)) + \lambda_1 u_1(s) + \lambda_2 u_2(s) + \cdots + \lambda_k u_k(s)]|ds \\
&\leq \sup_{t\in I} \int_0^T G(t,s) \cdot \sum_{i=1}^k \lambda_i \cdot \ \varphi(u_1(s) - u_0(s), u_2(s) - u_1(s), \cdots, u_k(s) - u_{k-1}(s))ds \\
&\leq \sum_{i=1}^k \lambda_i \cdot \ \varphi(d(u_1, u_0), d(u_2, u_1), \cdots, d(u_k, u_{k-1})) \cdot \sup_{t\in I} \int_0^T G(t,s) \\
&= \varphi(d(u_1, u_0), d(u_2, u_1), \cdots, d(u_k, u_{k-1})).
\end{aligned}$$

Finally, let $\alpha(t)$ be a lower solution for (3.1) and we will show that $\alpha \leq F(\underbrace{\alpha, \alpha, \cdots, \alpha}_{k-times})$.

Indeed,

$$\alpha'(t) + \sum_{i=1}^k \lambda_i.\alpha(t) \leq f(t, \alpha(t), \cdots, \alpha(t)) + \sum_{i=1}^k \lambda_i. \ \alpha(t)$$

for $t \in I$. Multiplying by $e^{\sum_{i=1}^k \lambda_i.t}$, we get

$$(\alpha(t)e^{\sum_{i=1}^k \lambda_i \cdot t})' \leq [f(t, \alpha(t), \cdots, \alpha(t)) + \sum_{i=1}^k \lambda_i \cdot \ \alpha(t)]e^{\sum_{i=1}^k \lambda_i \cdot \ t}$$

for $t \in I$ and this gives us

$$\alpha(t)e^{\sum_{i=1}^k \lambda_i \cdot \ t} \leq \alpha(0) + \int_0^t [f(s, \alpha(s), \cdots, \alpha(s)) + \sum_{i=1}^k \lambda_i \cdot \ \alpha(s)]e^{\sum_{i=1}^k \lambda_i \cdot \ s} ds \quad (3.2)$$

for $t \in I$, which implies that

$$\begin{aligned}
\alpha(0)e^{\sum_{i=1}^k \lambda_i \cdot \ T} &\leq \alpha(T)e^{\sum_{i=1}^k \lambda_i \cdot T} \\
&\leq \alpha(0) + \int_0^T [f(s, \alpha(s), \cdots, \alpha(s)) + \sum_{i=1}^k \lambda_i \cdot \ \alpha(s)]e^{\sum_{i=1}^k \lambda_i \cdot s} ds
\end{aligned}$$

and so

$$\alpha(0) \leq \int_0^T \frac{e^{\sum_{i=1}^k \lambda_i \cdot s}}{e^{\sum_{i=1}^k \lambda_i \cdot T} - 1} \ [f(s, \alpha(s), \cdots, \alpha(s)) + \sum_{i=1}^k \lambda_i \cdot \ \alpha(s)]ds.$$

From this inequality and (3.2), we obtain

$$\alpha(t)e^{\sum_{i=1}^k \lambda_i \cdot t} \le \int_0^t \frac{e^{\sum_{i=1}^k \lambda_i \cdot (T+s)}}{e^{\sum_{i=1}^k \lambda_i \cdot T} - 1} [f(s, \alpha(s), \cdots, \alpha(s)) + \sum_{i=1}^k \lambda_i \cdot \ \alpha(s)] ds + \int_t^T \frac{e^{\sum_{i=1}^k \lambda_i \cdot s}}{e^{\sum_{i=1}^k \lambda_i \cdot T} - 1} [f(s, \alpha(s), \cdots, \alpha(s)) + \sum_{i=1}^k \lambda_i \cdot \ \alpha(s)] ds$$

and so

$$\alpha(t) \le \int_0^t \frac{e^{\sum_{i=1}^k \lambda_i \cdot (T+s-t)}}{e^{\sum_{i=1}^k \lambda_i \cdot T} - 1} [f(s, \alpha(s), \cdots, \alpha(s)) + \sum_{i=1}^k \lambda_i \cdot \ \alpha(s)] ds + \int_t^T \frac{e^{\sum_{i=1}^k \lambda_i \cdot (s-t)}}{e^{\sum_{i=1}^k \lambda_i \cdot T} - 1} [f(s, \alpha(s), \cdots, \alpha(s)) + \sum_{i=1}^k \lambda_i \cdot \ \alpha(s)] ds.$$

Hence

$$\alpha(t) \le \int_0^T G(t,s)[f(s, \alpha(s), \cdots, \alpha(s)) + \sum_{i=1}^k \lambda_i \cdot \ \alpha(s)] ds = F(\underbrace{\alpha, \cdots, \alpha}_{k-times})(t)$$

for $t \in I$.

Finally, it follows from Theorems 2.2 and 2.3 that F has a unique fixed point.

Acknowledgments. C. Park was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2012R1A1A2004299) and D. Y. Shin was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2010-0021792).

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Received: September 05, 2012; Accepted: December 13, 2012