Fixed Point Theory, 15(2014), No. 2, 449-462
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

THE FIXED POINT PROPERTY IN c_0 WITH THE ALPHA NORM

BERTA GAMBOA DE BUEN* AND FERNANDO NÚÑEZ-MEDINA**

*Centro de Investigación en Matemáticas (CIMAT)
Apdo. Postal 402, 36000 Guanajuato, Gto., México
E-mail: gamboa@cimat.mx

** Departamento de Matemticas
Universidad de Guanajuato
Guanajuato Gto., Mxico. CP 36240
E-mail: fernandonm@ugto.mx

Abstract. We study the fixed point and the weak fixed point property in the Banach space $c_{0\alpha} = \left(c_0, \|\cdot\|_\alpha\right)$, where $\|(x_i)\|_\alpha = \sup_i |x_i| + \alpha \sum_i \frac{|x_i|}{2^i}$. It is known that $c_{0\alpha}$ has the weak fixed point property for every $\alpha \geq 0$. We prove that if K is a nonempty, convex, closed and bounded subset of $c_{0\alpha}$, then K is weakly compact if and only if every nonempty, convex and closed subset of K has the FPP.

Key Words and Phrases: Fixed point property, space $c_{0\alpha}$.
2010 Mathematics Subject Classification: Primary 46B20, 47H09, 47H10.

1. Introduction

In the last years the converse of Maurey theorem [MAU]: If $K \subset c_0$ is nonempty, convex, closed, bounded and has the fixed point property (FPP) then K is ω-compact, has been an active research theme. Llorens-Fuster and Sims proved in 1988 in [LFS] that some nonempty, convex, closed and bounded subsets of c_0 which are compact in a locally convex topology very similar to the weak topology of c_0 do not have the FPP. This fact led them to the following conjecture: Let K be a nonempty, convex, closed and bounded subset of c_0; then K has the FPP if and only if K is ω-compact. In 2003, Dowling, Lennard and Turett answered partially this conjecture in [DLT], specifically they defined the asymptotically isometric c_0 summing basic sequences (aisbc_0 sequences), showed that if $K \subset c_0$ is a nonempty, convex, closed and bounded set which is not weakly compact, then it contains an aisbc_0 sequence and using it, they construct a subset of K without the FPP. Later in 2004 the same authors proved in [DOL] that, as conjectured by Llorens-Fuster and Sims, the converse of Maurey's theorem is true.

In 1992 in [JIM], Jiménez-Melado A., used the space $c_{0\alpha}$ to prove that two properties which imply the weak fixed point property are not equivalent. In [FGB], Fetter H., and Gamboa de Buen B., showed that for all $\alpha \geq 0$, the space $c_{0\alpha}$ has the ω-FPP and that for $\alpha \in [0,1)$, the space c_α has the ω-FPP, where $c_{0\alpha}$ and c_α are the c_0 and c spaces respectively with the equivalent alpha norm. The following question arises naturally. If $K \subset c_{0\alpha}$ is nonempty, convex, closed, bounded and has the FPP, is K ω-compact? The same question can be asked in c_α, $\alpha \in [0,1)$.

In this article, first we show that in $c_{0\alpha}$ there exist nonempty, convex, closed and bounded subsets which are not ω-compact and without aisbc_0 sequences with the norm $||.||_\alpha$.

Then we define the corresponding asymptotically isometric $c_{0\alpha}$ summing basic sequences (aisb$c_{0\alpha}$ sequences). We give an example of a nonempty, convex, closed and bounded subset of c_0 which is not ω-compact and without aisb$c_{0\alpha}$ sequences with the norm $||.||_\infty$, proving that the families of aisbc_0 and aisb$c_{0\alpha}$ sequences are different.

Next we show that if K is a nonempty, convex, closed and bounded subset of a Banach space X that contains an aisb$c_{0\alpha}$ sequence, then we can construct $C \subset K$ nonempty, convex, closed and without the FPP.

Finally we show that if K is a nonempty, convex, closed and bouded subset of $c_{0\alpha}$ which is not ω-compact, then there exist $\{x_n\} \subset K$ and $L > 0$ such that $\{Lx_n\}$ is an aisb$c_{0\alpha}$ sequence, proving then that if K is a nonempty, convex, closed and bouded subset of $c_{0\alpha}$, then K is weakly compact if and only if every nonempty, convex and closed subset of K has the FPP. We also prove a similar result in the space c_α, for $\alpha \in (0,1]$.

As we said, one of the key points of the Dowling, Lennard and Turett work in [DLT] is to construct, in a nonempty, convex and closed subset K of c_0, a sequence with a "similar" behavior to the c_0 summing basis. To do that, they define the aisbc_0 sequences. Recall that c_{00} is the set of all eventually zero sequences in $\mathbb{K}$.

Definition 1.1. *Let $\{x_n\}$ be a sequence in a Banach space X. We say that $\{x_n\}$ is an asymptotically isometric c_0-summing basic sequence, aisbc$_0$ sequence for short, if there exists $\{\varepsilon_n\} \subset (0,\infty)$ with $\varepsilon_n \to 0$ such that*

$$\sup_{n\in\mathbb{N}} (1+\varepsilon_n)^{-1} \left|\sum_{j=n}^{\infty} t_j\right| \leq \left\|\sum_{n=1}^{\infty} t_n x_n\right\| \leq \sup_{n\in\mathbb{N}} (1+\varepsilon_n) \left|\sum_{j=n}^{\infty} t_j\right|, \tag{1.1}$$

for all $\{t_n\} \in c_{00}$. If $L > 0$, we say that $\{x_n\}$ is a L-aisbc$_0$ sequence if $\{Lx_n\}$ is an aisbc$_0$ sequence.

Remark 1.1. *In the previous definition we can replace c_{00} by l^1.*
An aisbc$_0$ sequence is a bounded basic sequence equivalent to the summing basis of c_0.

A subsequence of an aisbc$_0$ sequence $\{x_n\}_{n=1}^\infty$ is again an aisbc$_0$ sequence. Moreover, if $\{\lambda_n\} \subset (0,\infty)$ and $\lambda_n \longrightarrow 0$, then we can select $\{x_{n_k}\}_{k=1}^\infty$ such that the associated new sequence $\{\varepsilon_n'\}$ satisfies that $\varepsilon_{n+1}' < \varepsilon_n'$ and $\varepsilon_n' < \lambda_n$, $n \in \mathbb{N}$.

In [DLT] the following results were proved.

Proposition 1.1. *Let K be a nonempty, convex, closed and bounded subset of c_0. Then K is ω-compact if and only if every nonempty, convex and closed subset of K has the FPP.*

Proposition 1.2. *Let K be a nonempty, convex, closed and bounded subset of c. Then K is ω-compact if and only if every nonempty, convex and closed subset of K has the FPP.*

Note that proposition 1.1 does not prove Llorens-Fuster´s and Sims´ conjecture. In 2004 the same authors, Dowling, Lennard and Turett, proved in [DOL] the following theorem.

Theorem 1.1. *Let K be a nonempty, convex, closed and bounded subset of c_0. Then K is ω-compact if and only if K has the FPP.*

The conjecture in the case of c remains still open.

2. The fixed point property in the $c_{0\alpha}$ space

Definition 2.1. *Let $\alpha \geq 0$ and l_α^∞ the space of all scalar bounded sequences endowed with the α-norm given by*

$$|| (x_n) ||_\alpha = || (x_n) ||_\infty + \alpha \left\| (x_n) \right\|_s , \quad (x_n) \in l^\infty,$$

where

$$\left\| (x_n) \right\|_s = \sum_{n=1}^{\infty} \frac{|x_n|}{2^n}.$$

Note that $||.||_\alpha$ and $||.||_\infty$ are equivalent, since

$$||x||_\infty \leq ||x||_\alpha \leq (1+\alpha) \left\| x \right\|_\infty , \ x \in l^\infty.$$

As we said, Fetter H., and Gamboa de Buen B., proved in [FGB] that for all $\alpha \in [0, 1)$, the space $c_\alpha = (c, ||.||_\alpha)$ has the ω-FPP and also that $c_{0\alpha} = (c_0, ||.||_\alpha)$ has the ω-FPP for all $\alpha > 0$.

In what follows we shall fix $\alpha > 0$.

Next we will see that in $c_{0\alpha}$ there exist nonempty, convex, closed and bounded subsets which are not ω-compact and without aisbc_0 sequences with the norm $||.||_\alpha$.

Example 2.1. *Let $\{\xi_n\}$ be the c_0 summing basis. Then*

$$C = \left\{ \sum_{n=1}^{\infty} \lambda_n \xi_n : \lambda_n \geq 0 \ \text{and} \ \sum_{n=1}^{\infty} \lambda_n = 1 \right\}$$

does not have aisbc_0 sequences with the norm $||.||_\alpha$.

Proof. It is clear that C is a nonempty, convex, closed and bounded subset of $c_{0\alpha}$ which is not ω-compact. Suppose that C contains an aisbc_0 sequence $\{y_n\}$ with the norm $||.||_\alpha$, for some sequence $\{\varepsilon_n\}$. By remark 1.1 we can suppose that $\{\varepsilon_n\}$ satisfies $\varepsilon_{n+1} \leq \varepsilon_n < \frac{\alpha}{2}$, $n \in \mathbb{N}$. Since $\{y_n\} \in C$ we have that $y_n = \sum_{i=1}^{\infty} \lambda_i^n \xi_i$ for some sequence $\{\lambda_i^n\}$ such that $\lambda_i^n \geq 0$ and $\sum_{i=1}^{\infty} \lambda_i^n = 1$. Take $m \in \mathbb{N}$ and define

$$(t_n) = e_m,$$

where $\{e_n\}$ is the canonical basis of c_0. Then

$$\sum_{n=1}^{\infty} t_n y_n = y_m$$

and

$$\left\|\sum_{n=1}^{\infty} t_n y_n\right\|_\alpha = \left\|\sum_{i=1}^{\infty} \lambda_i^m \xi_i\right\|_\alpha = 1 + \alpha \sum_{k=1}^{\infty} \frac{\left|\sum_{j=k}^{\infty} \lambda_j^m\right|}{2^k} \geq 1 + \frac{\alpha}{2}. \tag{2.1}$$

On the other hand, since $\{y_n\}$ is an aisbc_0 sequence with the norm $||.||_\alpha$ we get

$$\left\|\sum_{n=1}^{\infty} t_n y_n\right\|_\alpha \leq \sup_{n\in\mathbb{N}}(1+\varepsilon_n)\left|\sum_{j=n}^{\infty} t_j\right| = (1+\varepsilon_1),$$

which contradicts (2.1), since $\varepsilon_1 < \frac{\alpha}{2}$. □

In view of example 2.1, in a similar way to the Dowling, Lennard and Turett definition given in [DLT] for c_0, we will define the asymptotically isometric summing basic sequences in the space $(c_0, ||.||_\alpha)$. First we recall the definition of convexly closed sequences given in [GBN].

Definition 2.2. *Let $\{x_n\}$ be a bounded basic sequence in a Banach space X. We say that $\{x_n\}$ is a convexly closed sequence if the set*

$$C = \left\{\sum_{n=1}^{\infty} t_n x_n : t_n \geq 0 \text{ and } \sum_{n=1}^{\infty} t_n = 1\right\}$$

is closed, that is, if $\overline{conv}\,\{x_n\} = C$.

Note that subsequences of convexly closed sequences are again convexly closed and that every basic sequence equivalent to a convexly closed sequence is convexly closed.

It is easy to see that the c_0 summing basis and the canonical basis of l^1 are convexly closed. An aisbc_0 sequence is also convexly closed, since it is equivalent to the c_0 summing basis.

Definition 2.3. *Let $\{x_n\}$ be a sequence in a Banach space X and $\alpha > 0$. We say that $\{x_n\}$ is an asymptotically isometric $c_{0\alpha}$-summing basic sequence, aisb$c_{0\alpha}$ sequence for short, if $\{x_n\}$ is convexly closed and there exist $\{\varepsilon_n\} \subset (0,\infty)$, $\{a_n\} \subset (0,\infty)$ and $\{\delta_n\} \subset \left(0, \sqrt{2}-1\right)$ such that $\varepsilon_n \to 0$, $\sum_{j=n+1}^{\infty} a_j \leq a_n$, $n \in \mathbb{N}$, $\delta_{n+1} \leq \delta_n$, $n \in \mathbb{N}$ and*

$$\sup_{n\in\mathbb{N}}(1+\varepsilon_n)^{-1}\left|\sum_{j=n}^{\infty} t_j\right| + \alpha\sum_{n=1}^{\infty}(1+\delta_n)^{-1}\left|\sum_{j=n}^{\infty} t_j\right| 2a_{n+1} \tag{2.2}$$

$$\leq \left\|\sum_{n=1}^{\infty} t_n x_n\right\| \leq \sup_{n\in\mathbb{N}}(1+\varepsilon_n)\left|\sum_{j=n}^{\infty} t_j\right| + \alpha\sum_{n=1}^{\infty}(1+\delta_n)\left|\sum_{j=n}^{\infty} t_j\right| a_n,$$

for all $\{t_n\} \in l^1$ with $\sum_{n=1}^{\infty} t_n = 0$. If $L > 0$, we say that $\{x_n\}$ is a L-aisb$c_{0\alpha}$ sequence if $\{Lx_n\}$ is an aisb$c_{0\alpha}$ sequence.

We saw in example 2.1 that there exist nonempty, convex, closed and bounded subsets of $c_{0\alpha}$ which are not ω-compact and without aisbc_0 sequences with the norm $||.||_\alpha$. The following example shows that there exist nonempty, convex, closed and bounded subsets of c_0 which are not ω-compact and without aisb$c_{0\alpha}$ sequences with the norm $||.||_\infty$.

Example 2.2. *Let $\{\xi_n\}$ be the c_0 summing basis. Then*

$$C = \left\{ \sum_{n=1}^{\infty} \lambda_n \xi_n : \lambda_n \geq 0 \text{ and } \sum_{n=1}^{\infty} \lambda_n = 1 \right\}$$

does not have aisbc$_{0\alpha}$ sequences with the norm $||.||_\infty$.

Proof. Suppose that C contains an aisbc$_{0\alpha}$ sequence $\{y_n\}$ with the norm $||.||_\infty$ for some sequences $\{\varepsilon_n\}$, $\{a_n\}$ and $\{\delta_n\}$. Since $\{y_n\} \in C$ we have that $y_n = \sum_{i=1}^{\infty} \lambda_i^n \xi_i$ for some sequence $\{\lambda_i^n\}_{i=1}^{\infty}$ such that $\lambda_i^n \geq 0$ and $\sum_{i=1}^{\infty} \lambda_i^n = 1$. Take $m \in \mathbb{N}$ with $m > 1$ and define

$$(t_n) = -e_1 + e_m,$$

where $\{e_n\}$ is the canonical basis of c_0. Then

$$\sum_{n=1}^{\infty} t_n y_n = -y_1 + y_m$$

and

$$\left(\sup_{1<n\leq m} \frac{1}{1+\varepsilon_n} \right) + \alpha \sum_{n=2}^{m} (1+\delta_n)^{-1} 2a_{n+1} \tag{2.3}$$

$$= \sup_{n\in\mathbb{N}} (1+\varepsilon_n)^{-1} \left| \sum_{j=n}^{\infty} t_j \right| + \alpha \sum_{n=1}^{\infty} (1+\delta_n)^{-1} \left| \sum_{j=n}^{\infty} t_j \right| 2a_{n+1} \leq \left\| \sum_{n=1}^{\infty} t_n y_n \right\|_\infty \leq 1.$$

Since (2.3) holds for all $m \in \mathbb{N}$, making $m \longrightarrow \infty$ in (2.3), we obtain

$$1 + \alpha \sum_{n=2}^{\infty} (1+\delta_n)^{-1} 2a_{n+1} \leq 1,$$

which contradicts the fact that $\alpha \sum_{n=2}^{\infty} (1+\delta_n)^{-1} 2a_{n+1} > 0$. □

The following proposition is an "analogous" of theorem 2 of [DLT]. In its proof, the set C and the operator T are constructed as in [DLT]. However, to prove that T is a nonexpansive mapping we have to do different estimations to those in [DLT].

Proposition 2.1. *Let K be a nonempty, convex, closed and bounded subset of a Banach space X. Fix $\{\varepsilon_n\} \subset (0,\infty)$ with $\varepsilon_n < 2^{-1}4^{-n}$, $n \geq 2$. If K contains an aisbc$_{0\alpha}$ sequence $\{x_n\}$ such that (2.2) holds with this $\{\varepsilon_n\}$ and some sequences $\{a_k\}$ and $\{\delta_k\}$, then there exist $C \subset K$ nonempty, convex and closed and $T : C \to C$ affine, nonexpansive and fixed point free. Moreover, T is contractive.*

Proof. Let $\{x_n\} \subset K$ be an aisbc$_{0\alpha}$ sequence with $\{\varepsilon_n\} \subset (0,\infty)$ such that $\varepsilon_n < 2^{-1}4^{-n}$, $n \geq 2$ and $\{a_k\} \subset (0,\infty)$ and $\{\delta_k\} \subset (0,\sqrt{2}-1)$ as in definition 2.3. Set

$$C = \overline{conv}\,\{x_n\} = \left\{ \sum_{n=1}^{\infty} t_n x_n : t_n \geq 0,\ n \in \mathbb{N} \text{ and} \sum_{n=1}^{\infty} t_n = 1 \right\} \subset K.$$

Thus C is nonempty, convex, closed and bounded. Define $Tx_n = \sum_{j=1}^{\infty} \frac{x_{n+j}}{2^j}$, $n \in \mathbb{N}$, and extend T linearly to C, that is, if $x = \sum_{n=1}^{\infty} t_n x_n \in C$ then define

$T\left(\sum_{n=1}^{\infty} t_n x_n\right) = \sum_{n=1}^{\infty} t_n T x_n$. It is clear that $T(C) \subset C$ and that T is an affine mapping. It is easy to see that T is fixed point free. See [DLT].

We only need to show that T is a contractive mapping. Let $x, y \in C$ with $x \neq y$. Then $x = \sum_{n=1}^{\infty} t_n x_n$ and $y = \sum_{n=1}^{\infty} s_n x_n$, with $t_n, s_n \geq 0$ and $\sum_{n=1}^{\infty} t_n = \sum_{n=1}^{\infty} s_n = 1$. Let $\beta_n = t_n - s_n$, $n \in \mathbb{N}$, so that $\sum_{n=1}^{\infty} \beta_n = 0$. Therefore,

$$\begin{aligned} T(x) - T(y) &= \sum_{n=1}^{\infty} \beta_n T(x_n) = \sum_{n=1}^{\infty} \beta_n \left(\sum_{j=1}^{\infty} \frac{x_{n+j}}{2^j} \right) \\ &= \left(\frac{\beta_1}{2} \right) x_2 + \left(\frac{\beta_1}{2^2} + \frac{\beta_2}{2} \right) x_3 + \left(\frac{\beta_1}{2^3} + \frac{\beta_2}{2^2} + \frac{\beta_3}{2} \right) x_4 + ... \end{aligned}$$

Define $B_1 = 0$ and $B_n = \frac{\beta_1}{2^{n-1}} + \frac{\beta_2}{2^{n-2}} + ... + \frac{\beta_{n-1}}{2}$, $n \geq 2$. Thus

$$T(x) - T(y) = \sum_{n=1}^{\infty} B_n x_n.$$

Since $\{x_n\}$ is an aisbc$_{0\alpha}$ we have

$$\|T(x) - T(y)\| = \left\| \sum_{n=1}^{\infty} B_n x_n \right\| \leq \sup_{k \in \mathbb{N}} (1 + \varepsilon_k) \left| \sum_{j=k}^{\infty} B_j \right| + \alpha \sum_{k=1}^{\infty} (1 + \delta_k) \left| \sum_{j=k}^{\infty} B_j \right| a_k.$$

By theorem 2 of [DLT] we obtain

$$\sup_{k \in \mathbb{N}} (1 + \varepsilon_k) \left| \sum_{j=k}^{\infty} B_j \right| < \sup_{k \in \mathbb{N}} (1 + \varepsilon_k)^{-1} \left| \sum_{j=k}^{\infty} \beta_j \right|. \tag{2.4}$$

On the other hand,

$$\sum_{j=1}^{\infty} B_j = \sum_{j=2}^{\infty} B_j = \sum_{j=1}^{\infty} \beta_j = 0$$

and

$$\begin{aligned} \sum_{j=n}^{\infty} B_j &= \left(\frac{\beta_1}{2^{n-1}} + \frac{\beta_2}{2^{n-2}} + ... + \frac{\beta_{n-1}}{2} \right) + \left(\frac{\beta_1}{2^n} + \frac{\beta_2}{2^{n-1}} + ... + \frac{\beta_n}{2} \right) \\ &\quad + \left(\frac{\beta_1}{2^{n+1}} + \frac{\beta_2}{2^n} + ... + \frac{\beta_{n+1}}{2} \right) + ... \\ &= \frac{\beta_1}{2^{n-2}} + \frac{\beta_2}{2^{n-3}} + ... + \frac{\beta_{n-2}}{2} + \sum_{j=n-1}^{\infty} \beta_j, \ n \geq 3. \end{aligned}$$

Since $\sum_{j=1}^{\infty} \beta_j = 0$, note that

$$\left| \sum_{j=3}^{\infty} B_j \right| = \left| \frac{\beta_1}{2} + \sum_{j=2}^{\infty} \beta_j \right| = \left| \frac{\beta_1}{2} - \beta_1 \right| = \left| \frac{\beta_1}{2} \right|,$$

$$\left|\sum_{j=4}^{\infty} B_j\right| = \left|\frac{\beta_1}{2^2}+\frac{\beta_2}{2}+\sum_{j=3}^{\infty}\beta_j\right| = \left|\frac{\beta_1}{2^2}+\frac{\beta_2}{2}-(\beta_1+\beta_2)\right|$$

$$\leq \left|\frac{\beta_1}{2}+\frac{\beta_2}{2}-(\beta_1+\beta_2)\right| + \frac{|\beta_1|}{2^2} = \frac{|\beta_1+\beta_2|}{2}+\frac{|\beta_1|}{2^2}.$$

In general, if $k \geq 3$ we have

$$\left|\sum_{j=k}^{\infty} B_j\right| \leq \frac{|\beta_1+\beta_2+...+\beta_{k-2}|}{2}+...+\frac{|\beta_1+\beta_2|}{2^{k-3}}+\frac{|\beta_1|}{2^{k-2}}.$$

Since $\delta_k < \sqrt{2}-1,\ k\in\mathbb{N}$, then $1+\delta_k < \frac{2}{1+\delta_k},\ k\in\mathbb{N}$. We also have that $\delta_{k+1} \leq \delta_k,\ k\in\mathbb{N}$; therefore

$$\begin{aligned}(1+\delta_k)\left|\sum_{j=k}^{\infty} B_j\right| a_k &\leq (1+\delta_k)a_k\left(\frac{|\beta_1+\beta_2+...+\beta_{k-2}|}{2}+...+\frac{|\beta_1|}{2^{k-2}}\right)\\ &\leq \sum_{j=1}^{k-2}(1+\delta_{k-j})a_k\left(\frac{|\beta_1+\beta_2+...+\beta_{k-1-j}|}{2^j}\right)\\ &< \sum_{j=1}^{k-2}(1+\delta_{k-j})^{-1}a_k\left(\frac{|\beta_1+\beta_2+...+\beta_{k-1-j}|}{2^{j-1}}\right)\end{aligned}$$

Thus

$$\begin{aligned}\sum_{k=3}^{\infty}(1+\delta_k)\left|\sum_{j=k}^{\infty} B_j\right| a_k &< \sum_{k=3}^{\infty}\sum_{j=1}^{k-2}(1+\delta_{k-j})^{-1}a_k\left(\frac{|\beta_1+\beta_2+...+\beta_{k-1-j}|}{2^{j-1}}\right)\\ &= \sum_{k=2}^{\infty}(1+\delta_k)^{-1}\left|\sum_{j=1}^{k-1}\beta_j\right|\left(a_{k+1}+\frac{a_{k+2}}{2}+\frac{a_{k+3}}{2^2}+...\right)\\ &\leq \sum_{k=2}^{\infty}(1+\delta_k)^{-1}\left|\sum_{j=k}^{\infty}\beta_j\right| 2a_{k+1}\\ &= \sum_{k=1}^{\infty}(1+\delta_k)^{-1}\left|\sum_{j=k}^{\infty}\beta_j\right| 2a_{k+1},\end{aligned}$$

since $a_{k+1} \leq a_k,\ k\in\mathbb{N}$ and $\sum_{n=1}^{\infty}\beta_n = 0$. We know that $\sum_{n=1}^{\infty}B_n = \sum_{n=2}^{\infty}B_n = 0$, so we get

$$\sum_{k=1}^{\infty}(1+\delta_k)\left|\sum_{j=k}^{\infty} B_j\right| a_k < \sum_{k=1}^{\infty}(1+\delta_k)^{-1}\left|\sum_{j=k}^{\infty}\beta_j\right| 2a_{k+1}. \tag{2.5}$$

Finally, from (2.4) and (2.5) we obtain

$$\|T(x)-T(y)\| = \left\|\sum_{k=1}^{\infty} B_k x_k\right\|$$

$$\leq \sup_{k\in\mathbb{N}}(1+\varepsilon_k)\left|\sum_{j=k}^{\infty} B_j\right| + \alpha\sum_{k=1}^{\infty}(1+\delta_k)\left|\sum_{j=k}^{\infty} B_j\right| a_k <$$

$$< \sup_{k\in\mathbb{N}}(1+\varepsilon_k)^{-1}\left|\sum_{j=k}^{\infty} \beta_j\right| + \alpha\sum_{k=1}^{\infty}(1+\delta_k)^{-1}\left|\sum_{j=k}^{\infty} \beta_j\right| 2a_{k+1} \leq$$

$$\leq \left\|\sum_{k=1}^{\infty} \beta_k x_k\right\| = \|x-y\|\,.$$

□

To simplify the proof of the below proposition we make use of the decomposition $||.||_\alpha = ||.||_\infty + \alpha\,\|.\|_s\,.$

Proposition 2.2. *Let K be a nonempty, convex, closed and bouded subset of $c_{0\alpha}$ which is not weakly compact. Then K has an L-aisbc$_{0\alpha}$ sequence $\{z_k\}$ such that (2.2) holds for some sequence $\{\varepsilon_k^{'}\} \subset (0,\infty)$ with $\varepsilon_k^{'} < 2^{-1}4^{-k},\ \ k \geq 2$.*

Proof. Suppose that K is a nonempty, convex, closed and bouded subset of $c_{0\alpha}$ which is not weakly compact. Since K is not ω-compact in c_0, by theorem 4 of [DLT], there exist $\{y_n\} \subset K \subset c_0$ and $L > 0$ such that $\{Ly_n\}$ is an aisbc_0 sequence and $y_n \xrightarrow{\omega^*} y \in l^\infty \backslash c_0$. Define $x_n = Ly_n,\ n \in \mathbb{N}$. Since $\{x_n\}$ is an aisbc_0 sequence then it is convexly closed and by remark 1.1 we can suppose that $\{x_n\}$ satisfies (1.1) for some sequence $\varepsilon_n \subset (0,\infty)$ with $\varepsilon_n < 2^{-1}4^{-n},\ \ n \geq 2$, that is

$$\sup_{n\in\mathbb{N}}(1+\varepsilon_n)^{-1}\left|\sum_{j=n}^{\infty} t_j\right| \leq \left\|\sum_{n=1}^{\infty} t_n x_n\right\|_\infty \leq \sup_{n\in\mathbb{N}}(1+\varepsilon_n)\left|\sum_{j=n}^{\infty} t_j\right|, \tag{2.6}$$

for all $\{t_n\} \in l^1$. Suppose that $x_n = \{\alpha_k^n\}_{k=1}^\infty$ and $x = \{\alpha_k\}_{k=1}^\infty$. Since $x_n \xrightarrow{\omega^*} x \in l^\infty \backslash c_0$ we have that x_n converges coordinatewise to x and

$$\sum_{m=i}^{\infty} \frac{|\alpha_m^n - \alpha_m|}{2^m} > 0,\ \ n, i \in \mathbb{N}. \tag{2.7}$$

Moreover, since x_n converges coordinatewise to x we also have that $||x_n - x||_s \longrightarrow 0$.

Fix $\{\delta_n\} \subset \left(0, \sqrt{2}-1\right)$ such that $\delta_{n+1} \leq \delta_n,\ n \in \mathbb{N}$. Since $||x_n - x||_s \longrightarrow 0$, there exists $B > 0$ such that $||x_n - x||_s < B$. Define $N_1 = 1$, $a_1 = B$ and $b_1 = 1$. Since $||x_n - x||_s \longrightarrow 0$, there exists $M_1 \in \mathbb{N}$ such that

$$\sum_{m=1}^{\infty} \frac{|\alpha_m^q - \alpha_m^r|}{2^m} = ||x_q - x_r||_s < \frac{1}{2^{N_1}(1+\delta_2)},\ \ q, r \geq M_1.$$

Take $n_1 > M_1$. From (2.7) we can select $N_2 \in \mathbb{N}$ such that $N_2 > N_1$,

$$0 < \sum_{m=1}^{N_2} \frac{|\alpha_m^{n_1} - \alpha_m|}{2^m} < a_1. \tag{2.8}$$

$$P_2 \equiv \sum_{m=N_1+1}^{N_2} \frac{|\alpha_m^{n_1} - \alpha_m|}{2^m} > 0, \tag{2.9}$$

$$P_2 - \frac{1}{2^{N_2}} > 0,$$

and

$$\sum_{m=N_2+1}^{\infty} \frac{|\alpha_m^{n_1} - \alpha_m|}{2^m} < P_2 \left(1 - \frac{1}{1+\delta_2}\right). \tag{2.10}$$

Since $||x_n - x||_s \longrightarrow 0$, there exists $M_2 \in \mathbb{N}$ such that

$$\sum_{m=1}^{\infty} \frac{|\alpha_m^q - \alpha_m^r|}{2^m} = ||x_q - x_r||_s < \frac{1}{2^{N_2}(1+\delta_3)}, \ q, r \geq M_2.$$

Since x_n converges coordinatewise to x, by (2.8), (2.9) and (2.10), there exists $n_2 > \max\{M_2, n_1\}$ such that

$$0 < a_2 \equiv \sum_{m=1}^{N_2} \frac{|\alpha_m^{n_1} - \alpha_m^{n_2}|}{2^m} < a_1,$$

$$b_2 \equiv \sum_{m=N_1+1}^{N_2} \frac{|\alpha_m^{n_1} - \alpha_m^{n_2}|}{2^m} > 0,$$

$$b_2 - \frac{1}{2^{N_2}} > 0,$$

$$\sum_{m=N_2+1}^{\infty} \frac{|\alpha_m^{n_1} - \alpha_m^{n_2}|}{2^m} < b_2 \left(1 - \frac{1}{1+\delta_2}\right),$$

and also that

$$0 < \sum_{m=1}^{\infty} \frac{|\alpha_m^{n_2} - \alpha_m|}{2^m} = \|x_{n_2} - x\|_s < \frac{1}{2}\left(b_2 - \frac{1}{2^{N_2}}\right).$$

Let $N_3 \in \mathbb{N}$ such that $N_3 > N_2$,

$$0 < \sum_{m=1}^{N_3} \frac{|\alpha_m^{n_2} - \alpha_m|}{2^m} < \frac{1}{2}\left(b_2 - \frac{1}{2^{N_2}}\right),$$

$$P_3 \equiv \sum_{m=N_2+1}^{N_3} \frac{|\alpha_m^{n_2} - \alpha_m|}{2^m} > 0,$$

$$P_3 - \frac{1}{2^{N_3}} > 0,$$

and

$$\sum_{m=N_3+1}^{\infty} \frac{|\alpha_m^{n_2} - \alpha_m|}{2^m} < P_3 \left(1 - \frac{1}{1+\delta_3}\right).$$

Since $||x_n - x||_s \longrightarrow 0$, there exists $M_3 \in \mathbb{N}$ such that

$$\sum_{m=1}^{\infty} \frac{|\alpha_m^q - \alpha_m^r|}{2^m} = ||x_q - x_r||_s < \frac{1}{2^{N_3}(1+\delta_4)}, \; q, r \geq M_3.$$

Since x_n converges coordinatewise to x, there exists $n_3 > \max\{M_3, n_2\}$ such that

$$0 < a_3 \equiv \sum_{m=1}^{N_3} \frac{|\alpha_m^{n_2} - \alpha_m^{n_3}|}{2^m} < \frac{1}{2}\left(b_2 - \frac{1}{2^{N_2}}\right),$$

$$b_3 \equiv \sum_{m=N_2+1}^{N_3} \frac{|\alpha_m^{n_2} - \alpha_m^{n_3}|}{2^m} > 0,$$

$$b_3 - \frac{1}{2^{N_3}} > 0$$

$$\sum_{m=N_3+1}^{\infty} \frac{|\alpha_m^{n_2} - \alpha_m^{n_3}|}{2^m} < b_3 \left(1 - \frac{1}{1+\delta_3}\right),$$

and also that

$$0 < \sum_{m=1}^{\infty} \frac{|\alpha_m^{n_3} - \alpha_m|}{2^m} < \frac{1}{2}\left(b_3 - \frac{1}{2^{N_3}}\right).$$

Continuing in this way we can construct sequences $\{a_k\}_{k=1}^{\infty} \subset (0, B]$ and $\{b_k\}_{k=1}^{\infty} \subset (0, 1]$ such that $a_1 = B, \; b_1 = 1$,

$$0 < a_k = \sum_{m=1}^{N_k} \frac{\left|\alpha_m^{n_{k-1}} - \alpha_m^{n_k}\right|}{2^m} < \frac{1}{2}\left(b_{k-1} - \frac{1}{2^{N_{k-1}}}\right), \; k \geq 3, \tag{2.11}$$

$$b_k = \sum_{m=N_{k-1}+1}^{N_k} \frac{\left|\alpha_m^{n_{k-1}} - \alpha_m^{n_k}\right|}{2^m} > 0, \; k \geq 2, \tag{2.12}$$

$$b_k - \frac{1}{2^{N_k}} > 0, \; k \geq 2,$$

$$2a_{k+1} < b_k - \frac{1}{2^{N_k}} < b_k < a_k, \; k \geq 2$$

and

$$\sum_{m=N_k+1}^{\infty} \frac{\left|\alpha_m^{n_{k-1}} - \alpha_m^{n_k}\right|}{2^m} < b_k \left(1 - \frac{1}{1+\delta_k}\right), \; k \geq 2. \tag{2.13}$$

Since $2a_{k+1} < a_k, \; k \geq 2$, then $a_{k+1} < a_k, \; k \geq 2$, and by construction we have that $a_2 < a_1$. Moreover, since $n_1 > M_1$ and $n_k > \max\{M_k, n_{k-1}\}, \; k \geq 2$, by construction we also have

$$\sum_{m=1}^{N_k} \frac{\left|\alpha_m^{n_{k+1}} - \alpha_m^{n_k}\right|}{2^m} < \frac{1}{2^{N_k}(1+\delta_{k+1})}, \; k \in \mathbb{N}. \tag{2.14}$$

Take now $\{t_k\} \in l^1$ such that $\sum_{k=1}^{\infty} t_k = 0$. Let $N_0 = 0$. Since

$$\sum_{k=1}^{\infty} t_k x_{n_k} = \left(\sum_{k=1}^{\infty} t_k \alpha_1^{n_k}, \sum_{k=1}^{\infty} t_k \alpha_2^{n_k}, ... \right),$$

then

$$\begin{aligned} \left\| \sum_{k=1}^{\infty} t_k x_{n_k} \right\|_s &= \sum_{m=1}^{\infty} \frac{1}{2^m} \left| \sum_{k=1}^{\infty} t_k \alpha_m^{n_k} \right| \\ &= \sum_{i=0}^{\infty} \sum_{m=N_i+1}^{N_{i+1}} \frac{1}{2^m} \left| \sum_{k=1}^{\infty} t_k \alpha_m^{n_k} \right|. \end{aligned}$$

Let $\alpha_m^{n_0} = \alpha_m$. Since $\sum_{k=1}^{\infty} t_k = 0$, we have

$$\begin{aligned} \sum_{k=1}^{\infty} t_k \alpha_m^{n_k} &= \sum_{k=1}^{\infty} t_k \left(\alpha_m^{n_k} - \alpha_m \right) \\ &= \sum_{k=1}^{\infty} \left(\sum_{i=k}^{\infty} t_i - \sum_{i=k+1}^{\infty} t_i \right) \left(\alpha_m^{n_k} - \alpha_m \right) \\ &= \sum_{i=1}^{\infty} \left(\sum_{k=i}^{\infty} t_k \right) \left(\alpha_m^{n_i} - \alpha_m^{n_{i-1}} \right). \end{aligned}$$

Thus,

$$\sum_{m=1}^{N_1} \frac{1}{2^m} \left| \sum_{k=1}^{\infty} t_k \alpha_m^{n_k} \right|$$

$$\geq \sum_{m=1}^{N_1} \frac{1}{2^m} \left(\left| \sum_{k=1}^{\infty} t_k \right| \left| \alpha_m^{n_1} - \alpha_m \right| - \sum_{p=2}^{\infty} \left| \sum_{k=p}^{\infty} t_k \right| \left| \alpha_m^{n_p} - \alpha_m^{n_{p-1}} \right| \right),$$

$$\sum_{m=N_1+1}^{N_2} \frac{1}{2^m} \left| \sum_{k=1}^{\infty} t_k \alpha_m^{n_k} \right|$$

$$\geq \sum_{m=N_1+1}^{N_2} \frac{1}{2^m} \left(\left| \sum_{k=2}^{\infty} t_k \right| \left| \alpha_m^{n_2} - \alpha_m^{n_1} \right| - \sum_{p \neq 2}^{\infty} \left| \sum_{k=p}^{\infty} t_k \right| \left| \alpha_m^{n_p} - \alpha_m^{n_{p-1}} \right| \right),$$

$$\sum_{m=N_2+1}^{N_3} \frac{1}{2^m} \left| \sum_{k=1}^{\infty} t_k \alpha_m^{n_k} \right|$$

$$\geq \sum_{m=N_2+1}^{N_3} \frac{1}{2^m} \left(\left| \sum_{k=3}^{\infty} t_k \right| \left| \alpha_m^{n_3} - \alpha_m^{n_2} \right| - \sum_{p \neq 3}^{\infty} \left| \sum_{k=p}^{\infty} t_k \right| \left| \alpha_m^{n_p} - \alpha_m^{n_{p-1}} \right| \right), ...$$

Therefore, from (2.14), (2.12) and (2.13) we obtain

$$\left\| \sum_{k=1}^{\infty} t_k x_{n_k} \right\|_s$$

$$\geq \left|\sum_{k=1}^{\infty} t_k\right| \left(\sum_{m=1}^{N_1} \frac{|\alpha_m^{n_1} - \alpha_m|}{2^m} - \sum_{m=N_1+1}^{\infty} \frac{|\alpha_m^{n_1} - \alpha_m|}{2^m}\right)$$

$$+ \left|\sum_{k=2}^{\infty} t_k\right| \left(-\sum_{m=1}^{N_1} \frac{|\alpha_m^{n_2} - \alpha_m^{n_1}|}{2^m} + \sum_{m=N_1+1}^{N_2} \frac{|\alpha_m^{n_2} - \alpha_m^{n_1}|}{2^m} - \sum_{m=N_2+1}^{\infty} \frac{|\alpha_m^{n_2} - \alpha_m^{n_1}|}{2^m}\right)$$

$$+ \left|\sum_{k=3}^{\infty} t_k\right| \left(-\sum_{m=1}^{N_2} \frac{|\alpha_m^{n_3} - \alpha_m^{n_2}|}{2^m} + \sum_{m=N_2+1}^{N_3} \frac{|\alpha_m^{n_3} - \alpha_m^{n_2}|}{2^m} - \sum_{m=N_3+1}^{\infty} \frac{|\alpha_m^{n_3} - \alpha_m^{n_2}|}{2^m}\right)$$

$$+ \left|\sum_{k=4}^{\infty} t_k\right| \left(-\sum_{m=1}^{N_3} \frac{|\alpha_m^{n_4} - \alpha_m^{n_3}|}{2^m} + \sum_{m=N_3+1}^{N_4} \frac{|\alpha_m^{n_4} - \alpha_m^{n_3}|}{2^m} - \sum_{m=N_4+1}^{\infty} \frac{|\alpha_m^{n_4} - \alpha_m^{n_3}|}{2^m}\right) + \ldots$$

$$\geq \left|\sum_{k=2}^{\infty} t_k\right| \left(-\frac{1}{2^{N_1}(1+\delta_2)} + b_2 - b_2\left(1 - \frac{1}{1+\delta_2}\right)\right)$$

$$+ \left|\sum_{k=3}^{\infty} t_k\right| \left(-\frac{1}{2^{N_2}(1+\delta_3)} + b_3 - b_3\left(1 - \frac{1}{1+\delta_3}\right)\right) +$$

$$+ \left|\sum_{k=4}^{\infty} t_k\right| \left(-\frac{1}{2^{N_3}(1+\delta_4)} + b_4 - b_4\left(1 - \frac{1}{1+\delta_4}\right)\right) + \ldots$$

$$= \sum_{k=2}^{\infty} \left|\sum_{j=k}^{\infty} t_j\right| \left(-\frac{1}{2^{N_k}(1+\delta_k)} + \frac{b_k}{1+\delta_k}\right)$$

$$= \sum_{k=2}^{\infty} \left|\sum_{j=k}^{\infty} t_j\right| \left(\frac{b_k - \frac{1}{2^{N_k}}}{1+\delta_k}\right) \geq \sum_{k=1}^{\infty} \left|\sum_{j=k}^{\infty} t_j\right| \frac{2a_{k+1}}{1+\delta_k},$$

since $2a_{k+1} \leq b_k - \frac{1}{2^{N_k}}$, $k \geq 2$. On the other hand, by (2.11) and (2.13) we have

$$\begin{aligned}
\left\|\sum_{k=1}^{\infty} t_k x_{n_k}\right\|_s &\leq \sum_{k=2}^{\infty} \left|\sum_{j=k}^{\infty} t_j\right| \left(a_k + b_k\left(1 - \frac{1}{1+\delta_k}\right)\right) \\
&\leq \sum_{k=2}^{\infty} \left|\sum_{j=k}^{\infty} t_j\right| \left(a_k + a_k\left(1 - \frac{1}{1+\delta_k}\right)\right) \\
&= \sum_{k=1}^{\infty} \left|\sum_{j=k}^{\infty} t_j\right| a_k \left(2 - \frac{1}{1+\delta_k}\right) \\
&< \sum_{k=1}^{\infty} \left|\sum_{j=k}^{\infty} t_j\right| a_k (1+\delta_k).
\end{aligned}$$

Then we obtain

$$\sum_{k=1}^{\infty}\left|\sum_{j=k}^{\infty}t_j\right|\frac{2a_{k+1}}{1+\delta_k}\leq\left\|\sum_{k=1}^{\infty}t_kx_{n_k}\right\|_s\leq\sum_{k=1}^{\infty}(1+\delta_k)\left|\sum_{j=k}^{\infty}t_j\right|a_k, \tag{2.15}$$

for all $\{t_k\}\in l^1$ with $\sum_{k=1}^{\infty}t_k=0$. Since $\{x_{n_k}\}$ is a subsequence of $\{x_k\}$, then $\{x_{n_k}\}$ is convexly closed with the norm $||.||_\infty$ and hence with the norm $||.||_\alpha$. Since $||.||_\alpha = ||.||_\infty+\alpha\,\|.\|_s$, by (2.6), (2.15) and remark 1.1 we obtain

$$\sup_{k\in\mathbb{N}}(1+\varepsilon_k^{'})^{-1}\left|\sum_{j=k}^{\infty}t_j\right|+\alpha\sum_{k=1}^{\infty}(1+\delta_k)^{-1}\left|\sum_{j=k}^{\infty}t_j\right|2a_{k+1}\leq$$
$$\leq\left\|\sum_{k=1}^{\infty}t_kx_{n_k}\right\|_\alpha\leq$$
$$\leq\sup_{k\in\mathbb{N}}(1+\varepsilon_k^{'})\left|\sum_{j=k}^{\infty}t_j\right|+\alpha\sum_{k=1}^{\infty}(1+\delta_k)\left|\sum_{j=k}^{\infty}t_j\right|a_k,$$

for all $\{t_n\}\in l^1$ with $\sum_{k=1}^{\infty}t_k=0$, for a new sequence $\varepsilon_k^{'}\subset(0,\infty)$ such that $\varepsilon_k^{'}< 2^{-1}4^{-k},\ \ k\geq 2$. Finally, define $z_k=y_{n_k},\ n\in\mathbb{N}$. Then $\{z_k\}$ is the desired sequence. □

From propositions 2.2 and 2.1 we get the following result.

Theorem 2.1. *Let K be a nonempty, convex, closed and bounded subset of $c_{0\alpha}$. If K is not weakly compact, then there exist $C\subset K$ nonempty, convex and closed and $T:C\to C$ affine, nonexpansive and fixed point free. Moreover, T is contractive.*

As we said, in [FGB], Fetter H., and Gamboa de Buen B., showed that for all $\alpha\geq 0$, the $c_{0\alpha}$ space has the ω-FPP, then from this result and theorem 2.1 we obtain the following theorem.

Theorem 2.2. *Let K be a nonempty, convex, closed and bouded subset of $c_{0\alpha}$. Then K is weakly compact if and only if every nonempty, convex and closed subset of K has the FPP.*

We think that the conjecture of Llorens-Fuster and Sims is also true in the $c_{0\alpha}$ setting.

Now we turn to the c_α space.

Define $\pi:c\longrightarrow\mathbb{K}$ by $\pi(\{x_k\})=\lim x_k$. Thus, $\pi\in c^*$. Take $a\in\mathbb{K}$ and consider the set $\pi^{-1}(\{a\})$. Define $U:\pi^{-1}(\{a\})\longrightarrow c_0$ by $U(\{x_i\})=\{x_i-a\}$. In the proof of corollary 7 given in [DLT], it was shown that U is an affine mapping and that

$$U:\left(\pi^{-1}(\{a\}),\sigma(c,c^*)_{|_{\pi^{-1}(\{a\})}}\right)\longrightarrow(c_0,\sigma(c_0,c_0^*))$$

is a homeomorphism. Since $\|.\|_\alpha$ and $\|.\|_\infty$ are equivalent, we also have that $\pi\in c_\alpha^*$ and that

$$U:\left(\pi^{-1}(\{a\}),\sigma(c_\alpha,c_\alpha^*)_{|_{\pi^{-1}(\{a\})}}\right)\longrightarrow(c_{0\alpha},\sigma(c_{0\alpha},c_{0\alpha}^*))$$

is a homeomorphism.

Consider $K \subset c_\alpha$ nonempty, convex, closed and bounded which is not ω-compact. Set

$$Q(K) = \left\{a \in \mathbb{K} : \pi^{-1}(\{a\}) \cap K \text{ is not } \omega\text{-compact in } c_\alpha\right\}.$$

Since K is not ω-compact in c_α, by lemma 6 of [DLT] we have that $Q(K) \neq \phi$.

Theorem 2.3. *Let K be a nonempty, convex, closed and bouded subset of c_α. Then*

i) *The set K is weakly compact if every nonempty, convex and closed subset of K has the FPP.*

ii) *For $\alpha \in [0, 1)$, the set K is weakly compact if and only if every nonempty, convex and closed subset of K has the FPP.*

Proof. i). Suppose that K is not weakly compact in c_α. Then there exists $a \in \mathbb{K}$ such that $\pi^{-1}(\{a\}) \cap K$ is not ω-compact in c_α and hence $U\left(\pi^{-1}(\{a\}) \cap K\right)$ is not ω-compact in $c_{0\alpha}$. By theorem 2.1, there exist $D \subset U\left(\pi^{-1}(\{a\}) \cap K\right)$ nonempty, convex and closed and $R : D \to D$ affine, nonexpansive and fixed point free. Moreover, R is contractive. Therefore $C = U^{-1}(D)$ is a nonempty, convex, closed and bounded subset of $\pi^{-1}(\{a\}) \cap K \subset K$ and the operator $T = U^{-1}RU : C \longrightarrow C$ is nonexpansive and fixed point free.

ii). This result follows by i) and that for $\alpha \in [0, 1)$, the space $c_\alpha = (c, ||.||_\alpha)$ has the ω-FPP. See [FGB]. □

Remark 2.1. *If in the definition of $\|.\|_s$ we take a sequence $\{u_n\} \subset (0, \infty)$ such that $\sum_{n=1}^{\infty} u_n < \infty$ and $\sum_{n=m}^{\infty} u_n \leq u_m$, $m \in \mathbb{N}$ instead of the sequence $\left\{\frac{1}{2^n}\right\}$, then we obtain analogous results to those given in this work.*

Acknowledgments. 1st Author Partially supported by SEP-Conacyt Grant 102380. 2nd Author Partially supported by Conacyt Scholarship 165342 and CIMAT Scholarship.

References

[MAU] B. Maurey, *Points fixes des contractions sur un convex fermé de L^1*, Seminaire d'Analyse fonctionnelle 80-81, Ecole Polytechnique, 1980.

[LFS] E. Llorens-Fuster, B. Sims, *The fixed point property in c_0*, Canad. Math. Bull., **41**(1998), 413-422.

[DLT] P.N. Dowling, C.J. Lennard, B. Turett, *Characterizations of weakly compact sets and new fixed point free maps in c_0*, Studia. Math., **154**(2003), 277-293.

[DOL] P.N. Dowling, C.J. Lennard, B. Turett, *Weak compactness is equivalent to the fixed point property in c_0*, Proc. Amer. Math. Soc., **132**(2004), 1659-1666.

[JIM] A. Jiménez-Melado, *Stability of weak normal structure in James quasi reflexive space*, Bull. Austral. Math. Soc., **46**(1992),367-372.

[FGB] H. Fetter, B. Gamboa de Buen, *3-space problem related to the fixed point property*, Bull. Austral. Math. Soc., **64**(2001), 51-56.

[GBN] B. Gamboa de Buen, Fernando Núñez-Medina, *The fixed point property in c_0 with an equivalent norm*, Abstract Appl. Anal., vol. 2011, Article ID 574614, 19 pages, 2011. doi: 10.1155/2011/574614

Received: September 20, 2012; Accepted: February 06, 2013