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SOLUTIONS OF A CLASS OF INTERNAL NONLOCAL CAUCHY PROBLEMS FOR THE DIFFERENTIAL EQUATION $x'(t) = f(t, x(t), x'(t))$

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Abstract. In this work, we study the existence of solutions for the Cauchy problem of the differential equation $x'(t) = f(t, x(t), x'(t))$ with some general class of internal nonlocal and integral conditions.
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1. Introduction

Problems with non-local conditions have been extensively studied by several authors in the last two decades. The reader is referred to [1]-[6] and [9]-[19] and references therein.
Here we are concerning with the Cauchy problem of the differential equation

$$x'(t) = f(t, x(t), x'(t)), \text{ for } a.e.\ t \in (0,1) \tag{1.1}$$

with the general internal nonlocal condition

$$\sum_{k=1}^{m} a_k x(\tau_k) = \alpha \sum_{j=1}^{n} b_j x(\eta_j) \tag{1.2}$$

where $\tau_k \in (a,c)$, $\eta_j \in (d,b)$, $0 < a < c \leq d < b < 1$ and α is parameter.

Our aim here is to study the existence of at least one absolutely continuous solution $x \in AC[0,1]$ for the problem (1.1)-(1.2) when the function f is measurable in $t \in [0,1]$ for any $(u_1, u_2) \in R^2$ and continuous in $(u_1, u_2) \in R^2$ for almost all $t \in [0,1]$. Also we deduce the existence of solutions for the Cauchy problem of equation (1) with the nonlocal integral condition

$$\int_a^c x(s)\, ds = \alpha \int_d^b x(s)\, ds. \tag{1.3}$$

It must be noticed that the following nonlocal and integral conditions are special cases of our nonlocal and integral conditions

$$x(\tau) \ = \ \beta \ x(\eta), \ \tau \in (a,c) \text{ and } \ \eta \in (d,b), \tag{1.4}$$

$$\sum_{k=1}^{m} \ a_k x(\tau_k) \ = \ \beta \ x(\eta), \ \tau_k \in (a,c) \text{ and } \ \eta \in (d,b), \tag{1.5}$$

$$\sum_{k=1}^{m} \ a_k x(\tau_k) \ = \ 0, \ \tau_k \in (a,c), \tag{1.6}$$

$$\int_a^c \ x(s) \ ds \ = \ \beta \ x(\eta), \ \eta \in (d,b), \tag{1.7}$$

and

$$\int_a^c \ x(s) \ ds \ = \ 0, \ (a,c). \tag{1.8}$$

The following theorems will be needed.

Theorem. (Kolmogorov Compactness Criterion) (see [8])

Let $\Omega \subseteq L^P(0,1), \ 1 \leq P < \infty$. *If*

(i) Ω *is bounded* $L^p(0,1)$,

(ii) $x_h \to x$ *as* $h \to 0$ *uniformly with respect to* $x \in \Omega$, *then* Ω *is relatively compact in* $L^P(0,1)$, *where*

$$x_h(t) = \frac{1}{h} \int_t^{t+h} \ x(s) \ ds.$$

Theorem. (Schauder) (see [14])

Let U *be a convex subset of a Banach space* X *, and* $T : U \to U$ *is compact, continuous map. Then* T *has at least one fixed point in* U *.*

2. Existence of solution

Consider the nonlocal problem (1)-(2). Let $\frac{dx(t)}{dt} = y(t)$, then

$$y(t) = f(t, x(t), y(t)) \tag{2.1}$$

and

$$x(t) = x(0) + \int_0^t \ y(s) \ ds. \tag{2.2}$$

Let $t = \tau_k$ in (2.2), we obtain

$$x(\tau_k) = x(0) + \int_0^{\tau_k} \ y(s) \ ds,$$

than

$$\sum_{k=1}^{m} \ a_k x(\tau_k) = \sum_{k=1}^{m} \ a_k x(0) + \sum_{k=1}^{m} \ a_k \int_0^{\tau_k} \ y(s) \ ds. \tag{2.3}$$

Let $t = \eta_j$ in (2.2), we can obtain

$$\alpha \sum_{j=1}^{n} \ b_j x(\eta_j) = \alpha \sum_{j=1}^{n} \ b_j x(0) + \alpha \sum_{j=1}^{n} \ b_j \int_0^{\eta_j} \ y(s) \ ds. \tag{2.4}$$

From (2.3) and (2.4), we obtain

$$x(0) = A(\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\ ds - \alpha \sum_{j=1}^{n} b_j \int_0^{\eta_j} y(s)\ ds). \tag{2.5}$$

Substitute from (2.5) into (2.2), we get

$$x(t) = A(\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\ ds - \alpha \sum_{j=1}^{n} b_j \int_0^{\eta_j} y(s)\ ds) + \int_0^t y(s)\ ds, \tag{2.6}$$

where y is the solution of the functional integral equation

$$y(t) = f(t, \{A(\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\ ds - \alpha \sum_{j=1}^{n} b_j \int_0^{\eta_j} y(s)\ ds) + \int_0^t y(s)\ ds\},\ y(t)). \tag{2.7}$$

Then we proved the following lemma.

Lemma 2.1. *Let* $\alpha \sum_{j=1}^{n} b_j \neq -\sum_{k=1}^{m} a_k$. *Then the solution of the nonlocal problem (1)-(2) can be expressed by the integral equation (2.6) where* y *is the solution of the functional integral equation (2.7).*

Consider the functional equation (2.7) with the following assumptions

(i) $f : [0,1] \times R^2 \to R$ *is measurable in* $t \in [0,1]$, *for any* $(u_1, u_2) \in R^2$ *and continuous in* $(u_1, u_2) \in R^2$, *for almost all* $t \in [0,1]$.

(ii) *There exists a function* $a \in L_1[0,1]$ *and constant* $b_i > 0$; $i = 1, 2$ *such that*

$$|f(t, u_1, u_2)| \leq |a(t)| + \sum_{i=1}^{2} b_i |u_i|;\ \forall (t, u_1, u_2) \in [0,1] \times R^2.$$

(iii) $(Ab_1(\sum_{k=1}^{m} a_k + \alpha \sum_{j=1}^{n} b_j) + b_1 + b_2) < 1$.

Now, we have the following theorem.

Theorem 2.1. *Assume that the assumptions (i) - (iii) are satisfied. Then the functional integral equation (2.7) has at least one solution* $y \in L_1[0,1]$.

Proof. Define the operator H by

$$Hy(t) = f(t, \{A(\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\ ds - \alpha \sum_{j=1}^{n} b_j \int_0^{\eta_j} y(s)\ ds) + \int_0^t y(s)\ ds\},\ y(t)). \tag{2.8}$$

Let

$$B_r = \{y :\ ||y||_{L_1} \leq r,\ r > 0\},$$

$$r = ||a||(1 - (Ab_1(\sum_{k=1}^{m} a_k + \alpha \sum_{j=1}^{n} b_j) + b_1 + b_2))^{-1}.$$

Clearly B_r is nonempty, convex and closed .

Let $y \in B_r$, then from assumptions (i) and (iii), we obtain

$$||Hy||_{L_1} = \int_0^1 |f(t, \{A(\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)ds - \alpha \sum_{j=1}^{n} b_j \int_0^{\eta_j} y(s)ds) + \int_0^t y(s)ds\}, y(t))|dt$$

$$\leq \int_0^1 (|a(t)| + b_1|A\sum_{k=1}^m a_k \int_0^{\tau_k} y(s)ds - A\alpha\sum_{j=1}^n b_j \int_0^{\eta_j} y(s)ds + \int_0^t y(s)ds| + b_2|y(t)|)dt$$

$$\leq \int_0^1 (|a(t)| + Ab_1\sum_{k=1}^m a_k \int_0^{\tau_k} |y(s)| + Ab_1\alpha\sum_{j=1}^n \int_0^{\eta_j} |y(s)|ds + b_1\int_0^t |y(s)|ds + b_2|y(t)|)dt$$

$$\leq ||a|| + Ab_1\sum_{k=1}^m a_k||y|| + Ab_1\alpha\sum_{j=1}^n b_j||y|| + b_1||y|| + b_2||y||$$

$$\leq ||a|| + (Ab_1\sum_{k=1}^m a_k + Ab_1\alpha\sum_{j=1}^n b_j + b_1 + b_2)||y||$$

$$\leq ||a|| + (Ab_1(\sum_{k=1}^m a_k + \alpha\sum_{j=1}^n b_j) + b_1 + b_2)||y||$$

$$\leq ||a|| + (Ab_1(\sum_{k=1}^m a_k + \alpha\sum_{j=1}^n b_j) + b_1 + b_2)r \leq r.$$

Then $||Hy||_{L_1} \leq r$, which implies that the operator H maps B_r into itself, *i.e* $H : B_r \to B_r$.
Assumption (ii) implies $f \in L_1 \to L_1$ and assumption (i) implies that H is continuous. To apply Schauder fixed point theorem it remains to show that H is compact Now, let Ω be a bounded subset of B_r , therefore $H(\Omega)$ is bounded in $L_1[0, 1]$, *i.e* condition (i) of Kolmogorov compactness criterion is satisfied, it remains to show that

$$(Hy)_h \to (Hy), \quad \text{in } L_1[0, 1].$$

Let $y \in \Omega \subset L_1[0, 1]$, we have the following estimation

$$||(Hy)_h - (Hy)||_{L_1} = \int_0^1 |(Hy)_h(t) - (Hy)(t)|\ dt$$

$$= \int_0^1 |\frac{1}{h}\int_t^{t+h} (Hy)(s)\ ds - (Hy)(t)|\ dt \leq \int_0^1 (\frac{1}{h}\int_t^{t+h} |(Hy)(s) - (Hy)(t)|\ ds)\ dt$$

$$\leq \int_0^1 \frac{1}{h}\int_t^{t+h} |f(s, A\sum_{k=1}^m a_k \int_0^{\tau_k} y(\tau)d\tau - \alpha A\sum_{j=1}^n b_j \int_0^{\eta_j} y(\tau)d\tau + \int_0^s y(\tau)d\tau,\ y(s))$$

$$- f(t, A\sum_{k=1}^m a_k \int_0^{\tau_k} y(s)ds - \alpha A\sum_{j=1}^n b_j \int_0^{\eta_j} y(s)ds + \int_0^t y(s)ds,\ y(t))|\ ds\ dt.$$

Now $f \in L_1 \to L_1$ and $y \in \Omega \subset L_1$ implies that $f \in L_1[0, 1]$ and

$$\frac{1}{h}\int_t^{t+h} |f(s, A\sum_{k=1}^m a_k \int_0^{\tau_k} y(\tau)d\tau$$

$$- \alpha A\sum_{j=1}^n b_j \int_0^{\eta_j} y(\tau)d\tau + \int_0^s y(\tau)d\tau,\ y(s)) - f(t, A\sum_{k=1}^m a_k \int_0^{\tau_k} y(s)ds$$

$$- \alpha A \sum_{j=1}^{n} b_j \int_0^{\eta_j} y(s)ds + \int_0^t y(s)ds,\ y(t))|\ ds \to 0\ as\ h \to 0,\ for\ t \in [0,1].$$

Therefore $(Hy)_h \to (Hy)$, uniformly as $h \to 0$. Then by Kolmogorov compactness criterion, $H(\Omega)$ is relatively compact. Hence H has a fixed point in B_r, then there exists at least one solution $y \in L_1[0,1]$ such that $y(t) = f(t, x(t), y(t))$, $t \in [0,1]$. □

Theorem 2.2. *Let the assumptions (i) - (iii) are satisfied. Then the nonlocal problem (1)-(2) has at least one an absolutely continuous solution* $x \in AC[0,1]$.

Proof. Form Theorem 2.1 and the integral equation (2.6) we deduce that there exists at least one absolutely continuous solution $x \in AC(0,1)$ of the integral equation (2.6). Therefore the integral equation (2.6) has at least one absolutely continuous solution $x \in AC(0,1)$. Now,

$$\lim_{t\to 0} x(t) = A(\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\ ds - \alpha \sum_{j=1}^{n} b_j \int_0^{\eta_j} y(s)\ ds) + \lim_{t\to 0} \int_0^t y(s)\ ds = x(0),$$

$$\lim_{t\to 1} x(t) = A(\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\ ds - \alpha \sum_{j=1}^{n} b_j \int_0^{\eta_j} y(s)\ ds) + \int_0^1 y(s)\ ds = x(1).$$

Then the integral equation (2.6) has at least one an absolutely continuous solution $x \in AC[0,1]$.

To complete the proof, we prove that the integral equation (2.6) satisfies nonlocal problem (1)-(2). Differentiating (2.6), we get

$$\frac{dx}{dt} = y(t) = f(t, x(t), \frac{dx}{dt}).$$

Let $t = \tau_k$ in (2.6) , we obtain

$$x(\tau_k) = A(\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\ ds - \alpha \sum_{j=1}^{n} b_j \int_0^{\eta_j} y(s)\ ds) + \int_0^{\tau_k} y(s)\ ds$$

$$= (A \sum_{k=1}^{m} a_k + 1) \int_0^{\tau_k} y(s)\ ds - \alpha a \sum_{j=1}^{n} b_j \int_0^{\eta_j} y(s)\ ds.$$

Then

$$\sum_{k=1}^{m} a_k x(\tau_k) = \sum_{k=1}^{m} a_k (A \sum_{k=1}^{m} a_k + 1) \int_0^{\tau_k} y(s)ds - \alpha A \sum_{k=1}^{m} a_k \sum_{j=1}^{n} b_j \int_0^{\eta_j} y(s)ds. \quad (2.9)$$

Also, let $t = \eta_j$ in (2.6), we obtain

$$x(\eta_j) = A(\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\ ds - \alpha \sum_{j=1}^{n} b_j \int_0^{\eta_j} y(s)\ ds) + \int_0^{\eta_j} y(s)\ ds$$

$$= A \sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\ ds + (1 - \alpha A \sum_{j=1}^{n} b_j) \int_0^{\eta_j} y(s)\ ds$$

and

$$\alpha\sum_{j=1}^{n} b_j x(\eta_j) = \alpha A\sum_{j=1}^{n} b_j \sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)ds + \alpha\sum_{j=1}^{n} b_j (1-\alpha A\sum_{j=1}^{n} b_j)\int_0^{\eta_j} y(s)ds. \tag{2.10}$$

Subtraction (2.10) from (2.9), we obtain

$$\sum_{k=1}^{m} a_k x(\tau_k) - \alpha\sum_{j=1}^{n} b_j x(\eta_j) = 0.$$

This completes the proof of the equivalent between the nonlocal problem (1)-(2) and the integral equation (2.6). This implies that there exists at least one absolutely continuous solution $x \in AC[0,1]$ of the nonlocal problem (1)-(2). □

Now letting $\alpha = 0$ in (2), then we can easily prove the following corollary .

Corollary 2.1. *Let the assumptions (i) - (iii) are satisfied. Then the nonlocal problem*

$$\frac{dx(t)}{dt} = f(t, x(t), \frac{dx(t)}{dt}), \ \textit{for a.e. } t \in (0,1),$$

$$\sum_{k=1}^{m} a_k x(\tau_k) = 0.$$

has at least one an absolutely continuous solution $x \in AC[0,1]$.

3. Nonlocal integral condition

Let $x \in [0,1]$ be a solution of the nonlocal problem (1)-(2). Let $a_k = t_k - t_{k-1}, \tau_k \in (t_{k-1}, t_k) = (a,c) \subset (0,1)$ and let $b_j = t_j - t_{j-1}, \eta_j \in (t_{j-1}, t_j) = (b,d) \subset (0,1)$, then the nonlocal condition (2) will be

$$\sum_{k=1}^{m} (t_k - t_{k-1})\ x(\tau_k) = \alpha\sum_{j=1}^{n} (t_j - t_{j-1})\ x(\eta_j).$$

From the continuity of the solution x of the nonlocal condition (2) we obtain

$$\lim_{m\to\infty}\sum_{k=1}^{m} (t_k - t_{k-1})\ x(\tau_k) = \lim_{n\to\infty}\alpha\sum_{j=1}^{n} (t_j - t_{j-1})\ x(\eta_j).$$

$i.e$ the nonlocal condition (2) transformed to the integral condition

$$\int_a^c x(s)\ ds = \alpha\int_b^d x(s)\ ds \tag{3.1}$$

and the solution of the integral equation (2.6) will be

$$x(t) = A\ (\int_a^c\int_0^t y(s)\ ds\ dt - \alpha\int_d^b\int_0^t y(s)\ ds\ dt) + \int_0^t y(s)\ ds. \tag{3.2}$$

Now, we have the following corollary.

Corollary 3.1. *Let the assumptions (i)-(iii) of Theorem 2.1 are satisfied. Then the nonlocal problem with the integral condition*

$$\frac{dx(t)}{dt} = f(t, x(t), \frac{dx(t)}{dt}), \ \textit{for a.e. } t \in (0, 1),$$

$$\int_a^c x(s)\ ds = \alpha \int_d^b x(s)\ ds$$

has at least one an absolutely continuous solution $x \in AC[0, 1]$.

Letting $\alpha = 0$ in (2.6), the we can easily prove the following corollary .

Corollary 3.2. *Let the assumptions (i) - (iii) are satisfied. Then the nonlocal problem*

$$\frac{dx(t)}{dt} = f(t, x(t), \frac{dx(t)}{dt}), \ \textit{for a.e. } t \in (0, 1),$$

$$\int_a^c x(s)\ ds = 0.$$

has at least one an absolutely continuous solution $x \in AC[0, 1]$.

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