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ITERATIVE APPROXIMATION OF SOLUTIONS OF GENERALIZED EQUATIONS OF HAMMERSTEIN TYPE

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Abstract. Let H be a real Hilbert space. For each $i = 1, 2, ...m$, let $F_i,\ K_i : H \to H$ be bounded and monotone mappings. Assume that the generalized Hammerstein equation $u + \sum_{i=1}^{m} K_i F_i u = 0$ has a solution in H. We construct a new explicit iterative sequence and prove strong convergence of the sequence to a solution of the generalized Hammerstein equation. Our iterative scheme in this paper seems far simpler than the iterative scheme used by Chidume and Ofoedu [C. E. Chidume, E. U. Ofoedu; Solution of nonlinear integral equations of Hammerstein type, Nonlinear Anal. 74 (2011), 4293-4299] and Chidume and Shehu [C.E. Chidume, Y. Shehu; Approximation of solutions of generalized equations of Hammerstein type, Comp. Math. Appl. 63 (2012), 966-974].
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1. Introduction

Let E be a real normed space and let $S := \{x \in E : \|x\| = 1\}$. E is said to have a *Gâteaux differentiable* norm (and E is called *smooth*) if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in S$; E is said to have a *uniformly Gâteaux differentiable* norm if for each $y \in S$ the limit is attained uniformly for $x \in S$. Further, E is said to be *uniformly smooth* if the limit exists uniformly for $(x, y) \in S \times S$. The *modulus of smoothness* of E is defined by

$$\rho_E(\tau) := sup\Big\{\frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1,\ \|y\| = \tau\Big\};\ \tau > 0.$$

E is equivalently said to be *smooth* if $\rho_E(\tau) > 0$, $\forall\, \tau > 0$. Let $q > 1$, E is said to be q-uniformly smooth (or to have a modulus of smoothness of power type $q > 1$) if there exists $c > 0$ such that $\rho_E(\tau) \le c\tau^q$. Hilbert spaces, $L_p(or\ \ l_p)$ spaces, $1 < p < \infty$, and the Sobolev spaces, W_m^p, $1 < p < \infty$, are q-uniformly smooth. Hilbert spaces are 2-uniformly smooth while

$$L_p(or\ \ell_p) \text{ or } W_m^p \text{ is} \begin{cases} p - \text{uniformly smooth if } 1 < p \leq 2 \\ 2 - \text{uniformly smooth if } p \geq 2. \end{cases}$$

Let E be a real normed space and let J_q, $(q > 1)$ denote the generalized duality mapping from E into 2^{E^*} given by

$$J_q(x) = \{f \in E^* : \langle x, f\rangle = ||x||^q \text{ and } ||f|| = ||x||^{q-1}\},$$

where E^* denotes the dual space of E and $\langle ., .\rangle$ denotes the generalized duality pairing. It is well known (see, for example, Xu [33]) that $J_q(x) = ||x||^{q-2}J(x)$ if $x \neq 0$. For $q = 2$, the mapping $J = J_2$ from E to 2^{E^*} is called normalized duality mapping. It is well known that if E is uniformly smooth, then J is single-valued (see, e.g., [33, 34]). A mapping $A : D(A) \subset E \to E$ is said to be *accretive* if $\forall x, y \in D(A)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y)\rangle \geq 0.$$

For some real number $\eta > 0$, A is called *η-strongly accretive* if $\forall x, y \in D(A)$, there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\langle Gx - Gy, j_q(x - y)\rangle \geq \eta||x - y||^q.$$

In Hilbert spaces, accretive operators are called *monotone*. The accretive operators were introduced independently in 1967 by Browder [1] and Kato [2]. Interest in such mappings stems from their firm connection with equations of evolution. For more on accretive/monotone mappings and connections with evolution equations, the reader may consult any of the following references Berinde [3, 4], Chidume [5], Cioranescu [6], Reich [7].

A nonlinear integral equation of Hammerstein type (see, e.g., Hammerstein [8]) is one of the form

$$u(x) + \int_\Omega k(x, y)f(y, u(y))dy = h(x) \tag{1.1}$$

where dy is a σ-finite measure on the measure space Ω; the real kernel k is defined on $\Omega \times \Omega$, f is a real-valued function defined on $\Omega \times \mathbb{R}$ and is, in general, nonlinear and h is a given function on Ω. If we now define an operator K by

$$Kv(x) = \int_\Omega k(x, y)v(y)dy;\ x \in \Omega,$$

and the so-called *superposition* or *Nemytskii* operator by $Fu(y) := f(y, u(y))$ then, the integral equation (1.1) can be put in the operator theoretic form as follows:

$$u + KFu = 0, \tag{1.2}$$

where, without loss of generality, we have taken $h \equiv 0$.

Interest in equation (1.2) stems mainly from the fact that several problems that arise in differential equations, for instance, elliptic boundary value problems whose linear parts possess Green's functions can, as a rule, be transformed into the form (1.2).

Among these, we mention the problem of the forced oscillations of finite amplitude of a pendulum (see, e.g., Pascali and Sburlan [9], Chapter IV).

Example 1.1. The amplitude of oscillation $v(t)$ is a solution of the problem

$$\begin{cases} \frac{d^2v}{dt^2} + a^2 \sin v(t) = z(t), \ t \in [0,1] \\ v(0) = v(1) = 0, \end{cases} \tag{1.3}$$

where the driving force $z(t)$ is periodical and odd. The constant $a \neq 0$ depends on the length of the pendulum and on gravity. Since the Green's function for the problem

$$v''(t) = 0, \ v(0) = v(1) = 0,$$

is the triangular function

$$k(t,x) = \begin{cases} t(1-x), \ 0 \le t \le x, \\ x(1-t), \ x \le t \le 1, \end{cases}$$

problem (1.3) is equivalent to the nonlinear integral equation

$$v(t) = -\int_0^1 k(t,x)[z(x) - a^2 \sin v(x)]dx. \tag{1.4}$$

If

$$\int_0^1 k(t,x)z(x)dx = g(t) \text{ and } v(t) + g(t) = u(t),$$

then (1.4) can be written as the Hammerstein equation

$$u(t) + \int_0^1 k(t,x)f(x,u(x))dx = 0,$$

where $f(x,u(x)) = a^2 \sin[u(x) - g(x)]$.
Equations of Hammerstein type play a crucial role in the theory of optimal control systems and in automation and network theory (see, e.g., Dolezale [24]). Several existence and uniqueness theorems have been proved for equations of the Hammerstein type (see, e.g., Brezis and Browder [10, 11, 12], Browder [1], Browder and De Figueiredo [14], Browder and Gupta [15], Chepanovich [17], De Figueiredo [23]). The Mann iteration scheme (see, e.g., Mann [30]) has successfully been employed (see, e.g., the recent monographs of Berinde [3] and Chidume [5]). The recurrence formulas used involved K^{-1} which is also assumed to be strongly monotone and this, apart from limiting the class of mappings to which such iterative schemes are applicable, is also not convenient in applications. Part of the difficulty is the fact that the composition of two monotone operators need not be monotone. For more recent results on Hammerstein equations, see, for example, [21, 22, 27, 35] and the references contained therein.
Recently, Chidume and Ofoedu [18] introduced a coupled explicit iterative scheme and proved the following strong convergence theorem for approximation of solution of a nonlinear integral equation of Hammerstein type in a 2-uniformly smooth real Banach space. In particular, they proved the following theorem.

Theorem 1.2. (Chidume and Ofoedu, [18]) Let E be a 2-uniformly smooth real Banach space. Let $F, K : E \to E$ be bounded and accretive mappings. Let $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ be sequences in E defined iteratively from arbitrary $u_1,\ v_1 \in E$ by

$$\begin{cases} u_{n+1} = u_n - \lambda_n\alpha_n(Fu_n - v_n) - \lambda_n\theta_n(u_n - u_1), \\ v_{n+1} = v_n - \lambda_n\alpha_n(Kv_n + u_n) - \lambda_n\theta_n(v_n - v_1), \end{cases} \tag{1.5}$$

where $\{\lambda_n\}_{n=1}^{\infty},\ \{\alpha_n\}_{n=1}^{\infty}$ and $\{\theta_n\}_{n=1}^{\infty}$ are real sequences in $(0,1)$ such that $\lambda_n = o(\theta_n),\ \alpha_n = o(\theta_n)$ and $\sum_{i=1}^{\infty}\lambda_n\theta_n = +\infty$. Suppose that $u + KFu = 0$ has a solution in E. Then, there exist real constants $\varepsilon_0,\ \varepsilon_1 > 0$ and a set $\Omega \subset W = E \times E$ such that if $\alpha_n \le \varepsilon_0\theta_n$ and $\lambda_n \le \varepsilon_1\theta_n, \forall n \ge n_0$, for some $n_0 \in \mathbb{N}$ and $w^* := (u^*, v^*) \in \Omega$ (where $v^* = Fu^*$), the sequence $\{u_n\}_{n=1}^{\infty}$ converges strongly to u^*.

A nonlinear integral equation of Urysohn type (see, e.g., Gupta [25, 26]) has the form

$$u(x) + \int_{\Omega} A(x, y, u(y))dy = h(x), \tag{1.6}$$

where Ω is the domain of a σ-finite measure in $\mathbb{R}$. The inhomogeneous term h and the unknown function u are measurable on Ω. The function $A : \Omega \times \Omega \times \mathbb{R} \to \mathbb{R}$ is called the *Urysohn kernel*. When

$$A(x, y, r) = k(x, y)f(y, r),$$

the Urysohn equation becomes a Hammerstein one.

In this paper, we shall discuss convergence results for Urysohn equations in which the kernels are of the form

$$A(x, y, r) = \sum_{i=1}^{m} k_i(x, y)f_i(y, r),\ \forall x, y \in \Omega, r \in \mathbb{R}. \tag{1.7}$$

The integral equations with these kernels can be regarded as generalized equations of Hammerstein type and integral equation (1.6) with kernel in equation (1.7) can be put in operator theoretic form:

$$u + \sum_{i=1}^{m} K_iF_iu = 0, \tag{1.8}$$

where, without loss of generality, we have taken $h \equiv 0$. The study of equation (1.8) was initiated by Browder [13] and further developed by Joshi [28] and Gupta [25, 26]. More recently, Chidume and Shehu [19] generalized the results of Chidume and Ofoedu [18] and proved the following strong convergence theorem for approximation of solution of generalized equation of Hammerstein equation in real Hilbert spaces.

Theorem 1.3. Let H be a real Hilbert space. For each $i = 1, 2, ...m$, let $F_i,\ K_i : H \to H$ be bounded and monotone mappings. Let $\{u_n\}_{n=1}^{\infty},\ \{v_{i,n}\}_{n=1}^{\infty},\ i = 1, 2, \dots, m$ be sequences in H defined iteratively by

$$\begin{cases} u_{n+1} = u_n - \lambda_n\alpha_n\Big(u_n + \sum_{i=1}^m K_i v_{i,n}\Big) - \lambda_n\theta_n(u_n - u_1),\\ v_{1,n+1} = v_{1,n} - \lambda_n\alpha_n(F_1u_n - v_{1,n}) - \lambda_n\theta_n(v_{1,n} - v_{1,1}),\\ v_{2,n+1} = v_{2,n} - \lambda_n\alpha_n(F_2u_n - v_{2,n}) - \lambda_n\theta_n(v_{2,n} - v_{2,1}),\\ \vdots\\ v_{m,n+1} = v_{m,n} - \lambda_n\alpha_n(F_mu_n - v_{m,n}) - \lambda_n\theta_n(v_{m,n} - v_{m,1}), \end{cases}$$

where $\{\lambda_n\}_{n=1}^\infty$, $\{\alpha_n\}_{n=1}^\infty$ and $\{\theta_n\}_{n=1}^\infty$ are real sequences in $(0,1)$ such that $\lambda_n = o(\theta_n)$, $\alpha_n = o(\theta_n)$ and $\sum_{i=1}^\infty \lambda_n\theta_n = +\infty$. Suppose that $u + \sum_{i=1}^m K_iF_iu = 0$ has a solution in H. Then, there exist real constants $\varepsilon_0,\ \varepsilon_1 > 0$ and a set $\Omega \subset W$ such that if $\alpha_n \le \varepsilon_0\theta_n$ and $\lambda_n \le \varepsilon_1\theta_n, \forall n \ge n_0$, for some $n_0 \in \mathbb{N}$ and $w^* := (u^*, x_1^*, x_2^*, \dots, x_m^*) \in \Omega$ (where $x_i^* = F_iu^*,\ i = 1,2,\dots,m$), the sequence $\{u_n\}_{n=1}^\infty$ converges strongly to u^*.
We remark here that conditions imposed on the iteration parameters in Theorem 1.2 and Theorem 1.3 are too strong compared to the conditions we shall imposed on our iteration parameters in Section 3.

It is our purpose in this paper to introduce *a new explicit iteration scheme* which converges strongly to a solution of generalized equations of Hammerstein type $u + \sum_{i=1}^m K_iF_iu = 0$ in *real Hilbert spaces* when $K_i,\ F_i,\ i = 1,2,\dots,m$ are bounded and monotone. Our iterative scheme in this paper seems far simpler than the iterative scheme used by Chidume and Shehu (Theorem 1.3) in real Hilbert spaces. Thus, our results improve the results of Chidume and Shehu (Theorem 1.3) in real Hilbert spaces. Furthermore, our results improve and generalize the results of Chidume and Ofoedu (Theorem 1.2) in real Hilbert spaces.

2. Preliminaries

We shall make use of the following lemmas in the sequel.

Lemma 2.1. *Let H be a real Hilbert space. Then*

$$||x+y||^2 \le ||x||^2 + 2\langle y, x+y\rangle,$$

for all $x, y \in H$.

Lemma 2.2. (see, e.g., [4, 32]) *Let $\{a_n\}_{n=1}^\infty$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \le (1-\alpha_n)a_n + \sigma_n,\ \ n \ge 1,$$

where $\{\alpha_n\}_{n=1}^\infty \subset [0,1]$ and $\{\sigma_n\}_{n=1}^\infty$ is a sequence in $\mathbb{R}$ such that $\sum_{n=1}^\infty \alpha_n = \infty$. Suppose that $\sigma_n = o(\alpha_n)$, $n \ge 1$ (i.e., $\lim_{n\to\infty}\frac{\sigma_n}{\alpha_n} = 0$) or $\sum_{n=1}^\infty |\sigma_n| < +\infty$ or $\limsup_{n\to\infty}\frac{\sigma_n}{\alpha_n} \le 0$. Then, $a_n \to 0$ as $n \to \infty$.

Lemma 2.3. (Shioji and Takahashi, [31]) *Let $(x_0, x_1, x_2, ...) \in l_\infty$ be such that $\mu_n x_n \le 0$ for all Banach limits μ. If $\limsup_{n\to\infty}(x_{n+1} - x_n) \le 0$, then $\limsup_{n\to\infty} x_n \le 0$.*

Lemma 2.4. (Lim and Xu, [29]) *Suppose E is a Banach space with uniform normal structure, K a nonempty bounded subset of E and $T : C \to C$ is a uniformly L-Lipschitzian mapping with $L < N(E)^{\frac{1}{2}}$. Suppose also there exists a nonempty bounded* *closed convex subset B of C with the following property (P):*

$$x \in B \text{ implies } \omega_w(x) \in B,$$

(where $\omega_w(x)$ is the ω-limit set of T at x, that is, the set $\{y \in E : y = \text{ weak } \omega - \lim T^{n_j}x \text{ for some } n_j \to \infty\}$). Then T has a fixed point in B.

3. Main results

Theorem 3.1. *Let H be a real Hilbert space. For each $i = 1, 2, ...m$, let $F_i,\ K_i : H \to H$ be bounded and monotone mappings. Let $\{u_n\}_{n=1}^{\infty},\ \{v_{i,n}\}_{n=1}^{\infty},\ i = 1, 2, \ldots, m$ be sequences in H defined iteratively from arbitrary $u_1,\ v_{i,1} \in H$ by*

$$\begin{cases} u_{n+1} = u_n - \beta_n^2\Big(u_n + \sum_{i=1}^m K_i v_{i,n}\Big) - \beta_n(u_n - u_1), \\ v_{1,n+1} = v_{1,n} - \beta_n^2(F_1u_n - v_{1,n}) - \beta_n(v_{1,n} - v_{1,1}), \\ v_{2,n+1} = v_{2,n} - \beta_n^2(F_2u_n - v_{2,n}) - \beta_n(v_{2,n} - v_{2,1}), \\ \vdots \\ v_{m,n+1} = v_{m,n} - \beta_n^2(F_mu_n - v_{m,n}) - \beta_n(v_{m,n} - v_{m,1}), \end{cases} \tag{3.1}$$

where $\{\beta_n\}_{n=1}^{\infty}$ is a real sequence in $(0,1)$ such that $\lim_{n\to\infty} \beta_n = 0$. Suppose that

$$u + \sum_{i=1}^m K_iF_iu = 0$$

has a solution in H. Then, the sequences $\{u_n\}_{n=1}^{\infty},\ \{v_{i,n}\}_{n=1}^{\infty},\ i = 1, 2, \ldots, m$ are bounded.

Proof. We exploit the method used in Chidume and Shehu [19]. Let

$$W := H \times H \times \ldots \times H$$

be the Cartesian product with $(m + 1)$ factors and the norm

$$||w||_W := \Big(||u||^2 + \sum_{i=1}^m ||x_i||^2\Big)^{\frac{1}{2}}.$$

Define the sequence $\{w_n\}_{n=1}^{\infty}$ in W by $w_n := (u_n, v_{1,n}, v_{2,n}, \ldots, v_{m,n})$. Let u^* be a solution of $u + \sum_{i=1}^m K_iF_iu = 0,\ x_1^* = F_1u^*,\ x_2^* = F_2u^*, \ldots, x_m^* = F_mu^*$ and $w^* = (u^*, x_1^*, x_2^*, \ldots, x_m^*)$. We observe that $u^* = -\sum_{i=1}^m K_ix_i^*$. It suffices to show that $\{w_n\}_{n=1}^{\infty}$ is bounded. For this, let $n_0 \in \mathbb{N}$, then there exists $r > 0$ sufficiently large such that $w_1 \in B(w^*, \frac{r}{2}),\ w_{n_0} \in B(w^*, r)$. Define $B := \overline{B(w^*, r)}$. Since $F_i,\ K_i$ are bounded, then

$$M_0 = \sup\Big\{\Big|\Big|u + \sum_{i=1}^m K_ix_i\Big|\Big|_H^2 + 2r^2 : (u, x_1, x_2, \ldots, x_m) \in B\Big\} < +\infty$$

$$M_i = \sup\{||F_iu - x_i||_H^2 + 2r^2 : (u, x_1, x_2, \ldots, x_m) \in B\} < +\infty,\ i = 1, 2, \ldots, m.$$

Let $M = \sum_{i=0}^{m} M_i$. We show that $w_n \in B$ for all $n > n_0$. We do this by induction. By construction, $w_{n_0} \in B$. Suppose $w_n \in B$ for $n > n_0$. We prove that $w_{n+1} \in B$. Observe that

$$||w_{n+1} - w^*||^2 = ||u_{n+1} - u^*||^2 + \sum_{i=1}^{m} ||v_{i,n+1} - x_i^*||^2.$$

Then, we obtain

$$\begin{aligned}
||u_{n+1} - u^*||^2 &= ||u_n - u^* - \beta_n^2\Big(u_n + \sum_{i=1}^{m} K_i v_{i,n}\Big) - \beta_n(u_n - u_1)||^2 \\
&= ||u_n - u^*||^2 - 2\Big\langle \beta_n^2\Big(u_n + \sum_{i=1}^{m} K_i v_{i,n}\Big) + \beta_n(u_n - u_1), u_n - u^*\Big\rangle \\
&\quad + \Big|\Big|\beta_n^2\Big(u_n + \sum_{i=1}^{m} K_i v_{i,n}\Big) + \beta_n(u_n - u_1)\Big|\Big|^2 \\
&\leq ||u_n - u^*||^2 - 2\beta_n^2\Big\langle u_n + \sum_{i=1}^{m} K_i v_{i,n}, u_n - u^*\Big\rangle \\
&\quad - 2\beta_n\langle u_n - u_1, u_n - u^*\rangle + 4\beta_n^2 M_0. \qquad (3.2)
\end{aligned}$$

But since K_i is monotone, we obtain

$$\begin{aligned}
&\Big\langle u_n + \sum_{i=1}^{m} K_i v_{i,n}, u_n - u^*\Big\rangle = \Big\langle u_n + \sum_{i=1}^{m} K_i v_{i,n} - u^* - \sum_{i=1}^{m} K_i x_i^*, u_n - u^*\Big\rangle \\
&= ||u_n - u^*||^2 + \langle K_1 v_{1,n} - K_1 x_1^*, (u_n - u^*) + (v_{1,n} - x_1^*) - (v_{1,n} - x_1^*)\rangle \\
&\quad + \langle K_2 v_{2,n} - K_2 x_2^*, (u_n - u^*) + (v_{2,n} - x_2^*) - (v_{2,n} - x_2^*)\rangle \\
&\quad + \ldots + \langle K_m v_{m,n} - K_m x_m^*, (u_n - u^*) + (v_{m,n} - x_m^*) - (v_{m,n} - x_m^*)\rangle \\
&\geq ||u_n - u^*||^2 + \langle K_1 v_{1,n} - K_1 x_1^*, (u_n - u^*) - (v_{1,n} - x_1^*)\rangle \\
&\quad + \langle K_2 v_{2,n} - K_2 x_2^*, (u_n - u^*) - (v_{2,n} - x_2^*)\rangle \\
&\quad + \ldots + \langle K_m v_{m,n} - K_m x_m^*, (u_n - u^*) - (v_{m,n} - x_m^*)\rangle
\end{aligned}$$

and

$$\langle u_n - u_1, u_n - u^*\rangle = ||u_n - u^*||^2 + \langle u^* - u_1, u_n - u^*\rangle.$$

Thus, substituting in (3.2), we obtain

$$\begin{aligned}
||u_{n+1} - u^*||^2 &\leq (1 - 2\beta_n)||u_n - u^*||^2 + 2\beta_n^2 A_0 \\
&\quad - 2\beta_n\langle u^* - u_1, u_n - u^*\rangle + 4\beta_n^2 M_0 \qquad (3.3)
\end{aligned}$$

where

$$\begin{aligned}
A_0 = \sup_{n \geq n_0} |&\langle K_1 v_{1,n} - K_1 x_1^*, (u_n - u^*) - (v_{1,n} - x_1^*)\rangle \\
&+ \langle K_2 v_{2,n} - K_2 x_2^*, (u_n - u^*) - (v_{2,n} - x_2^*)\rangle \\
&+ \ldots + \langle K_m v_{m,n} - K_m x_m^*, (u_n - u^*) - (v_{m,n} - x_m^*)\rangle|.
\end{aligned}$$

Also,

$$\begin{aligned}||v_{1,n+1}-x_1^*||^2 &= ||v_{1,n}-x_1^*-\beta_n^2(F_1u_n-v_{1,n})-\beta_n(v_{1,n}-v_{1,1})||^2\\ &= ||v_{1,n}-x_1^*||^2-2\Big\langle\beta_n^2(F_1u_n-v_{1,n})+\beta_n(v_{1,n}-v_{1,1}),v_{1,n}-x_1^*\Big\rangle\\ &\quad +\beta_n^2||\beta_n(F_1u_n-v_{1,n})+(v_{1,n}-v_{1,1})||^2\\ &\leq ||v_{1,n}-x_1^*||^2-2\beta_n^2\langle F_1u_n-v_{1,n},v_{1,n}-x_1^*\rangle\\ &\quad -2\beta_n||v_{1,n}-x_1^*||^2-2\beta_n\langle x_1^*-v_{1,1},v_{1,n}-x_1^*\rangle+4\beta_n^2M_1. \qquad (3.4)\end{aligned}$$

Observe that

$$\langle F_1u_n-v_{1,n},v_{1,n}-x_1^*\rangle=\langle F_1u_n-F_1v_{1,n}+F_1v_{1,n}-F_1x_1^*+F_1x_1^*-v_{1,n},v_{1,n}-x_1^*\rangle$$

$$=\langle F_1v_{1,n}-F_1x_1^*,v_{1,n}-x_1^*\rangle+\langle F_1u_n-F_1v_{1,n}+F_1x_1^*-v_{1,n},v_{1,n}-x_1^*\rangle$$

$$\geq\langle F_1u_n-F_1v_{1,n}+F_1x_1^*-v_{1,n},v_{1,n}-x_1^*\rangle.$$

Substituting this into (3.4), we have

$$\begin{aligned}||v_{1,n+1}-x_1^*||^2 &\leq (1-2\beta_n)||v_{1,n}-x_1^*||^2+2\beta_n^2A_1\\ &\quad -2\beta_n\langle x_1^*-v_{1,1},v_{1,n}-x_1^*\rangle+4\beta_n^2M_1 \qquad (3.5)\end{aligned}$$

where $A_1=\sup_{n\geq n_0}|\langle F_1u_n-F_1v_{1,n}+F_1x_1^*-v_{1,n},v_{1,n}-x_1^*\rangle|$. Continuing, we have for each $i=2,3,\ldots,m$ that

$$\begin{aligned}||v_{i,n+1}-x_1^*||^2 &\leq (1-2\beta_n)||v_{i,n}-x_i^*||^2+2\beta_n^2A_i\\ &\quad -2\beta_n\langle x_i^*-v_{i,1},v_{i,n}-x_i^*\rangle+4\beta_n^2M_i \qquad (3.6)\end{aligned}$$

where $A_i=\sup_{n\geq n_0}|\langle F_iu_n-F_iv_{1,n}+F_ix_i^*-v_{1,n},v_{i,n}-x_i^*\rangle|$. Let $A=\sum_{i=0}^m A_i$. Then, from (3.3), (3.5) and (3.6), we obtain

$$\begin{aligned}||w_{n+1}-w^*||^2 &\leq (1-2\beta_n)||w_n-w^*||^2+2\beta_n^2A\\ &\quad -2\beta_n\Big[\langle u^*-u_1,u_n-u^*\rangle\\ &\quad +\sum_{i=1}^m\langle x_i^*-v_{i,1},v_{i,n}-x_i^*\rangle\Big]+4\beta_n^2M \qquad (3.7)\\ &\leq (1-2\beta_n)||w_n-w^*||^2+2\beta_n^2A+4\beta_n^2M\\ &\quad +\beta_n\Big[||w_1-w^*||^2+||w_n-w^*||^2\Big]\\ &= (1-\beta_n)||w_n-w^*||^2+\beta_n||w_1-w^*||^2+2\beta_n^2A+4\beta_n^2M\\ &< (1-\beta_n)r^2+\beta_n\frac{r^2}{4}+2\beta_n^2A+4\beta_n^2M\end{aligned}$$

Since $\lim_{n\to\infty}\beta_n = 0$, then $\beta_n < \min\left\{\frac{r^2}{8(A+1)}, \frac{r^2}{16(M+1)}\right\}$, $\forall n \geq n_0$. Then, we obtain

$$\begin{aligned} ||w_{n+1} - w^*||^2 &< (1-\beta_n)r^2 + \beta_n\frac{r^2}{4} + 2\beta_n^2 A + 4\beta_n^2 M \\ &< (1-\beta_n)r^2 + \frac{\beta_n r^2}{4} + \frac{\beta_n A r^2}{4(A+1)} + \frac{\beta_n r^2 M}{4(M+1)} \\ &\leq (1-\beta_n)r^2 + \frac{3\beta_n r^2}{4} \\ &= r^2 - \frac{\beta_n r^2}{4} < r^2. \end{aligned}$$

Hence, $w_{n+1} \in B$.

Thus, by induction, $\{w_n\}_{n=1}^{\infty}$ is bounded and so are $\{u_n\}_{n=1}^{\infty}$, $\{v_{i,n}\}_{n=1}^{\infty}$, $i = 1, 2, \ldots, m$. This completes the proof.

Theorem 3.2. *Let H be a real Hilbert space. For each $i = 1, 2, ...m$, let F_i, $K_i : H \to H$ be bounded and monotone mappings. Let $\{u_n\}_{n=1}^{\infty}$, $\{v_{i,n}\}_{n=1}^{\infty}$, $i = 1, 2, \ldots, m$ be sequences in H defined iteratively by (3.1), where $\{\beta_n\}_{n=1}^{\infty}$ is a real sequence in $(0,1)$ such that $\lim_{n\to\infty}\beta_n = 0$ and $\sum_{n=1}^{\infty}\beta_n = +\infty$. Suppose that $u + \sum_{i=1}^{m} K_i F_i u = 0$ has a solution in H. Let $W := H \times H \times \ldots \times H$ be the Cartesian product with $(m+1)$ factors. Then, there exists a set $\Omega \subset W$ (defined below) such that if $w^* := (u^*, x_1^*, x_2^*, \ldots, x_m^*) \in \Omega$ (where $x_i^* = F_i u^*$, $i = 1, 2, \ldots, m$), then sequence $\{u_n\}_{n=1}^{\infty}$ converges strongly to u^*.*

Proof. Since, by Theorem 3.1, we have that $\{u_n\}_{n=1}^{\infty}$, $\{v_{i,n}\}_{n=1}^{\infty}$, $i = 1, 2, \ldots, m$ are bounded, there exists $R > 0$ sufficiently large such that $u_n \in \overline{B_H(u^*, R)}$, $v_{i,n} \in \overline{B_H(x_i^*, R)}$, $\forall n \geq 1$, $i = 1, 2, \ldots, m$. Furthermore, the sets $\overline{B_H(u^*, R)}$ and $\overline{B_H(x_i^*, R)}$ are bounded, closed, convex and nonempty subsets of H. For each $i = 0, 1, \ldots, m$, define the maps $\varphi_i : H \to \mathbb{R}$ by

$$\varphi_0(x) := \mu_n ||u_n - x||_H^2, \ \varphi_i(y) := \mu_n ||v_{i,n} - y||_H^2, \ i = 1, \ldots, m$$

(where μ is a Banach limit). Then for each $i = 0, 1, \ldots, m$, φ_i is continuous, convex and coercive. Since H is reflexive, there exist $x^* \in \overline{B_H(u^*, R)}$ and $y_i^* \in \overline{B_H(x_i^*, R)}(i = 1, \ldots, m)$ such that

$$\varphi_0(x^*) = \min\{\varphi_0(x) : x \in \overline{B_H(u^*, R)}\}$$

and

$$\varphi_i(y_i^*) = \min\{\varphi_i(y) : y \in \overline{B_H(x_i^*, R)}\}.$$

So, the sets

$$\Omega_0 := \left\{u \in \overline{B_H(u^*, R)} : \varphi_0(u) = \min_{x \in \overline{B_H(u^*, R)}} \varphi_0(x)\right\}$$

and

$$\Omega_i := \left\{x_i \in \overline{B_H(x_i^*, R)} : \varphi_i(x_i) = \min_{y \in \overline{B_H(x_i^*, R)}} \varphi_i(y)\right\}$$

are nonempty sets.

Let $t \in (0,1)$, $\Omega := \Omega_0 \times \Omega_1 \times \ldots \times \Omega_m$ and let $w^* := (u^*, x_1^*, x_2^*, \ldots, x_m^*) \in \Omega$. Then, by convexity of $\overline{B_H(u^*,R)}$, we have that $(1-t)u^* + tu_1 \in \overline{B_H(u^*,R)}$. Thus, $\mu_n||u_n - u^*||^2 \leq \mu_n||u_n - (1-t)u^* - tu_1||^2$. Moreover, we have, by Lemma 2.1 that

$$||u_n - u^* - t(u_1 - u^*)||^2 \leq ||u_n - u^*||^2 - 2t\langle u_1 - u^*, u_n - u^* - t(u_1 - u^*)\rangle.$$

This implies that $\mu_n\langle u_1 - u^*, u_n - u^* - t(u_1 - u^*)\rangle \leq 0$. Furthermore, we obtain that

$$\lim_{t\to 0}\Big(\langle u_1 - u^*, u_n - u^*\rangle - \langle u_1 - u^*, u_n - u^* - t(u_1 - u^*)\rangle\Big) = 0.$$

Thus, given $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that for all $t \in (0, \delta_\epsilon)$ and $\forall n \in \mathbb{N}$,

$$\langle u_1 - u^*, u_n - u^*\rangle < \epsilon + \langle u_1 - u^*, u_n - u^* - t(u_1 - u^*)\rangle.$$

Taking Banach limit on both sides of this inequality, we obtain

$$\mu_n\langle u_1 - u^*, u_n - u^*\rangle \leq \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have

$$\mu_n\langle u_1 - u^*, u_n - u^*\rangle \leq 0.$$

Furthermore, since $\{u_n\}_{n=1}^\infty$, $\{v_{i,n}\}_{n=1}^\infty$, F_i, K_i, $i = 1, 2, \ldots, m$ are all bounded, we have from (3.1) that

$$||u_{n+1} - u_n|| \leq \beta_n\Big[\beta_n(||u_n|| + \sum_{i=1}^m ||K_i v_{i,n}||) + ||u_n - u_1||\Big] \leq \beta_n K_0,$$

for some constant $K_0 > 0$. Thus, $\lim_{n\to\infty}||u_{n+1} - u_n|| = 0$. Again, we have

$$\lim_{n\to\infty}\Big(\langle u_1 - u^*, u_{n+1} - u^*\rangle - \langle u_1 - u^*, u_n - u^*\rangle\Big) = 0.$$

Thus, the sequence $\{\langle u_1 - u^*, u_n - u^*\rangle\}$ satisfies the conditions of Lemma 2.3. Hence, we obtain that

$$\limsup_{n\to\infty}\langle u_1 - u^*, u_n - u^*\rangle \leq 0.$$

Following the same line of arguments, we obtain

$$\limsup_{n\to\infty}\langle v_{i,1} - x_i^*, v_{i,n} - x_i^*\rangle \leq 0,\ i = 1, 2, \ldots, m.$$

Now, define

$\sigma_n := \max\{\langle u_1 - u^*, u_n - u^*\rangle, 0\}$ and $\xi_{i,n} := \max\{\langle v_{i,1} - x_i^*, v_{i,n} - x_i^*\rangle, 0\}$, $(i = 1, \ldots, m)$

then, $\lim_{n\to\infty}\sigma_n = 0 = \lim_{n\to\infty}\xi_{i,n}$, $i = 1, 2, \ldots, m$. Furthermore,

$$\langle u_1 - u^*, u_n - u^*\rangle \leq \sigma_n$$

$$\langle v_{i,1} - x_i^*, v_{i,n} - x_i^*\rangle \leq \xi_{i,n},\ i = 1, 2, \ldots, m.$$

From (3.7), we have

$$\begin{aligned}||w_{n+1}-w^*||^2 &\le (1-2\beta_n)||w_n-w^*||^2+2\beta_n^2A\\ &\quad -2\beta_n\Big[\langle u^*-u_1,u_n-u^*\rangle+\sum_{i=1}^m\langle x_i^*-v_{i,1},v_{i,n}-x_i^*\rangle\Big]+4\beta_n^2M\\ &\le (1-2\beta_n)||w_n-w^*||^2+2\beta_n^2A+2\beta_n\Big(\sigma_n+\sum_{i=1}^m\xi_{i,n}\Big)+4\beta_n^2M\\ &= (1-2\beta_n)||w_n-w^*||^2+2\beta_n\gamma_n,\end{aligned}$$

where $\gamma_n=\beta_nA+\Big(\sigma_n+\sum_{i=1}^m\xi_{i,n}\Big)+2\beta_nM$. Hence, by Lemma 2.2, we have that $w_n\to w^*$ as $n\to\infty$. But $w_n=(u_n,v_{1,n},v_{2,n},\dots,v_{m,n})$ and $w^*=(u^*,x_1^*,x_2^*,\dots,x_m^*)$. This implies that $u_n\to u^*$. This completes the proof.

Definition 3.3. Let E be a real linear space. A mapping $T:D(T)\subset E\to E$ is said to be *generalized Lipschitz* if there exists $L>0$ such that

$$||Tx-Ty||\le L(1+||x-y||)\ \forall x,y\in D(T).$$

Clearly, every Lipschitz map is generalized Lipschitz. Furthermore, any map with bounded range is a generalized Lischitz map. The following example (see, e.g., [16]) shows that the class of generalized Lipschitz maps properly includes the class of Lipschitz maps and that of mappings with bounded range.

Example 3.4. Let $E=(-\infty,+\infty)$ and $T:E\to E$ be defined by

$$Tx=\begin{cases}x-1,\ \ x\in(-\infty,-1),\\ x-\sqrt{1-(x+1)^2},\ \ x\in[-1,0),\\ x+\sqrt{1-(x-1)^2},\ \ x\in[0,1],\\ x+1,\ \ x\in(1,+\infty).\end{cases}$$

Then, T is a generalized Lipschitz map which is not Lipschitz and whose range is not bounded.
Clearly, every generalized Lipschitz map is bounded. So, we obtain the following corollaries.

Corollary 3.5. *Let H be a real Hilbert space. For each $i=1,2,...m$, let $F_i,\ K_i:H\to H$ be generalized Lipschitz and monotone mappings. Let $\{u_n\}_{n=1}^\infty$, $\{v_{i,n}\}_{n=1}^\infty$, $i=1,2,\dots,m$ be sequences in H defined iteratively by (3.1), where $\{\beta_n\}_{n=1}^\infty$ is a real sequence in $(0,1)$ such that $\lim_{n\to\infty}\beta_n=0$ and $\sum_{n=1}^\infty\beta_n=+\infty$. Suppose that*

$$u+\sum_{i=1}^m K_iF_iu=0$$

has a solution in H and $w^*:=(u^*,x_1^*,x_2^*,\dots,x_m^*)\in\Omega$ (where $x_i^*=F_iu^*$, $i=1,2,\dots,m$) with Ω as defined in the proof of Theorem 3.2, the sequence $\{u_n\}_{n=1}^\infty$ converges strongly to u^*.

Corollary 3.6. *Let H be a real Hilbert space. For each $i = 1, 2, ...m$, let $F_i : H \to H$ be generalized Lipschitz, monotone mapping and $K_i : H \to H$ bounded, monotone mapping. Let $\{u_n\}_{n=1}^{\infty}$, $\{v_{i,n}\}_{n=1}^{\infty}$, $i = 1, 2, \ldots, m$ be sequences in H defined iteratively by* (3.1)*, where $\{\beta_n\}_{n=1}^{\infty}$ is a real sequence in* $(0, 1)$ *such that $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = +\infty$. Suppose that*

$$u + \sum_{i=1}^{m} K_i F_i u = 0$$

has a solution in H and $w^* := (u^*, x_1^*, x_2^*, \ldots, x_m^*) \in \Omega$ (where $x_i^* = F_i u^*$, $i = 1, 2, \ldots, m$) with Ω as defined in the proof of Theorem 3.2, the sequence $\{u_n\}_{n=1}^{\infty}$ converges strongly to u^*.

We make the following remark concerning how the condition "there exists a set $\Omega \subset W$ (defined below) such that if $w^* := (u^*, x_1^*, x_2^*, \ldots, x_m^*) \in \Omega$ (where $x_i^* = F_i u^*$, $i = 1, 2, \ldots, m$) " made in Theorem 3.2 can be checked.

Remark 3.7. Let $m = 1$. Let H be a real Hilbert space and $F, K : H \to H$ be monotone mappings with $D(F) = H = D(K)$. Let $W := H \times H$ and $A : W \to W$ be a mapping defined by

$$Aw = (Fu - v, Kv + u), \ \forall w = (u, v) \in W.$$

Then by the results of Chidume and Shehu [20], A is monotone. Suppose $u + KFu = 0$ in H and $A : H \times H \to H \times H$ is defined by $Aw = (Fu - v, Kv + u)$, $\forall w = (u, v) \in H \times H$. Observe that u^* in H is a solution of $u + KFu = 0$ if and only if $w^* = (u^*, v^*)$ is a solution of $Aw = 0$ in $H \times H$ for $v^* = Fu^*$. Let $A^{-1}(0) := \{w^* \in W = H \times H : Aw^* = 0\}$.

Let $I - A$ be a nonexpansive mapping on $W = H \times H$. Then $\Omega = \Omega_0 \times \Omega_1$ is a nonempty, closed and convex subset of $W = H \times H$ (which is a Hilbert space) that has property (P). By Lemma 2.4, it follows (since every nonexpansive mapping is asymptotically nonexpansive with $k_n \equiv 1 < N(W)^{\frac{1}{2}} = 2^{\frac{1}{4}}, \forall n \geq 1$) that $\Omega \cap A^{-1}(0) \neq \emptyset$.

Thus, if $I - A$ is a nonexpansive mapping on $W = H \times H$, then $\Omega \cap A^{-1}(0) \neq \emptyset$.

Remark 3.8. It is easy to see that the iterative scheme studied in this paper seems far simpler than the iterative scheme used by Chidume and Shehu [19] in the sense that the conditions $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = +\infty$ on our iteration parameter are natural and not too strong compared to the conditions imposed on iteration parameters by these authors. Prototype for our iteration parameter is, for example, $\beta_n = \frac{1}{n+1}$, $n \geq 1$.

Remark 3.9. In Hilbert spaces, the results of Chidume and Shehu [19] generalize the results of Chidume and Ofoedu [18] and our results in this paper improve the results of Chidume and Shehu [19]. Thus, our results in this paper improve and generalize the results of Chidume and Ofoedu [18] in real Hilbert spaces.

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References

[1] F.E. Browder, *Nonlinear mappings of nonexpansive and accretive type in Banach spaces*, Bull. Amer. Math. Soc., **73**(1967), 875-882.

[2] T. Kato, *Nonlinear semigroups and evolution equations*, J. Math. Soc. Japan, **19**(1967), 508-520.

[3] V. Berinde, *Iterative Approximation of Fixed Points*, Springer Verlag Series: Lecture Notes in Mathematics, vol. 1912, 2007.

[4] V. Berinde, *Iterative Approximation of Fixed Points*, Ed. Efemeride, Baia Mare, 2002.

[5] C.E. Chidume, *Geometric Properties of Banach Spaces and Nonlinear Iterations*, Springer Verlag Series: Lecture Notes in Mathematics, 2009.

[6] I. Cioranescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, Kluwer Academic Publ., Dordrecht, 1990.

[7] S. Reich, *Strong convergence theorems for resolvents of accretive operators in Banach spaces*, J. Math. Anal. Appl., **183**(1994), 118-120.

[8] A. Hammerstein, *Nichtlineare integralgleichungen nebst anwendungen*, Acta Math., **54**(1930), 117-176.

[9] D. Pascali, S. Sburlan, *Nonlinear Mappings of Monotone Type*, Ed. Academiei, Bucharest, Romania, 1978.

[10] H. Brezis, F.E. Browder, *Some new results about Hammerstein equations*, Bull. Amer. Math. Soc., **80**(1974), 567-572.

[11] H. Brezis, F.E. Browder, *Existence theorems for nonlinear integral equations of Hammerstein type*, Bull. Amer. Math. Soc., **81**(1975), 73-78.

[12] H. Brezis, F.E. Browder, *Nonlinear integral equations and system of Hammerstein type*, Advances in Math., **18**(1975), 115-147.

[13] F.E. Browder, *Nonlinear functional analysis and nonlinear integral equations of Hammerstein and Urysohn type*, Contributions to Nonlinear Functional Analysis, Academic Press, 1971, 425-500.

[14] F.E. Browder, D.G. Figueiredo, P. Gupta, *Maximal monotone operators and a nonlinear integral equations of Hammerstein type*, Bull. Amer. Math. Soc., **76**(1970), 700-705.

[15] F.E. Browder, P. Gupta, *Monotone operators and nonlinear integral equations of Hammerstein type*, Bull. Amer. Math. Soc., **75**(1969), 1347-1353.

[16] S.S. Chang, Y.J. Cho, H. Zhou, *Iterative Methods for Nonlinear Operator Equations in Banach Spaces,* Nova Science Publishers, Inc,. Huntington, NY, 2002.

[17] R.Sh. Chepanovich, *Nonlinear Hammerstein equations and fixed points*, Publ. Inst. Math., Beograd, N.S., **35**(1984), 119-123.

[18] C.E. Chidume, E.U. Ofoedu, *Solution of nonlinear integral equations of Hammerstein type*, Nonlinear Anal., **74**(2011), 4293-4299.

[19] C.E. Chidume, Y. Shehu, *Approximation of solutions of generalized equations of Hammerstein type*, Comput. Math. Appl., **63**(2012), 966-974.

[20] C.E. Chidume, Y. Shehu, *Strong convergence theorem for approximation of solutions of equations of Hammerstein type*, Nonlinear Anal., **75**(2012), 5664-5671.

[21] C.E. Chidume, N. Djitte, *Approximation of solutions of Hammerstein equations with bounded strongly accretive nonlinear operators*, Nonlinear Anal., **70**(2009), 4071-4078.

[22] C.E. Chidume, N. Djitte, *Iterative approximation of solutions of nonlinear equations of Hammerstein type*, Nonlinear Anal., **70**(2009), 4086-4092.

[23] D.G. De Figueiredo, C.P. Gupta, *On the variational methods for the existence of solutions to nonlinear equations of Hammerstein type*, Bull. Amer. Math. Soc., **40**(1973), 470-476.

[24] V. Dolezale, *Monotone Operators and its Applications in Automation and Network Theory*, Studies in Automation and Control, Elesevier Science Publ., New York, 1979.

[25] C.P. Gupta, *Nonlinear equations of Urysohn's type in a Banach space*, Comm. Math. Univ. Carolinae, **16**(1975), 377-386.

[26] C.P. Gupta, *On a class of nonlinear integral equations of Urysohn's type*, J. Math. Anal. Appl., **58**(1977), 344-360.

[27] G. Infante, P. Pietramala, *Existence and multiplicity of non-negative solutions for systems of pertubed Hammerstein integral equations*, Nonlinear Anal., **71**(2009), 1301-1310.

[28] M. Joshi, *Existence theorem for a generalized Hammerstein type equation*, Comm. Math. Univ. Carolinae, **15**(1974), 283-291.

[29] T.C. Lim, H.K. Xu, *Fixed point theorems for asymptotically nonexpansive mappings*, Nonlinear Anal., **2**(1994), 1345-1355.

[30] W.R. Mann, *Mean value methods in iterations*, Bull. Amer. Math. Soc., **4**(1953), 506-510.

[31] S. Shioji, Takahashi, W., *Strong convergence of approximated sequences for nonexpansive mappings in Banach spaces*, Proc. Amer. Math. Soc., **125**(1997), 3641-3645.

[32] H.K. Xu, *Iterative algorithm for nonlinear operators*, J. London Math. Soc., **66**(2)(2002), 1-17.

[33] H.K. Xu, *Inequality in Banach spaces with applications*, Nonlinear Anal., **16**(1991), 1127-1138.

[34] Z.B. Xu, G.F. Roach, *Characteristic inequalities of uniformly smooth Banach spaces*, J. Math. Anal. Appl., **157**(1991), 189-210.

[35] Z. Yang, D. O'Regan, *Positive solvability of systems of nonlinear Hammerstein integral equations*, J. Math. Anal. Appl., **311**(2005), 600-614.

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