Fixed Point Theory, 15(2014), No. 2, 399-426
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

PROPERTIES OF A CLASS OF APPROXIMATELY SHRINKING OPERATORS AND THEIR APPLICATIONS

ANDRZEJ CEGIELSKI* AND RAFAŁ ZALAS**

*Faculty of Mathematics, Computer Science and Econometrics
University of Zielona Góra, ul. Szafrana 4a, 65-516 Zielona Góra, Poland
E-mail: a.cegielski@wmie.uz.zgora.pl

**Faculty of Mathematics, Computer Science and Econometrics
University of Zielona Góra, ul. Szafrana 4a, 65-516 Zielona Góra, Poland
E-mail: r.zalas@wmie.uz.zgora.pl

Abstract. In this paper we present an application of a class of quasi-nonexpansive operators to iterative methods for solving the following variational inequality problem $VIP(F,C)$: Find $\bar{u} \in C$ such that $\langle F\bar{u}, z-\bar{u}\rangle \geq 0$ for all $z \in C$, where C is a closed and convex subset of a Hilbert space $\mathcal{H}$ and $F : \mathcal{H} \to \mathcal{H}$ is strongly monotone and Lipschitz continuous. A classical method for $VIP(F,C)$ is the gradient projection (GP) method $x^{k+1} = P_C(x^k - \mu F x^k)$ which generates sequences converging to the unique solution of $VIP(F,C)$ if $\mu > 0$ is sufficiently small. Unfortunately, in many optimization problems the GP method cannot be applied, because it requires an explicit computation of $P_C u^k$ in each iteration, where $u^k = x^k - \mu F x^k$. To overcome this disadvantage of the GP method, one can replace the operator P_C employed in the k-th iteration of the method by a quasi-nonexpansive operator T_k and a constant μ by $\lambda_k \geq 0$, $k \geq 0$, satisfying $\bigcap_{k=0}^{\infty} \operatorname{Fix} T_k \supseteq \operatorname{Fix} T$ and $\lim_k \lambda_k = 0$. The new method can be presented equivalently as a so called general hybrid steepest descent (GHSD) method in the form $u^{k+1} = T_k u^k - \lambda_k F T_k u^k$. One should, however, suppose something more on the operators T_k in order to guarantee the convergence of u^k to the solution of $VIP(F,C)$. In this paper we introduce a class of approximately shrinking operators, prove the closedness of this class with respect to compositions and convex combinations and apply the operators from this class to a general hybrid steepest descent method for solving $VIP(F,C)$. We give sufficient conditions for the convergence of the GHSD method as well as present several examples of methods which satisfy these conditions. In particular, we apply the results in the case, when $C = \bigcap_{i=1}^{m} \operatorname{Fix} U_i$ and $U_i : \mathcal{H} \to \mathcal{H}$ are quasi-nonexpansive operators having a common fixed point, $i = 1, 2, ..., m$.
Key Words and Phrases: variational inequality problem, fixed point, quasi-nonexpansive operator, hybrid steepest descent method, string-averaging.
2010 Mathematics Subject Classification: 47H05, 47H09, 47H10, 65K15.

1. Introduction

Let $\mathcal{H}$ be a real Hilbert space equipped with an inner product $\langle \cdot, \cdot \rangle$ and with the corresponding norm $\|\cdot\|$. In this paper we consider the following variational inequality problem (VIP): Given a nonlinear operator $F : \mathcal{H} \to \mathcal{H}$ and a closed convex subset $C \subseteq \mathcal{H}$, find $\bar{u} \in C$ such that

$$\langle F\bar{u}, z - \bar{u}\rangle \geq 0 \text{ for all } z \in C. \tag{1.1}$$

This problem, denoted by $\mathrm{VIP}(F, C)$, is a fundamental problem in optimization theory, because many optimization problems can be translated into VIPs. The VIP problem was intensively studied in the last decades; see, e.g., the two-volume book by Facchinei and Pang [22], the book of Kinderlehrer and Stampacchia [29] and papers by Yamada [38], Yamada and Ogura [39] and Hirstoaga [28]. In the whole paper we suppose that F is Lipschitz continuous and strongly monotone which guarantees the existence and the uniqueness of a solution of $VIP(F, C)$ (see, e.g., [40, Theorem 46.C]). The standard method for $VIP(F, C)$ is the gradient projection (GP) method

$$x^{k+1} = P_C(x^k - \mu F x^k), \tag{1.2}$$

where $P_C : \mathcal{H} \to \mathcal{H}$ denotes the metric projection onto C (see [25]). If μ is sufficiently small, then the operator $P_C(\mathrm{Id} - \mu F)$ is a contraction, consequently, the Banach fixed point theorem yields the convergence of sequences generated by the GP method to the unique solution of $VIP(F, C)$. Denoting $u^k = x^k - \mu F x^k$ one can present the method in an equivalent form

$$u^{k+1} = P_C u^k - \mu F P_C u^k. \tag{1.3}$$

The method can be efficiently applied if the metric projection $P_C u^k$ can be easily computed. Otherwise, the method can be essentially affected, because $P_C u^k$ should be computed in each iteration. Some researchers overcome this obstacle by replacing the metric projection P_C by a nonexpansive or a quasi-nonexpansive operator T with $\operatorname{Fix} T = C$ and the constant $\mu > 0$ by a sequence λ_k with $\lim_k \lambda_k = 0$ (see, e.g., [27, 36, 38, 39]). This leads to the method

$$u^{k+1} = T u^k - \lambda_k F T u^k \tag{1.4}$$

called by Yamada a hybrid steepest descent method (see [38]). Other researchers apply different nonexpansive or quasi-nonexpansive operators T_k with $\operatorname{Fix} T_k \supseteq C$ in particular iterations (see [1, 10, 28, 30]). This leads to the method

$$u^{k+1} = T_k u^k - \lambda_k F T_k u^k \tag{1.5}$$

called in [10] the generalized hybrid descent method.

In this paper we study generalized hybrid descent method (1.5) for solving $\mathrm{VIP}(F, C)$, where T_k are quasi-nonexpansive operators satisfying $\bigcap_{k=0}^{\infty} \operatorname{Fix} T_k \supseteq C$. Roughly spoken, we suppose that the method generates sequences for whose the displacement $\|T_k u^k - u^k\|$ cannot be too small for those u^k which are far from C (see also [39] and [28] for related assumptions). The paper is organized as follows. Section 2 contains preliminaries on nonexpansive, firmly nonexpansive, quasi-nonexpansive as well as demi-closed operators. The properties recalled in Section 2 will be applied in the rest part of the paper. In Sections 3 and 4 we introduce the notion of approximately shrinking (AS) operators, present some basic examples of these operators and give properties of AS operators, in particular, we prove that the family of AS operators is closed under composition and convex combination. These properties are applied in Section 5, where we extend the notion and the properties of AS operators to the uniformly approximately shrinking (UAS) families of operators. Section 6 contains one of the main results of the paper (Theorem 6.3), where we show that an application of UAS operators in (1.5), satisfying some additional conditions guarantees the convergence of $\{u^k\}_{k=0}^{\infty}$ to a unique solution of VIP(F, C). In Section 7 we present several examples of GHSD method for VIP(F, C) with $C = \bigcap_{i=1}^{m} \operatorname{Fix} U_i$, where $U_i : \mathcal{H} \to \mathcal{H}$ are quasi-nonexpansive operators having a common fixed point. We apply the results of Sections 3–6 in order to prove that the sequences generated by particular methods converge to a unique solution of VIP(F, C). The methods are combined with a a string-averaging scheme which is presented in Section 7. Moreover, we present an independent convergence result (Theorem 7.6) dedicated to this specific scheme. In the Appendix we prove the equivalence of some conditions for real valued functions (Lemma 8.1). This result is used in the proof of the equivalence of various definitions of approximately shrinking operators as well as in the proof of the equivalence of various definitions of boundedly regular operators.

2. Preliminaries

2.1. **Nonexpansive and firmly nonexpansive operators.** We say that U is η-strongly monotone, where $\eta > 0$, if $\langle Ux - Uy, x - y\rangle \geq \eta\|x - y\|^2$ for all $x, y \in \mathcal{H}$. We say that U is *κ-Lipschitz continuous*, where $\kappa \geq 0$, if $\|Ux - Uy\| \leq \kappa\|x - y\|$ for all $x, y \in \mathcal{H}$. If $\kappa = 1$, then U is called a *nonexpansive* (NE) operator. If $\kappa < 1$ then we say that U is a *contraction*. An example of a nonexpansive and monotone operator is a *firmly nonexpansive* (FNE) one, i.e., an operator U satisfying $\langle Ux - Uy, x - y\rangle \geq \|Ux - Uy\|^2$ for all $x, y \in \mathcal{H}$ (apply the Cauchy–Schwarz inequality). A *relaxation* $U_\alpha := \operatorname{Id} + \alpha(U - \operatorname{Id})$ of an FNE operator, where $\alpha \in [0, 2]$, is called an *α-relaxed firmly nonexpansive* (α-RFNE or RFNE) operator. If $\alpha \in (0, 2)$ then the relaxation U_α of an FNE operator is called a *strictly relaxed firmly nonexpansive* operator. Furthermore, U is FNE if and only if $S := 2U - \operatorname{Id}$ is NE (see [23, Chapter 12]). Equivalently, an operator $S : \mathcal{H} \to \mathcal{H}$ is NE if and only if $U := \frac{1}{2}(S + \operatorname{Id})$ is FNE.

Proposition 2.1. *Let $U_i : \mathcal{H} \to \mathcal{H}$ be (strictly) relaxed firmly nonexpansive, $i \in I := \{1, 2, \ldots, m\}$. Then*

(i) *a composition $U := U_m \ldots U_1$ and*
(ii) *a convex combination $U := \sum_{\in I} \omega_i U_i$, where $\omega_i \geq 0, i \in I$ and $\sum_{\in I} \omega_i = 1$*

are (strictly) relaxed firmly nonexpansive.

Proof. See [33, Theorem 3(b)], [5, Lemma 2.2 and Proposition 2.1] or [7, Theorems 2.2.35, 2.2.42]. □

Let $C \subseteq \mathcal{H}$ and $x \in \mathcal{H}$. If there is $y \in C$ such that $\|y - x\| \leq \|z - x\|$ for all $z \in C$ then y is called a *metric projection* of x onto C and is denoted by $P_C x$. Let C be nonempty, closed and convex. Then for any $x \in \mathcal{H}$ the metric projection $y := P_C x$ is uniquely defined and characterized by the following inequality $\langle z - y, x - y\rangle \leq 0$ (see, e.g., [20, Theorems 3.4(2) and 4.1]). A function $d(\cdot, C) : \mathcal{H} \to \mathbb{R}$ defined by $d(x, C) := \|P_C x - x\|$ is continuous and convex, consequently, it is weakly lower semi-continuous (see [21, Chapter I, Corollary 2.2]).

2.2. **Quasi-nonexpansive operators.** Let $U : \mathcal{H} \to \mathcal{H}$ be an operator having a fixed point. We say that U is *quasi-nonexpansive* (QNE) if $\|Ux - z\| \le \|x - z\|$ for all $x \in \mathcal{H}$ and all $z \in \operatorname{Fix} U$. We say that U is *γ-strongly quasi-nonexpansive* (γ-SQNE), where $\gamma \ge 0$, if

$$\|Ux - z\|^2 \le \|x - z\|^2 - \gamma \|Ux - x\|^2$$

for all $x \in \mathcal{H}$ and all $z \in \operatorname{Fix} U$. If $\gamma_1 < \gamma_2$ and U is γ_2-SQNE then it is γ_1-SQNE. If $\gamma > 0$ then U is called strongly quasi-nonexpansive (SQNE). We say that U is a *cutter* if $\langle z - Ux, x - Ux \rangle \le 0$ for all $x \in \mathcal{H}$ and all $z \in \operatorname{Fix} U$. A NE operator having a fixed point is QNE. The fixed point set of a QNE operator is closed and convex (see [4, Proposition 2.6(ii)]). U is a cutter if and only if its *α-relaxation* is $\frac{2-\alpha}{\alpha}$-SQNE, where $\alpha \in (0, 2]$ (see [19, Proposition 2.3(ii)] and [7, Theorem 2.1.39]). In particular, U is a cutter if and only if U is 1-SQNE. Furthermore, U is QNE if and only if $\frac{1}{2}(U + \mathrm{Id})$ is a cutter (see [4, Proposition 2.3(v)⇔(vi)] or [7, Corollary 2.1.33(ii)])

A FNE operator having a fixed point is a cutter (see [24, pp. 43-44] or [7, Theorem 2.2.5]).

Proposition 2.2. *Let $U_i : \mathcal{H} \to \mathcal{H}$ be ρ_i-strongly quasi-nonexpansive, $i \in I := \{1, 2, \dots, m\}$, with $\bigcap_{i \in I} \operatorname{Fix} U_i \ne \emptyset$ and $\rho := \min_{i \in I} \rho_i > 0$. Then:*

(i) *a composition $U := U_m \dots U_1$ is $\frac{\rho}{m}$- strongly quasi-nonexpansive;*
(ii) *a convex combination $U := \sum_{\in I} \omega_i U_i$, where $\omega_i > 0, i \in I$ and $\sum_{\in I} \omega_i = 1$, is ρ-strongly quasi-nonexpansive.*

Moreover, in both cases,

$$\operatorname{Fix} U = \bigcap_{i \in I} \operatorname{Fix} U_i.$$

Proof. See [7, Theorems 2.1.26, 2.1.42, 2.1.48 and 2.1.50] □

A comprehensive review of the properties of NE, QNE, SQNE and FNE operators can be found in [7, Chapter 2].

2.3. **Demi-closed operators.** We say that an operator $U : \mathcal{H} \to \mathcal{H}$ is *demi-closed* at 0 if for any sequence $\{x^k\}_{k=0}^{\infty} \subseteq \mathcal{H}$ converging weakly to $y \in \mathcal{H}$ and such that $Ux^k \to 0$ it holds $Uy = 0$. If we replace the weak convergence by the strong one then we obtain the definition of the closedness of U at 0. If U is nonexpansive then $U - \mathrm{Id}$ is demi-closed at 0 (see [34, Lemma 2]). It is clear that in a finite dimensional Hilbert space the notions of a demi-closed operator and a closed one coincide. Note that a continuous operator defined on a finite dimensional Hilbert space is closed.

3. Approximately shrinking operators and basic examples

In this Section we introduce some subclasses of quasi-nonexpansive operators $U : \mathcal{H} \to \mathcal{H}$ with the property that the displacement $\|Ux - x\|$ cannot be too small for those x which are far from $\operatorname{Fix} U$. This property is important for the convergence analysis of methods for solving variational inequality problem over $\operatorname{Fix} U$. These convergence analysis will be presented in Sections 6 and 7.

Definition 3.1. We say that a quasi-nonexpansive operator $U:\mathcal{H}\to\mathcal{H}$ is *approximately shrinking* (AS) on a subset $D\subseteq\mathcal{H}$ if for any sequence $\{x^k\}_{k=0}^{\infty}\subseteq D$ it holds

$$\lim_k \|Ux^k - x^k\| = 0 \Longrightarrow \lim_k d(x^k, \operatorname{Fix} U) = 0. \tag{3.1}$$

We say that U is *approximately shrinking* if U is AS on any bounded subset $D\subseteq\mathcal{H}$. If U is AS then we say that the AS property holds for U.

Proposition 3.2. *Let $U:\mathcal{H}\to\mathcal{H}$ be quasi-nonexpansive. Then the following conditions are equivalent:*

(i) *U is approximately shrinking.*
(ii) *For any bounded sequence $\{x^k\}_{k=0}^{\infty}\subseteq\mathcal{H}$ and for any $\eta>0$ there are $\gamma>0$ and $k_0\geq 0$ such that for all $k\geq k_0$ it holds*

$$\|Ux^k - x^k\| < \gamma \Longrightarrow d(x^k, \operatorname{Fix} U) < \eta. \tag{3.2}$$

(iii) *For any bounded subset $S\subseteq\mathcal{H}$ and for any $\eta>0$ there is $\gamma>0$ such that for any $x\in S$ it holds*

$$\|Ux - x\| < \gamma \Longrightarrow d(x, \operatorname{Fix} U) < \eta. \tag{3.3}$$

Proof. The proposition follows from Lemma 8.1 (see Appendix) with $f,g\colon\mathcal{H}\to[0,\infty)$ defined as

$$f(x) := \|x - Ux\|, \quad g(x) := d(x, \operatorname{Fix} U).$$

□

Definition 3.3. We say that a quasi-nonexpansive operator $U:\mathcal{H}\to\mathcal{H}$ is *linearly shrinking* (LS) if there is $\delta>0$ such that for any $x\in\mathcal{H}$ it holds

$$\|Ux - x\| \geq \delta d(x, \operatorname{Fix} U). \tag{3.4}$$

It is clear that if U is linearly shrinking then it is approximately shrinking.

Conditions which play a similar role to (3.1) or (3.4) appear in many applications (see [3, 6, 8, 26, 28, 31, 32, 34, 35, 39]).

Proposition 3.4. *Let $U:\mathcal{H}\to\mathcal{H}$ be quasi-nonexpansive. If U is AS (LS) then its relaxation U_α is AS (LS) for all $\alpha\in[0,1]$. If, furthermore, U is a cutter then its relaxation U_α is AS (LS) for all $\alpha\in[0,2]$.*

Proof. Let U be AS (LS). If $\alpha=0$ then $U_\alpha=\mathrm{Id}$ and the claim is obvious. We have $U_\alpha x - x = \alpha(Ux - x)$ for all $x\in\mathcal{H}$ and all $\alpha\in\mathbb{R}$, therefore, $\operatorname{Fix} U_\alpha = \operatorname{Fix} U$ for any $\alpha>0$. If $\alpha\in(0,1]$ then the convexity of the norm and the quasi nonexpansivity of U yield

$$\begin{aligned}\|U_\alpha x - z\| &= \|(1-\alpha)(x-z)+\alpha(Ux-z)\| \leq (1-\alpha)\|x-z\|+\alpha\|Ux-z\|\\ &\leq \|x-z\|\end{aligned}$$

for all $x\in\mathcal{H}$ and $z\in\operatorname{Fix} U_\alpha$, i.e., U_α is quasi-nonexpansive. If U is a cutter and $\alpha\in(0,2]$, then U_α is quasi-nonexpansive. In both cases we have $\|U_\alpha x - x\| = \alpha\|Ux - x\|$, consequently, U_α is AS (LS). □

Example 3.5. Let $C \subseteq \mathcal{H}$ be a nonempty, closed and convex subset. By the characterization of the metric projection, P_C is a cutter. Furthermore,

$$\operatorname{Fix} P_C = C.$$

Since a cutter is QNE and $\|P_C x - x\| = d(x, C)$, P_C is linearly shrinking. By Proposition 3.4, an α-relaxation of P_C, where $\alpha \in (0, 2]$, is linearly shrinking.

Example 3.6. Let $\dim \mathcal{H} < \infty$, $f : \mathcal{H} \to \mathbb{R}$ be a convex function and $P_f : \mathcal{H} \to \mathcal{H}$ be a subgradient projection relative to f i.e.,

$$P_f(x) := \begin{cases} x - \frac{f(x)_+}{\|g_f(x)\|^2} g_f(x) & \text{if } g_f(x) \neq 0, \\ x & \text{otherwise,} \end{cases}$$

where $g_f(x)$ denotes a subgradient of f at $x \in \mathcal{H}$. The existence of $g_f(x)$ and, consequently, of $P_f(x)$ follows from [3, Corollary 7.9]. Suppose that the sublevel set

$$S(f, 0) := \{x \in | \, f(x) \leq 0\} \neq \emptyset.$$

Then $\operatorname{Fix} P_f = S(f, 0)$ (see [7, Lemma 4.2.5]) and P_f is a a cutter (see [7, Corollary 4.2.6]). By [10, Lemma 24], P_f is approximately shrinking. By Proposition 3.4, an α-relaxation of P_f, where $\alpha \in (0, 2]$, is approximately shrinking.

Example 3.7. Let $C \subseteq \mathcal{H}$ be a nonempty, closed and convex subset and $\delta \in (0, 1]$. Define set valued mappings $\mathcal{T}_C, \mathcal{U}_C, \mathcal{C}_\delta : \mathcal{H} \to 2^{\mathcal{H}}$ by

$$\mathcal{T}_C(x) := \{y \in \mathcal{H} \mid \|y - z\| \leq \|x - z\| \text{ for all } z \in C\},$$

$$\mathcal{U}_C(x) := \{y \in \mathcal{H} \mid \langle x - y, z - y \rangle \leq 0 \text{ for all } z \in C\}$$

and

$$\mathcal{C}_\delta(x) := \{y \in \mathcal{H} \mid \|y - x\| \geq \delta \|P_C x - x\|\}.$$

Further, define set valued mappings $\mathcal{T}_C^\delta, \mathcal{U}_C^\delta : \mathcal{H} \to 2^{\mathcal{H}}$ by

$$\mathcal{T}_C^\delta(x) := \mathcal{T}_C(x) \cap \mathcal{C}_\delta(x) \text{ and } \mathcal{U}_C^\delta(x) := \mathcal{U}_C(x) \cap \mathcal{C}_\delta(x), \ x \in \mathcal{H}.$$

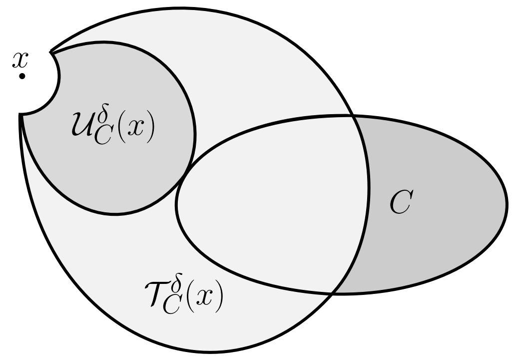
It follows from the characterization of the metric projection that

$$P_C(x) \in \mathcal{U}_C^\delta(x), \ x \in \mathcal{H}.$$

Furthermore, $\mathcal{U}_C^\delta(x) \subseteq \mathcal{T}_C^\delta(x)$, $x \in \mathcal{H}$, because a cutter is a quasi-nonexpansive operator. Therefore, $\mathcal{T}_C^\delta(x) \neq \emptyset$, $x \in \mathcal{H}$. Note that an operator T with $\operatorname{Fix} T = C$ is quasi-nonexpansive (a cutter) if and only if T is a selection of $\mathcal{T}_C$ (of $\mathcal{U}_C$). Therefore, it follows from [7, Corollary 2.1.33(ii)] that

$$\mathcal{U}_C(x) = \frac{1}{2}(\mathcal{T}_C(x) + x), \ x \in \mathcal{H}.$$

By definition, any selection $T : \mathcal{H} \to \mathcal{H}$ of $\mathcal{T}_C^\delta$ is linearly shrinking. In particular, any selection $U : \mathcal{H} \to \mathcal{H}$ of $\mathcal{U}_C^\delta$ is linearly shrinking. In Figure 1 there are presented the subsets $\mathcal{T}_C^\delta(x), \mathcal{U}_C^\delta(x)$ for a closed convex subset $C \subseteq \mathcal{H}$ and for some $x \in \mathcal{H}$.



Subsets $\mathcal{T}_C^\delta(x)$ and $\mathcal{U}_C^\delta(x)$

Note that there are quasi-nonexpansive operators which are not approximately shrinking, e.g., the operator $U : \mathcal{H} \to \mathcal{H}$ defined by

$$Ux := \begin{cases} P_D x & \text{if } x \notin D \\ P_C x & \text{if } x \in D, \end{cases}$$

where $C := \{x \in \mathcal{H} \mid \|x\| \le 1\}$, $D := \{x \in \mathcal{H} \mid \|x\| \le 2\}$.

4. Properties of approximately shrinking operators

In this section we present further properties of approximately shrinking operators.

Proposition 4.1. *Let $U : \mathcal{H} \to \mathcal{H}$, with $\operatorname{Fix} U \neq \emptyset$, be quasi-nonexpansive. Then the following hold:*

(i) *If U is approximately shrinking, then $U - \operatorname{Id}$ is demi-closed at 0.*
(ii) *If $\dim \mathcal{H} < \infty$ and $U - \operatorname{Id}$ is closed at zero, then U is approximately shrinking.*

Proof. (i) Let U be AS and $\{x^k\}_{k=0}^\infty \subseteq \mathcal{H}$ be bounded. Assume that $x^k \rightharpoonup x$ and $\|Ux^k - x^k\| \to 0$. Let $\{x^{n_k}\}_{k=0}^\infty \subseteq \{x^k\}_{k=0}^\infty$ be such that $\liminf_{k\to\infty} d(x^k, \operatorname{Fix} U) = \lim_{k\to\infty} d(x^{n_k}, \operatorname{Fix} U)$. Then, by the AS property of U and by the weak lower semi continuity of $d(\cdot, \operatorname{Fix} U)$, we have

$$0 = \lim_{k\to\infty} d(x^{n_k}, \operatorname{Fix} U) = \liminf_{k\to\infty} d(x^k, \operatorname{Fix} U) \ge d(x, \operatorname{Fix} U).$$

The closedness of $\operatorname{Fix} U$ (see Subsection 2.2) yields now $x \in \operatorname{Fix} U$ which proves the demi closedness of U.

(ii) Assume that $\dim \mathcal{H} < \infty$ and $U - \operatorname{Id}$ is closed at 0. Let $\{x^k\}_{k=0}^\infty \subseteq \mathcal{H}$ be bounded. Hence, there is a subsequence $\{x^{n_k}\}_{k=0}^\infty$ of $\{x^k\}_{k=0}^\infty$, which converges to $x \in \mathcal{H}$. Without loss of generality we can assume that $\limsup_{k\to\infty} d(x^k, \operatorname{Fix} U) = \lim_{k\to\infty} d(x^{n_k}, \operatorname{Fix} U)$. Suppose that $\|Ux^k - x^k\| \to 0$. The continuity of $d(\cdot, \operatorname{Fix} U)$ implies that

$$0 \le \limsup_{k\to\infty} d(x^k, \operatorname{Fix} U) = \lim_{k\to\infty} d(x^{n_k}, \operatorname{Fix} U) = d(x, \operatorname{Fix} U).$$

By the closedness of $U - \operatorname{Id}$ at 0 we get that $x \in \operatorname{Fix} U$, i.e., $d(x, \operatorname{Fix} U) = 0$. Therefore, $\lim_{k\to\infty} d(x^k, \operatorname{Fix} U) = 0$, which completes the proof. □

Because a continuous operator is closed, Proposition 4.1 yields the following result.

Corollary 4.2. *If a quasi-nonexpansive operator* $U : \mathbb{R}^n \to \mathbb{R}^n$ *is continuous (e.g., a nonexpansive one), then* U *is approximately shrinking.*

An important subclass of quasi-nonexpansive operators with similar properties to approximately shrinking operators was introduced by Yamada and Ogura [39, Definition 1]. Below, we present an equivalent definition. A proof of equivalence can be found in [9, Proposition 2.9].

Definition 4.3. We say that a quasi-nonexpansive operator U is *quasi-shrinking* on a subset $C \subseteq \mathcal{H}$ if for any sequence $\{u^k\}_{k=0}^{\infty} \subseteq C$ the following implication holds

$$\lim_k (d(u^k, \operatorname{Fix} U) - d(Uu^k, \operatorname{Fix} U)) = 0 \Longrightarrow \lim_k d(u^k, \operatorname{Fix} U) = 0. \tag{4.1}$$

If U is quasi-shrinking on $\mathcal{H}$, then U is just called *quasi-shrinking.*

Proposition 4.4. *If an operator* $U : \mathcal{H} \to \mathcal{H}$ *is quasi-shrinking then it is approximately shrinking. Moreover, if* $\dim \mathcal{H} < \infty$ *and* U *is approximately shrinking and strongly quasi-nonexpansive, then* U *is quasi-shrinking.*

Proof. Suppose that $C \subseteq \mathcal{H}$ is bounded, U is quasi-shrinking on C and denote $S := P_{\operatorname{Fix} U}$. Let $\{u^k\}_{k=0}^{\infty} \subseteq C$ and $\lim_k \|Uu^k - u^k\| = 0$. Since the metric projection is nonexpansive, we have

$$0 \le \lim_k \|SUu^k - Su^k\| \le \lim_k \|Uu^k - u^k\| = 0,$$

i.e.,

$$\lim_k \|SUu^k - Su^k\| = 0. \tag{4.2}$$

By the quasi nonexpansivity of U and by the triangle inequality,

$$0 \le \|Su^k - u^k\| - \|Su^k - Uu^k\| \le \|Uu^k - u^k\|,$$

consequently,

$$\lim_k (\|Su^k - u^k\| - \|Su^k - Uu^k\|) = 0. \tag{4.3}$$

Applying again the triangle inequality, (4.2) and (4.3), we obtain

$$\begin{aligned} 0 &\le \limsup_k (\|Su^k - u^k\| - \|SUu^k - Uu^k\|) \\ &\le \limsup_k [\|Su^k - u^k\| - (\|Su^k - Uu^k\| - \|SUu^k - Su^k\|)] \\ &= \limsup_k (\|Su^k - u^k\| - \|Su^k - Uu^k\|) = 0, \end{aligned}$$

i.e.,

$$\lim_k (\|Su^k - u^k\| - \|SUu^k - Uu^k\|) = 0.$$

Since U is quasi-shrinking, we have $\lim_k d(u^k, \operatorname{Fix} U) = \lim_k \|Su^k - u^k\| = 0$ which completes the first part of the proof. Let now $\dim \mathcal{H} < \infty$ and U be approximately shrinking and strongly quasi-nonexpansive. By Proposition 4.1(i), $U - \mathrm{Id}$ is closed at 0. Consequently, [9, Proposition 2.11] yields, that U is quasi-shrinking. □

Let $\mathcal{U} := \{U_i : \mathcal{H} \to \mathcal{H} \mid i \in I\}$ be a finite family of operators, where $I := \{1, \dots, m\}$. The following result is a simple extension of [7, Theorem 4.8.2], where the operators $U_i : \mathcal{H} \to \mathcal{H}$ are assumed to be cutters (or, equivalently, 1-SQNE), $i \in I$. The proof goes along lines of [7, Theorem 4.8.2] with some simple modifications.

Proposition 4.5. *Let $U_i : \mathcal{H} \to \mathcal{H}$ be ρ_i-SQNE, $\rho_i > 0$, $i \in I$, with $\bigcap_{i\in I} \operatorname{Fix} U_i \neq \emptyset$, $U := \sum_{i\in I} \omega_i U_i$, where $\omega_i \geq 0, i \in I$, and $\sum_{\in I} \omega_i = 1$, and let $x \in \mathcal{H}$ and $z \in \bigcap_{i\in I} \operatorname{Fix} U_i$ be arbitrary. Then*

$$\|Ux - z\|^2 \leq \|x - z\|^2 - \sum_{i\in I} \omega_i \rho_i \|U_i x - x\|^2. \tag{4.4}$$

Consequently,

$$\frac{1}{2R} \sum_{i\in I} \omega_i \rho_i \|U_i x - x\|^2 \leq \|Ux - x\| \tag{4.5}$$

for any positive $R \geq \|x - z\|$.

Proof. By [7, Corollary 2.1.43], U_i is an α_i-relaxed cutter with $\alpha_i := \frac{2}{1+\rho_i} \in (0, 2)$. Hence,

$$\alpha_i \langle U_i x - x, x - z\rangle \leq -\|U_i x - x\|^2$$

(see [7, Remark 2.1.31]). Therefore, the triangle inequality and the convexity of the function $\|\cdot\|^2$ yield

$$\begin{aligned}
\|Ux - z\|^2 &= \|x + \sum_{i\in I} \omega_i (U_i x - x) - z\|^2 \\
&\leq \|x - z\|^2 + \|\sum_{i\in I} \omega_i (U_i x - x)\|^2 + 2\sum_{i\in I} \omega_i \langle x - z, U_i x - x\rangle \\
&\leq \|x - z\|^2 + \sum_{i\in I} \omega_i \|U_i x - x\|^2 - \sum_{i\in I} \frac{2\omega_i}{\alpha_i} \|U_i x - x\|^2 \\
&= \|x - z\|^2 - \sum_{i\in I} \omega_i \left(\frac{2}{\alpha_i} - 1\right) \|U_i x - x\|^2 \\
&= \|x - z\|^2 - \sum_{i\in I} \omega_i \rho_i \|U_i x - x\|^2,
\end{aligned}$$

i.e., (4.4) is satisfied. Let $R > 0$ be such that $\|x - z\| \leq R$. Inequality (4.5) is clear for $x \in \bigcap_{i\in I} \operatorname{Fix} U_i$. Suppose that $x \notin \bigcap_{i\in I} \operatorname{Fix} U_i$. Then $x \notin \operatorname{Fix} U$ (see Proposition 2.2) and $\|Ux - x\| > 0$. By the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
\|Ux - z\|^2 &= \|Ux - x\|^2 + \|x - z\|^2 + 2\langle Ux - x, x - z\rangle \\
&\geq \|Ux - x\|^2 + \|x - z\|^2 - 2R\|Ux - x\|.
\end{aligned} \tag{4.6}$$

Now (4.4) and (4.6) yield

$$\frac{1}{2R} \sum_{i\in I} \omega_i \rho_i \|U_i x - x\|^2 \leq \|Ux - x\|$$

which completes the proof. □

Proposition 4.6. *Let $U_i : \mathcal{H} \to \mathcal{H}$ be ρ_i-SQNE, $\rho_i > 0$, $i \in I$, with $\bigcap_{i\in I} \operatorname{Fix} U_i \neq \emptyset$, $U := U_m U_{m-1} ... U_1$, and let $x \in \mathcal{H}$ and $z \in \bigcap_{i\in I} \operatorname{Fix} U_i$ be arbitrary. Then*

$$\|Ux - z\|^2 \leq \|x - z\|^2 - \sum_{i=1}^{m} \rho_i \|S_i x - S_{i-1} x\|^2, \tag{4.7}$$

where $S_i := U_i U_{i-1} \dots U_1$, $i \in I$, $S_0 := \mathrm{Id}$. Consequently,

$$\frac{1}{2R} \sum_{i=1}^{m} \rho_i \|S_i x - S_{i-1} x\|^2 \leq \|Ux - x\|. \tag{4.8}$$

for any positive $R \geq \|x - z\|$.

Proof. By the definition of an SQNE operator and by the definition of S_i, $i = 0, 1, ..., m$, we have

$$\begin{aligned} \|Ux - z\|^2 &= \|U_m U_{m-1} ... U_1 x - z\|^2 \\ &\leq \|S_{m-1} x - z\|^2 - \rho_m \|S_m x - S_{m-1} x\|^2 \\ &\leq \|S_{m-2} x - z\|^2 - \rho_{m-1} \|S_{m-1} x - S_{m-2} x\|^2 - \rho_m \|S_m x - S_{m-1} x\|^2 \\ &\leq \ ... \leq \|x - z\|^2 - \sum_{i=1}^{m} \rho_i \|S_i x - S_{i-1} x\|^2, \end{aligned}$$

i.e., (4.7) is satisfied. The rest of the proof is similar to the proof of Proposition 4.5, with an application of (4.7) instead of (4.4). □

Before we formulate two most important results of this section, we recall a definition of a boundedly regular family of operators. Let $C_i \subseteq \mathcal{H}$, $i \in I := \{1, 2, ..., m\}$, be closed convex subsets with $C := \bigcap_{i\in I} C_i \neq \emptyset$. Denote $\mathcal{C} := \{C_i \mid i \in I\}$.

Definition 4.7. (cf. [18, Definition 5.1]) We say that the family $\mathcal{C}$ is *boundedly regular* (BR) if for any bounded sequence $\{x^k\}_{k=0}^{\infty} \subseteq \mathcal{H}$ the following implication holds

$$\lim_k \max_{i\in I} d(x^k, C_i) = 0 \Longrightarrow \lim_k d(x^k, C) = 0.$$

Similarly to the definition of an approximately shrinking operator, the notion of bounded regularity has also equivalent forms which are presented in the proposition below. We will apply this equivalence in the sequel.

Proposition 4.8. *The following conditions are equivalent:*

(i) *$\mathcal{C}$ is boundedly regular.*

(ii) *For any bounded sequence $\{x^k\}_{k=0}^{\infty} \subseteq \mathcal{H}$ and for any $\eta > 0$ there are $\gamma > 0$ and $k_0 \geq 0$ such that for all $k \geq k_0$ it holds*

$$\max\{d(x^k, C_i) \mid i \in I\} < \delta \Longrightarrow d(x^k, C) < \eta.$$

(iii) *(cf.* [3, Definition 5.1]*) For any bounded subset $S \subseteq \mathcal{H}$ and for any $\eta > 0$, there is $\delta > 0$ such that for any $x \in S$ it holds*

$$\max\{d(x, C_i) \mid i \in I\} < \delta \Longrightarrow d(x, C) < \eta.$$

Proof. The proposition follows from Lemma 8.1 (see Appendix) with $f, g\colon \mathcal{H} \to [0, \infty)$ defined as $f(x) := \max\{d(x, C_i) \mid i \in I\}$, and $g(x) := d(x, C)$. □

The following proposition gives sufficient conditions for a family $\mathcal{C}$ to be boundedly regular. The proof of the proposition can be found in [3, Proposition 5.4 (iii), Corollary 5.14 and Corollary 5.22].

Proposition 4.9. *Let $C_i \subseteq \mathcal{H}$, $i \in I := \{1, 2, ..., m\}$, be closed convex with $C := \bigcap_{i\in I} C_i \neq \emptyset$. If one of the following conditions is satisfied:*

(i) $\dim \mathcal{H} < \infty$,

(ii) $C_m \cap \operatorname{int} \bigcap_{i=1}^{m-1} C_i \neq \emptyset$,

(iii) C_i *are half-spaces,* $i \in I$

then the family $\mathcal{C} := \{C_i \mid i \in I\}$ is boundedly regular.

An example of a family which in not boundedly regular can be found in [2, Example 5.5]. Note that a subfamily of a BR family $\mathcal{C}$ needs not to be BR. To see it consider two closed convex subsets $A, B \subseteq \mathcal{H}$ having a common point such that the family $\{A, B\}$ is not BR. It follows from the definition that the family $\{A, B, A \cap B\}$ is boundedly regular.

For quasi-nonexpansive operators U_i, $i \in I := \{1, 2, ..., m\}$, denote $\mathcal{F} := \{\operatorname{Fix} U_i \mid i \in I\}$.

Theorem 4.10. *Let $U_i : \mathcal{H} \to \mathcal{H}$ be ρ_i-SQNE, $\rho_i > 0$, $i \in I$, with $\bigcap_{i\in I} \operatorname{Fix} U_i \neq \emptyset$. If U_i, $i \in I$, are approximately shrinking and the family $\mathcal{F}$ is boundedly regular, then the operator $U := \sum_{i\in I} \omega_i U_i$, where $\omega_i > 0, i \in I$, $\sum_{i\in I} \omega_i = 1$, is approximately shrinking.*

Proof. Let $\{x^k\}_{k=0}^{\infty} \subseteq \mathcal{H}$ be bounded. Assume that $\|Ux^k - x^k\| \to 0$. Then, by (4.5) with $R \geq \|x^k - z\|$ for all $k \geq 0$ and some $z \in \operatorname{Fix} U$, it holds

$$\lim_k \|U_i x^k - x^k\| = 0, \text{ for all } i \in I.$$

The AS property of U_i implies that

$$\lim_k d(x^k, \operatorname{Fix} U_i) = 0, \text{ for all } i \in I.$$

Since $\{x^k\}_{k=0}^{\infty}$ is bounded and the family $\mathcal{F}$ is boundedly regular, we have

$$\lim_k d(x^k, \operatorname{Fix} U) = 0,$$

which completes the proof. □

Theorem 4.11. *Let $U_i : \mathcal{H} \to \mathcal{H}$ be ρ_i-SQNE, $\rho_i > 0$, $i \in I$, with $\bigcap_{i\in I} \operatorname{Fix} U_i \neq \emptyset$. If U_i, $i \in I$, are approximately shrinking and the family $\mathcal{F}$ is boundedly regular, then the operator $U := U_m U_{m-1} \dots U_1$ is approximately shrinking.*

Proof. Let $\{x^k\}_{k=0}^{\infty} \subseteq \mathcal{H}$ be bounded. Since $S_i := U_i U_{i-1} ... U_1$ is QNE as a composition of SQNE operators (see Proposition 2.2), the sequence $\{S_i x^k\}_{k=0}^{\infty}$ is also bounded, $i \in I$. Therefore, by the AS property of U_i, it holds

$$\lim_k \|U_i(S_{i-1}x^k) - S_{i-1}x^k\| = 0 \Longrightarrow \lim_k d(S_{i-1}x^k, \operatorname{Fix} U_i) = 0, \tag{4.9}$$

$i \in I$. Assume that $\|Ux^k - x^k\| \to 0$. By (4.8) with $R \geq \|x^k - z\|$ for all $k \geq 0$ and some $z \in \operatorname{Fix} U$, we have

$$\lim_k \|U_i(S_{i-1}x^k) - S_{i-1}x^k\| = 0, \tag{4.10}$$

$i \in I$. Moreover, the definition of the metric projection and the triangle inequality yield

$$\begin{aligned} d(x^k, \operatorname{Fix} U_i) &= \|x^k - P_{\operatorname{Fix} U_i} x^k\| \leq \|x^k - P_{\operatorname{Fix} U_i} S_{i-1}x^k\| \\ &\leq \sum_{j=1}^{i-1} \|S_j x^k - S_{j-1}x^k\| + \|S_{i-1}x^k - P_{\operatorname{Fix} U_i} S_{i-1}x^k\|. \end{aligned}$$

Therefore, (4.9) and (4.10) imply $\lim_k d(x^k, \operatorname{Fix} U_i) = 0$, for all $i \in I$. By the bounded regularity of $\mathcal{F}$, we have $\lim_k d(x^k, \operatorname{Fix} U) = 0$ which completes the proof. $\square$

5. Uniformly approximately shrinking families of operators

In this section we extend the notion and the properties of an approximately shrinking operator to a family of operators. The most important results of this section are Theorems 5.4 and 5.6, where we present conditions under which a uniformly approximately shrinking family of operators is closed under convex combination and composition. This property is applied in Section 7, where we present several methods for solving variational inequality problem over the common fixed point set of quasi-nonexpansive operators.

Let $\mathcal{U} := \{U_i \mid \mathcal{H} \to \mathcal{H}, i \in I\}$ be a family of QNE operators. In this section we do not suppose that I is finite.

Definition 5.1. We say that the family $\mathcal{U}$ is *uniformly approximately shrinking* (UAS) if for any bounded sequence $\{x^k\}_{k=0}^\infty \subseteq \mathcal{H}$ and for any $\eta > 0$ there are $\gamma_\mathcal{U} > 0$ and $k_\mathcal{U} \geq 0$, such that for all $k \geq k_\mathcal{U}$ and for all $i \in I$ it holds

$$\|U_i x^k - x^k\| < \gamma_\mathcal{U} \Longrightarrow d(x^k, \operatorname{Fix} U_i) < \eta.$$

It is clear that if $\mathcal{U}$ is a finite family of AS operators then $\mathcal{U}$ is uniformly approximately shrinking.

Example 5.2. Let $C \subseteq \mathcal{H}$ be a nonempty, closed and convex subset and $\delta \in (0, 1]$. Let $\mathcal{U}$ be the family of all selections of the set valued mapping $\mathcal{T}_C^\delta$ (see Example 3.7). Then $\mathcal{U}$ is uniformly approximately shrinking. To verify this fact, it suffices to choose $\gamma_\mathcal{U} = \eta\delta$ and $k_\mathcal{U} = 0$. Note that $\gamma_\mathcal{U}$ and $k_\mathcal{U}$ do not dependent on $\{x^k\}_{k=0}^\infty$.

Lemma 5.3. *Let $\mathcal{U}$ be a uniformly approximately shrinking family of cutters and $\varepsilon \in (0, 2]$. Then the family $\mathcal{U}_\alpha := \{U_\alpha \mid U \in \mathcal{U}, \alpha \in [\varepsilon, 2]\}$ is uniformly approximately shrinking.*

Proof. Let $\{x^k\}_{k=0}^\infty$ be bounded, $\eta > 0$ and let $\gamma_\mathcal{U} > 0$ and $k_\mathcal{U} \geq 0$ be such that for any $k \geq k_\mathcal{U}$ and for any $U \in \mathcal{U}$, it holds $d(x^k, \operatorname{Fix} U) < \eta$ whenever $\|Ux^k - x^k\| < \gamma_\mathcal{U}$. Let $U \in \mathcal{U}$ and $\alpha \in [\varepsilon, 2]$. Then U_α is QNE and $\operatorname{Fix} U_\alpha = \operatorname{Fix} U$. Put $\gamma_0 := \varepsilon\gamma_\mathcal{U}$, $k_0 := k_\mathcal{U}$ and let $k \geq k_0$. Suppose that $\|U_\alpha x^k - x^k\| < \gamma_0$. Then

$$\varepsilon\gamma_\mathcal{U} = \gamma_0 > \|U_\alpha x^k - x^k\| = \alpha\|Ux^k - x^k\| \geq \varepsilon\|Ux^k - x^k\|$$

and, consequently,

$$d(x^k, \operatorname{Fix} U_\alpha) = d(x^k, \operatorname{Fix} U) < \eta.$$

□

Suppose that $\bigcap_{i\in I} \operatorname{Fix} U_i \neq \emptyset$. For a subset $J \subseteq I$ denote

$$\mathcal{F}_J := \{\operatorname{Fix} U_j \mid j \in J\}$$

and $\mathcal{F} := \mathcal{F}_I$. Suppose that $\mathcal{F}$ is finite. Note that $\mathcal{F}$ can be finite even if I is infinite. Further, for a fixed $m \in \mathbb{N}$ and $\varepsilon \in (0, \frac{1}{m}]$ denote

$$\mathcal{J}_m := \{J \subseteq I \mid |J| = m \text{ and } \mathcal{F}_J \text{ is BR}\}$$

and

$$\mathcal{V}_m := \left\{ V := \sum_{j\in J} \omega_j U_j \mid J \in \mathcal{J}_m, \omega_j \in [\varepsilon, 1], j \in J, \sum_{j\in J} \omega_j = 1 \right\}.$$

Note that if $\mathcal{J}_m \neq \emptyset$ and the operators U_i are SQNE, $i \in I$, then any element of $\mathcal{V}_m$ is SQNE (see Proposition 2.2(ii)).

Theorem 5.4. *Let $U_i : \mathcal{H} \to \mathcal{H}$ be ρ_i-SQNE, $i \in I$, $\rho := \inf_{i\in I} \rho_i > 0$ and $\bigcap_{i\in I} \operatorname{Fix} U_i \neq \emptyset$. Assume that $\mathcal{F}$ is finite, $m \leq |I|$ and $\mathcal{J}_m$ is nonempty. If $\mathcal{U}$ is uniformly approximately shrinking then $\mathcal{V}_m$ is uniformly approximately shrinking.*

Proof. We have to show that for every bounded sequence $\{x^k\}_{k=0}^\infty \subseteq \mathcal{H}$ and for any $\eta > 0$ there exist $\gamma_{\mathcal{V}} > 0$ and $k_{\mathcal{V}} \geq 0$ such that for all $V \in \mathcal{V}_m$ and all $k \geq k_{\mathcal{V}}$ it holds

$$\|Vx^k - x^k\| < \gamma_{\mathcal{V}} \Longrightarrow d(x^k, \operatorname{Fix} V) < \eta,$$

Let $\eta > 0$, $\{x^k\}_{k=0}^\infty \subseteq \mathcal{H}$ be bounded and $V \in \mathcal{V}_m$, i.e., $V = \sum_{j\in J} \omega_j U_j$, for some $J \in \mathcal{J}_m$, where $\omega_j \in [\varepsilon, 1], j \in J, \sum_{j\in J} \omega_j = 1$. By Proposition 2.2, $\operatorname{Fix} V = \bigcap_{j\in J} \operatorname{Fix} U_j$. Let $z \in \operatorname{Fix} V$ and $R > 0$ be such that $\|x^k - z\| \leq R$ for all $k \geq 0$. By Proposition 4.5, we have

$$\frac{\varepsilon\rho}{2R} \|U_i x^k - x^k\|^2 \leq \frac{1}{2R} \sum_{i\in J} \omega_j \rho_j \|U_j x^k - x^k\|^2 \leq \|Vx^k - x^k\| \tag{5.1}$$

for all $i \in J$ and all $k \geq 0$. Since the sequence $\{x^k\}_{k=0}^\infty$ is bounded and $\mathcal{F}_J$ is boundedly regular, there exist $\delta > 0$ and $k_0 \geq 0$ such that

$$\max_{j\in J} d(x^k, \operatorname{Fix} U_j) < \delta \Longrightarrow d(x^k, \operatorname{Fix} V) < \eta \tag{5.2}$$

for all $k \geq k_0$. Note that δ and k_0 may be chosen independently of $J \in \mathcal{J}_m$. To prove this, consider the families $\mathcal{F}_J$, $J \in \mathcal{J}_m$. For any $\mathcal{F}_J$ we can find $\delta_J > 0$ and $k_J \geq 0$ such that (5.2) holds with $\delta := \delta_J$ and $k_0 := k_J$ (note that $\mathcal{F}_J$ is BR). It is easily seen that δ_J, $J \in \mathcal{J}_m$ and $k_J \geq 0$, can be chosen such that $\delta := \inf_{J\in\mathcal{J}_m} \delta_J > 0$ and $k_0 := \sup_{J\in\mathcal{J}_m} k_J < +\infty$, because $\mathcal{F}$ is finite.

The family $\mathcal{U}$ is uniformly AS, hence, we can find $\gamma_{\mathcal{U}} > 0$ and $k_{\mathcal{U}} \geq 0$ such that for all $j \in I$ and all $k \geq k_{\mathcal{U}}$ it holds

$$\|U_j x^k - x^k\| < \gamma_{\mathcal{U}} \Longrightarrow d(x^k, \operatorname{Fix} U_j) < \delta. \tag{5.3}$$

Let $k_{\mathcal{V}} := \max\{k_{\mathcal{U}}, k_0\}$, $\gamma_{\mathcal{V}} := \varepsilon\gamma_{\mathcal{U}}^2\rho/2R$ and let $k \geq k_{\mathcal{V}}$. Assume that $\|Vx^k - x^k\| < \gamma_{\mathcal{V}}$. Then, by (5.1), $\|U_j x^k - x^k\| < \gamma_{\mathcal{U}}$ for all $j \in J$. Therefore, (5.3) yields $\max_{j\in J} d(x^k, \operatorname{Fix} U_j) < \delta$, which, by (5.2), completes the proof. □

Let U be AS and $\eta > 0$. Consider a family of bounded sequences $\mathcal{X} := \{\{x_n^k\}_{k=0}^\infty \mid n \in I\}$, where I is finite. By the AS property of U, it is clear that for any sequence $\{x_n^k\}_{k=0}^\infty$, $n \in I$, there are $\gamma_n > 0$ and $k_n \geq 0$, such that for all $k \geq k_n$ it holds

$$\|Ux_n^k - x_n^k\| < \gamma_n \Longrightarrow d(x_n^k, \operatorname{Fix} U) < \eta.$$

If we put $\gamma := \min_{n\in I} \gamma_n$ and $k_0 := \max_{n\in I} k_n$ then for all $k \geq k_0$ and for all $n \in I$ it holds

$$\|Ux_n^k - x_n^k\| < \gamma \Longrightarrow d(x_n^k, \operatorname{Fix} U) < \eta.$$

Such an approach cannot be applied in the case when I is infinite. Nevertheless, if I is countable we can omit such inconveniences by presenting another approach for choosing γ and k_0. Moreover, such an approach can be presented not only for one AS operator U, but for a family of UAS operators.

Lemma 5.5. *Let $\mathcal{U}$ be uniformly approximately shrinking and let $\mathcal{X} := \{\{x_n^k\}_{k=0}^\infty \mid n \in \mathbb{N}\} \subseteq \mathcal{H}$ be bounded. Then, for any $\eta > 0$ there are $\gamma > 0$ and $k_0 \geq 0$ such that for all $U \in \mathcal{U}$, $n \in \mathbb{N}$ and $k \geq k_0$ it holds*

$$\|Ux_n^k - x_n^k\| < \gamma \Longrightarrow d(x_n^k, \operatorname{Fix} U) < \eta.$$

Proof. Let $\eta > 0$ be fixed. The family $\mathcal{X}$ is countable, hence there is a sequence $\{u^l\}_{l=0}^\infty$ which consists of all x_n^k, $k, n \geq 0$, and maintains the order of indices, i.e., if $k_1 \leq k_2$, $n_1 \leq n_2$ and $x_{n_1}^{k_1}, x_{n_2}^{k_2}$ have the positions l_1, l_2 in $\{u^l\}_{l=0}^\infty$, respectively, then $l_1 \leq l_2$ (one can apply, e.g., a well known construction which shows that $\mathbb{N}^2$ is countable). By the boundedness of $\mathcal{X}$, the sequence $\{u^l\}_{l=0}^\infty$ is also bounded. By the UAS property of $\mathcal{U}$, there are $\gamma_{\mathcal{U}} > 0$ and $l_{\mathcal{U}} \geq 0$, such that for all $U \in \mathcal{U}$ and $l \geq l_{\mathcal{U}}$ it holds

$$\|Uu^l - u^l\| < \gamma_{\mathcal{U}} \Longrightarrow d(u^l, \operatorname{Fix} U) < \eta.$$

Let $\gamma := \gamma_{\mathcal{U}}$ and k_0 be the smallest integer such that $x_1^{k_0}$ has a position $l_0 \geq l_{\mathcal{U}}$ in $\{u^l\}_{l=0}^\infty$. Suppose that $k \geq k_0$, $n \geq 0$, $\|Ux_n^k - x_n^k\| < \gamma$ and x_n^k is situated at the position $l_{k,n}$ in the sequence $\{u^l\}_{k=0}^\infty$, i.e., $x_n^k = u^{l_{k,n}}$. Then it follows from the properties of the sequence $\{u^l\}_{l=0}^\infty$ that x_n^k has a position $l_{k,n} \geq l_{\mathcal{U}}$ in $\{u^l\}_{l=0}^\infty$. Hence,

$$d(x_n^k, \operatorname{Fix} U) = d(u^{l_{k,n}}, \operatorname{Fix} U) < \eta$$

which completes the proof. □

For a fixed $m \in \mathbb{N}$ denote

$$\mathcal{J}_m := \{J = (j_1, \ldots, j_m) \mid j_i \in I, i = 1, 2, ..., m, \text{ and } \mathcal{F}_J \text{ is BR}\} \tag{5.4}$$

and

$$\mathcal{S}_m := \{S = U_{j_m} \ldots U_{j_1} \mid J \in \mathcal{J}_m\}.$$

We identify $J := (j_1, \ldots, j_m)$ in (5.4) wit a subset $\{j_1, \ldots, j_m\} \subseteq I$ and, for simplicity, we write $J = \{j_1, \ldots, j_m\}$. Note that if $\mathcal{J}_m \neq \emptyset$ and the operators U_i are SQNE then any element of $\mathcal{S}_m$ is SQNE (see Proposition 2.2(i)).

Theorem 5.6. *Let $U_i : \mathcal{H} \to \mathcal{H}$ be ρ_i-SQNE for all $i \in I$, $\rho := \inf_{i \in I} \rho_i > 0$ and $\bigcap_{i \in I} \operatorname{Fix} U_i \neq \emptyset$. Assume that $\mathcal{U}$ is countable, $\mathcal{F}$ is finite, $m \in \mathbb{N}$ is fixed and $\mathcal{J}_m$ is nonempty. If $\mathcal{U}$ is uniformly approximately shrinking then $\mathcal{S}_m$ is uniformly approximately shrinking.*

Proof. Let $\eta > 0$ and $\{x^k\}_{k=0}^\infty$ be bounded. We have to show that there exist $\gamma_{\mathcal{S}} > 0$ and $k_{\mathcal{S}} \geq 0$ such that for all $S \in \mathcal{S}_m$ and all $k \geq k_{\mathcal{S}}$ it holds

$$\|Sx^k - x^k\| < \gamma_{\mathcal{S}} \Longrightarrow d(x^k, \operatorname{Fix} S) < \eta. \tag{5.5}$$

Denote

$$\mathcal{J} := \{J = (j_1, \ldots, j_t) \mid j_i \in I, i = 1, 2, ..., t, 1 \leq t \leq m\},$$
$$\mathcal{S} := \{S = U_{j_t} \ldots U_{j_1} \mid J = (j_1, \ldots, j_t) \in \mathcal{J}\},$$

and

$$\mathcal{Y} := \{\{y^k\}_{k=0}^\infty := \{Sx^k\}_{k=0}^\infty \mid S \in \mathcal{S}\} \cup \{\{x^k\}_{k=0}^\infty\}.$$

We divide the proof in 5 steps.

Step 1: Let $z \in \bigcap_{i \in I} \operatorname{Fix} U_i$ and $R > 0$ be such that $\|x^k - z\| \leq R$, for all $k \geq 0$ and let $S \in \mathcal{S}$. The operator S is QNE, hence $\|Sx^k - z\| \leq \|x^k - z\| \leq R$, for all $k \geq 0$ and $z \in \operatorname{Fix} S = \bigcap_{i \in I} \operatorname{Fix} U_i$ (see Proposition 2.2). Therefore, $\mathcal{Y}$ is bounded.

Step 2: Let $S \in \mathcal{S}_m$, i.e., $S = U_{j_m} \ldots U_{j_1}$ for some $J = (j_1, \ldots, j_m) \in \mathcal{J}_m$. Assume, for simplicity, that $J = (1, \ldots, m)$. Let $z \in \operatorname{Fix} S$ and $R > 0$ be such that $\|x^k - z\| \leq R$. By Proposition 4.6, we have

$$\|Sx^k - x^k\| \geq \frac{\rho}{2R} \sum_{j=1}^{m} \|S_j x^k - S_{j-1} x^k\|^2, \tag{5.6}$$

where $S_j := U_j \ldots U_1$, $j = 1, 2, ..., m$ and $S_0 := \operatorname{Id}$.

Step 3: Since $\mathcal{Y}$ is bounded and $\mathcal{F}_J$ is boundedly regular, there exist $\delta > 0$ and $k_0 \geq 0$ such that it holds

$$\max_{j \in J} d(x^k, \operatorname{Fix} U_j) < \delta \Longrightarrow d(x^k, \operatorname{Fix} S) < \eta \tag{5.7}$$

for all $k \geq k_0$. Note that δ and k_0 may be chosen independently of $J \in \mathcal{J}_m$, by similar arguments to those made in the proof of Theorem 5.4.

Step 4: The family $\mathcal{U}$ is countable, hence, by definition, $\mathcal{Y}$ consists of a countable number of sequences. By Step 1, $\mathcal{Y}$ is bounded and, by assumption, $\mathcal{U}$ is uniformly approximately shrinking. Therefore, it follows from Lemma 5.5 that there are $\gamma_{\mathcal{U}} > 0$ and $k_{\mathcal{U}} \geq 0$ such that for all $U \in \mathcal{U}$, $\{y^k\}_{k=0}^\infty \in \mathcal{Y}$ and $k \geq k_{\mathcal{U}}$ it holds

$$\|Uy^k - y^k\| < \gamma_{\mathcal{U}} \Longrightarrow d(y^k, \operatorname{Fix} U) < \delta/2. \tag{5.8}$$

By (5.8) with $U = U_j$ and $y^k = S_{j-1}x^k$, we have

$$\|U_j(S_{j-1}x^k) - S_{j-1}x^k\| < \gamma_{\mathcal{U}} \Longrightarrow d(S_{j-1}x^k, \operatorname{Fix} U_j) < \delta/2, \tag{5.9}$$

for all $j \in J$.

Step 5: Now we go to the main part of the proof. Let $0 < \gamma_{\mathcal{U}} \leq \delta/2m$,

$$k_{\mathcal{S}} := \max\{k_{\mathcal{U}}, k_0\},\ \gamma_{\mathcal{S}} := \rho\gamma_{\mathcal{U}}^2/2R \tag{5.10}$$

and suppose that $\|Sx^k - x^k\| < \gamma_{\mathcal{S}}$ for some $k \geq k_{\mathcal{S}}$. Hence, by (5.6),

$$\|U_j(S_{j-1}x^k) - S_{j-1}x^k\| < \gamma_{\mathcal{U}}, \text{ for all } j \in J.$$

By (5.9), the definition of the metric projection and the triangle inequality, we get

$$\begin{aligned} d(x^k, \operatorname{Fix} U_j) &= \|x^k - P_{\operatorname{Fix} U_j} x^k\| \leq \|x^k - P_{\operatorname{Fix} U_j} S_{j-1} x^k\| \\ &\leq \sum_{l=1}^{j-1} \|S_l x^k - S_{l-1} x^k\| + \|S_{j-1} x^k - P_{\operatorname{Fix} U_j} S_{j-1} x^k\| \\ &< \sum_{l=1}^{m} \gamma_{\mathcal{U}} + \delta/2 \leq \delta. \end{aligned}$$

By the bounded regularity (see (5.7)) $d(x^k, \operatorname{Fix} S) < \eta$, which completes the proof. □

The following two Corollaries follow immediately from Theorems 5.4 and 5.6.

Corollary 5.7. *Let $U_i : \mathcal{H} \to \mathcal{H}$, $i \in I := \{1, \ldots, m\}$, be approximately shrinking cutters with a common fixed point. Let $\varepsilon \in (0, \frac{1}{m}]$ be arbitrary. If $\mathcal{F}$ is boundedly regular, then the family*

$$\mathcal{V} := \left\{ V = \sum_{i \in I} \omega_i U_{i,\alpha_i} \mid \alpha_i \in [\varepsilon, 2 - \varepsilon], \omega_i \in [\varepsilon, 1], i \in I, \sum_{i \in I} \omega_i = 1 \right\}$$

is uniformly approximately shrinking.

Corollary 5.8. *Let $U_i : \mathcal{H} \to \mathcal{H}$, $i \in I := \{1, \ldots, m\}$, be approximately shrinking cutters with a common fixed point. If $\mathcal{F}$ is boundedly regular and $A \subseteq [\varepsilon, 2 - \varepsilon]$, where $\varepsilon \in (0, 1)$, is a countable subset, then the family*

$$\mathcal{S} = \{S = U_{m,\alpha_m} \ldots U_{1,\alpha_1} \mid \alpha_1, \ldots, \alpha_m \in A\}$$

is uniformly approximately shrinking.

6. General method for VIP over the common fixed point set of quasi-nonexpansive operators

Let $U_i : \mathcal{H} \to \mathcal{H}$, $i \in I := \{1, 2, ..., m\}$, be quasi-nonexpansive with $C := \bigcap_{i \in I} \operatorname{Fix} U_i \neq \emptyset$ and $F : \mathcal{H} \to \mathcal{H}$ be κ-Lipschitz continuous and η-strongly monotone, where $0 < \eta \leq \kappa$. It is clear that C is a closed convex subset, because $\operatorname{Fix} U_i$, $i \in I$, are closed convex. Consider the following variational inequality problem, called VIP(F, C): Find $u^* \in C$ such that

$$\langle Fu^*, u - u^* \rangle \geq 0 \text{ for all } u \in C. \tag{6.1}$$

The existence and uniqueness of a solution of VIP(F, C) follows from [40, Theorem 46.C]. In this Section we will analyze the convergence properties of the following method, which we call a *generalized hybrid steepest descent* (GHSD) *method:*

$$\begin{array}{ll} u^0 \in \mathcal{H} & \text{– arbitrary} \\ u^{k+1} := T_k u^k - \lambda_k F T_k u^k, & \end{array} \tag{6.2}$$

where $\{T_k\}_{k=0}^{\infty}$ is a sequence of quasi-nonexpansive operators $T_k : \mathcal{H} \to \mathcal{H}$ satisfying $C \subseteq \bigcap_{k\geq 0} \operatorname{Fix} T_k$ and $\{\lambda_k\}_{k=0}^{\infty} \subseteq [0, 2\eta/\kappa^2]$ (cf. [10, Section 3]). Before we present our main result of this Section, we recall a definition of an s-approximately shrinking method (cf. [10, Definition 11]).

Definition 6.1. Let $T_k : \mathcal{H} \to \mathcal{H}$ be ρ_k-strongly quasi-nonexpansive, where $\rho_k \geq 0$, $k \geq 0$. We say that method (6.2) is *s-approximately shrinking* with respect to a subset $D \subseteq \mathcal{H}$ if for any $\eta > 0$ there are $\gamma > 0$ and $k_0 \geq 0$, such that for all $k \geq k_0$ it holds

$$\sum_{l=0}^{s-1} \rho_{k-l} \|T_{k-l}u^{k-l} - u^{k-l}\|^2 < \gamma \Longrightarrow d(u^k, D) < \eta. \tag{6.3}$$

We say that the method is approximately shrinking with respect to D if it is 1-approximately shrinking with respect to D.

The following result follows from [10, Lemma 9 and Theorem 12].

Theorem 6.2. *Let $\{u^k\}_{k=0}^{\infty}$ be generated by the GHSD method* (6.2). *If the following conditions are satisfied:*

(i) *the operators T_k are ρ_k-SQNE with $\rho_k \geq 0$, $k \geq 0$,*
(ii) $\lim_k \lambda_k = 0$ *and* $\sum_{k=0}^{\infty} \lambda_k = +\infty$,

then $\{u^k\}_{k=0}^{\infty}$ and $\{FT_k u^k\}_{k=0}^{\infty}$ are bounded. Moreover, if

(iii) *the method is s-approximately shrinking with respect to C for some $s \geq 1$,*

then $\{u^k\}_{k=0}^{\infty}$ converges in norm to a unique solution of VIP(F, C).

The most difficult problem in applications of Theorem 6.2 is to guarantee that the method is s-AS with respect to C. In the theorem below, which is the main theorem of this section, we present sufficient conditions for the method to be s-AS. These conditions correspond directly to the quasi-nonexpansive operators T_k, $k \geq 0$, and U_i, $i \in I$, and not to the method. Such conditions are in most applications easier to verify than the condition that the method is s-AS (see Section 7).

One can prove that the iteration in (6.2) is equivalent to the following one

$$u^{k+1} = (1 - \alpha_k)T_k u^k + \alpha_k Q T_k u^k, \tag{6.4}$$

where $Q : \mathcal{H} \to \mathcal{H}$ is a contraction. Recently, Hirstoaga in [28] proposed a method for solving VIP(F, C), where $C := \bigcap_{k\geq 0} \operatorname{Fix} T_k$, which is slightly more general than (6.4). Hirstoaga supposed that the method has the following property: if $\|u^{k+1} - T_k u^k\| \to 0$ then any weak cluster point of $\{u^k\}_{k=0}^{\infty}$ belongs to C. Furthermore, Hirstoaga gave several examples of methods satisfying this condition, where T_k, $k \geq 0$, are assumed to be nonexpansive (see [28, Section 3]). In Section 7 we will present, however, examples of method (6.2), where T_k, $k \geq 0$, are quasi-nonexpansive. In many applications these operators are simpler to evaluate than the nonexpansive ones with corresponding properties. As an example we give here the subgradient projection P_f relative to a convex function $f : \mathcal{H} \to \mathbb{R}$ defined on a finite dimensional Hilbert space $\mathcal{H}$ with $S(f, 0) := \{x \in \mathcal{H} \mid f(x) \leq 0\} \neq \emptyset$ (see Example 3.6). In general, $P_f x$ is simpler to evaluate than $P_{S(f,0)}x$ for $x \in \mathcal{H}$ with $f(x) > 0$.

Theorem 6.3. *Let $\{u^k\}_{k=0}^\infty$ be generated by method* (6.2) *and the following conditions be satisfied:*

(i) *the operators T_k are ρ_k-SQNE, $k \geq 0$, with $\rho := \inf_k \rho_k > 0$,*
(ii) $\lim_k \lambda_k = 0$ *and* $\sum_{k=0}^\infty \lambda_k = +\infty$,
(iii) *there is $s \geq m$, such that for any $i \in I$ and $k \geq s-1$, there is $l_{k,i} \in \{0, \dots, s-1\}$ with $\operatorname{Fix} T_{k-l_{k,i}} \subseteq \operatorname{Fix} U_i$,*
(iv) *the family $\mathcal{T} := \{T_k \mid k \geq 0\}$ is uniformly approximately shrinking,*
(v) *the family $\mathcal{F} := \{\operatorname{Fix} U_i \mid i \in I\}$ is boundedly regular.*

Then the method is s-approximately shrinking with respect to C, consequently $\{u^k\}_{k=0}^\infty$ converges in norm to a unique solution of VIP(F, C).

Proof. Let $\eta > 0$ be fixed and $\{u^k\}_{k=0}^\infty$ be a sequence generated by method (6.2). By Theorem 6.2, the sequences $\{u^k\}_{k=0}^\infty$ and $\{FT_k u^k\}_{k=0}^\infty$ are bounded. By the bounded regularity of $\mathcal{F}$, there are $\delta > 0$ and $k_1 \geq 0$, such that for any $k \geq k_1$it holds

$$\max_{i \in I} d(u^k, \operatorname{Fix} U_i) < \delta \Longrightarrow d(u^k, C) < \eta. \tag{6.5}$$

By assumption, $\mathcal{T}$ is uniformly AS, consequently, there are $\gamma_{\mathcal{T}} > 0$ and $k_{\mathcal{T}} \geq 0$ such that for all $k \geq k_{\mathcal{T}}$ it holds

$$\|T_k u^k - u^k\| < \gamma_{\mathcal{T}} \Longrightarrow d(u^k, \operatorname{Fix} T_k) < \delta/2. \tag{6.6}$$

Put $\gamma := \rho \min\{\gamma_{\mathcal{T}}^2, (\delta/4s)^2\}$ and $k_0 := \max\{k_{\mathcal{T}}, k_1, k_2\} + s$, where k_2 is such that for all $k \geq k_2$

$$\sum_{l=0}^{s-2} \lambda_{k-l} \|FT_{k-l} u^{k-l}\| < \delta/4.$$

The existence of k_2 follows from the assumption $\lim_k \lambda_k = 0$ and from the boundedness of $\{FT_k u^k\}_{k=0}^\infty$. Assume that $k \geq k_0$ and

$$\sum_{l=0}^{s-1} \rho_{k-l} \|T_{k-l} u^{k-l} - u^{k-l}\|^2 < \gamma. \tag{6.7}$$

Hence, $\|T_{k-l} u^{k-l} - u^{k-l}\| < \gamma_{\mathcal{T}}$, for all $l \in \{0, \dots, s-1\}$. In particular, this inequality holds for $l = l_{k,i}$ where $l_{k,i}$ is the smallest integer from $\{0, \dots, s-1\}$ such that $\operatorname{Fix} T_{k-l_{k,i}} \subseteq \operatorname{Fix} U_i$, $i \in I$. By (6.6), we have

$$d(u^{k-l_{k,i}}, \operatorname{Fix} T_{k-l_{k,i}}) < \delta/2. \tag{6.8}$$

Furthermore, (6.7) yields

$$\|T_{k-l} u^{k-l} - u^{k-l}\| < \frac{\delta}{4s}$$

for any $l = 0, 1, ..., s-1$. Now, for any $i \in I$ we have

$$\begin{aligned}\|u^k - u^{k-l_{k,i}}\| &\le \sum_{l=0}^{l_{k,i}-1} \|u^{k-l} - u^{k-l-1}\| \le \sum_{l=0}^{s-2} \|u^{k-l} - u^{k-l-1}\| \\ &\le \sum_{l=0}^{s-2} \|T_{k-l-1}u^{k-l-1} - u^{k-l-1}\| + \sum_{l=0}^{s-2} \lambda_{k-l-1}\|FT_{k-l-1}u^{k-l-1}\| \\ &< \delta/4 + \delta/4 = \delta/2.\end{aligned}$$

Consequently, (iii), the definition of the metric projection, the triangle inequality and (6.8) yield for any $i \in I$

$$\begin{aligned}\|u^k - P_{\operatorname{Fix} U_i} u^k\| &\le \|u^k - P_{\operatorname{Fix} T_{k-l_{k,i}}} u^k\| \le \|u^k - P_{\operatorname{Fix} T_{k-l_{k,i}}} u^{k-l_{k,i}}\| \\ &\le \|u^k - u^{k-l_{k,i}}\| + \|u^{k-l_{k,i}} - P_{\operatorname{Fix} T_{k-l_{k,i}}} u^{k-l_{k,i}}\| \\ &< \delta/2 + \delta/2 = \delta.\end{aligned}$$

By implication (6.5), we obtain $d(u^k, C) < \eta$, consequently, method (6.2) is s-approximately shrinking with respect to C. Now, Theorem 6.2 yields the convergence of u^k to a unique solution of VIP(F, C). □

7. Examples

In this Section we present several examples of GHSD method with various constructions of operators T_k basing on a finite family of operators and analyze their convergence properties by applying Theorem 6.3.

Let $U_i : \mathcal{H} \to \mathcal{H}$, $i \in I := \{1, 2, ..., m\}$, be quasi-nonexpansive with $C := \bigcap_{i \in I} \operatorname{Fix} U_i \neq \emptyset$ and $F : \mathcal{H} \to \mathcal{H}$ be κ-Lipschitz continuous and η-strongly monotone, where $0 < \eta \le \kappa$. We assume without loss of generality that all U_i, $i \in I$, are cutters, because any QNE operator $U : \mathcal{H} \to \mathcal{H}$ can be treated as a relaxation of a cutter $S : \mathcal{H} \to \mathcal{H}$ and $\operatorname{Fix} U = \operatorname{Fix} S$ (see Subsection 2.2). We consider the GHSD method (6.2) for solving VIP(F, C), where $\{T_k\}_{k=0}^{\infty}$ is a sequence of quasi-nonexpansive operators $T_k : \mathcal{H} \to \mathcal{H}$ satisfying $C \subseteq \bigcap_{k \ge 0} \operatorname{Fix} T_k$ and $\{\lambda_k\}_{k=0}^{\infty} \subseteq [0, 2\eta/\kappa^2]$ is a sequence satisfying $\lim_k \lambda_k = 0$ and $\sum_{k=1}^{\infty} \lambda_k = \infty$. Denote $\mathcal{U} := \{U_i \mid i \in I\}$ and $\mathcal{F} := \{\operatorname{Fix} U_i \mid i \in I\}$.

At the beginning we present a general procedure of a construction of operators T_k basing on the family $\mathcal{U}$. This procedures is related to the so called *string-averaging* scheme introduced by Censor et al. in [12] and studied in [13, 14, 15, 16]. The scheme has the following form.

Procedure 7.1 (String-Averaging Scheme)**.** Let $k \ge 0$ and $r \in \mathbb{N}$.

Step 1. For $p = 1, 2, ..., t_k$, where $t_k \le m$, choose nonempty, ordered subsets I_k^p of elements taken from I, called strings, of the form

$$I_k^p = (i_{k1}^p, i_{k2}^p, ..., i_{k,m_k^p}^p),$$

where $i_{kj}^p \in I$, $j = 1, 2, ..., m_k^p$ and $m_k^p \leq r$. Further, choose vectors a_k^p of corresponding relaxation parameters of the form

$$a_k^p = (\alpha_{k1}^p, \alpha_{k2}^p, ..., \alpha_{k,m_k^p}^p),$$

where $\alpha_{kj}^p \in [\varepsilon, 2-\varepsilon]$, $j = 1, 2, ..., m_k^p$, and $\varepsilon \in (0, 1]$ is a constant.

Step 2. For $p = 1, 2, ..., t_k$, and $j = 1, 2, ..., m_k^p$, denote by V_{kj}^p the relaxation of $U_{i_{kj}^p}$ with a relaxation parameter α_{kj}^p, i.e.,

$$V_{kj}^p := \mathrm{Id} + \alpha_{kj}^p (U_{i_{kj}^p} - \mathrm{Id}),$$

and create operators

$$S_{kj}^p := V_{kj}^p V_{k,j-1}^p ... V_{k1}^p,$$

and

$$V_k^p := S_{k,m_k^p}^p = V_{k,m_k^p}^p V_{k,m_k^p-1}^p ... V_{k1}^p \tag{7.1}$$

Step 3. For $p = 1, 2, ..., t_k$ choose weights $\omega_k^p \geq \omega$ satisfying $\sum_{p=1}^{t_k} \omega_k^p = 1$, where $\omega \in (0, \frac{1}{m}]$ is a constant, and define

$$T_k := \sum_{p=1}^{t_k} \omega_k^p V_k^p. \tag{7.2}$$

In the original string-averaging scheme there is supposed that the strings do not depend of k, i.e., $t_k = t$ and $m_k^p = m^p$ for all k, and that $\alpha_{kj}^p = 1$, $j = 1, 2, ..., m^p$, $p = 1, 2, ..., t$, consequently, $V_k^p = V^p$ and $T_k = T$ (cf. [13]).

First, we present some properties of the operators defined above, basing on the facts given in Subsection 2.2. It follows from Proposition 2.2 that for all $k \geq 0$, $p = 1, 2, ..., t_k$ and $j = 1, 2, ..., m_k^p$ the operators defined in Procedure 7.1 have the following properties:

(1) V_{kj}^p is $\frac{\varepsilon}{2}$-SQNE because V_{kj}^p is an α_{kj}^p-relaxation of a cutter $U_{i_{kj}^p}$ and $\varepsilon \leq \alpha_{kj}^p \leq 2$;
(2) S_{kj}^p is $\frac{\varepsilon}{2r}$-SQNE because S_{kj}^p is a composition of j operators V_{ki}^p, $i = 1, 2, ..., j$, which are $\frac{\varepsilon}{2}$-SQNE and $j \leq m_k^p \leq r$;
(3) V_k^p is $\frac{\varepsilon}{2r}$-SQNE because $V_k^p = S_{k,m_k^p}^p$;
(4) T_k is $\frac{\varepsilon}{2r}$-SQNE because T_k is a convex combination of operators V_k^p which are $\frac{\varepsilon}{2r}$-SQNE.

Consider the GHSD method combined with the string-averaging scheme of the construction of operators T_k. This approach gives a very flexible framework related to a lot of results known in literature.

Because the operators T_k are ρ_k-SQNE, with $\rho := \inf_k \rho_k > 0$, Theorem 6.3 implies that only conditions (iii), (iv) and (v) have to be checked in order to guarantee the convergence of sequences generated by the GHSD method. In the examples below we present several constructions of T_k satisfying these conditions, which are special cases of the string-averaging scheme.

Example 7.2. (*Hybrid steepest descent method*) Let $m = 1$. Then the family $\mathcal{U}$ consists of only one operator $U : \mathcal{H} \to \mathcal{H}$ which is a cutter. Trivially $\mathcal{F}$ is BR. Assume that U is AS. GHSD method (6.2) combined with Procedure 7.1 has the form:

$$u^{k+1} = U_{\alpha_k} u^k - \lambda_k F U_{\alpha_k} u^k,$$

where U_{α_k} is an α_k-relaxation of a cutter U with $\alpha_k \in [\varepsilon, 2-\varepsilon]$ and $\varepsilon \in (0,1]$. Condition (iii) of Theorem 6.3 is trivial and condition (iv) follows from Lemma 5.3. Consequently, u^k converges in norm to a unique solution of the VIP$(F, \operatorname{Fix} U)$. If we take $\alpha_k = \alpha \in [0,2]$, $k \geq 0$, then $T := U_\alpha$ is quasi-nonexpansive and we obtain the hybrid steepest descent method

$$u^{k+1} = Tu^k - \lambda_k F T u^k$$

studied by Yamada in [38] and developed by Yamada and Ogura in [39]. In [38] T is supposed to be nonexpansive and the convergence was proved under an additional assumption $\lim_k (\lambda_k - \lambda_{k+1})/\lambda_{k+1}^2 = 0$ (see [38, Theorem 3.2]). Recall that if $\mathcal{H}$ is finite dimensional, then a nonexpansive operator is approximately shrinking (see Corollary 4.2). In [39, Theorem 4] the convergence was proved under the assumption that T is quasi-shrinking. This assumption implies that T is approximately shrinking (see Proposition 4.4). Therefore, for $\alpha \in (0,2)$ the convergence follows from Theorem 6.3.

In the next examples we assume that $m > 1$. Furthermore, we assume that U_i, $i \in I$, are approximately shrinking and the family $\mathcal{F}$ is boundedly regular.

Example 7.3. (*GHSD with successive projections*) Consider GHSD method (6.2) combined with Procedure 7.1, where $t_k = 1$, $m_k := m_k^1 = 1$, $I_k := I_k^1 = i_k$ and $\alpha_k := \alpha_{k1}^1$, $k \geq 0$. The method has the form

$$u^{k+1} = U_{i_k,\alpha_k} u^k - \lambda_k F U_{i_k,\alpha_k} u^k.$$

Condition (iv) of Theorem 6.3 follows from Lemma 5.3, because $\alpha_k \in [\varepsilon, 2-\varepsilon]$, $k \geq 0$, for a constant $\varepsilon \in (0,1]$. The sequence $\{i_k\}_{k=0}^\infty$ is called a control. We consider two types of control:

(a) (*almost cyclic control*) Suppose that there is $s \geq m$ such that $I \subseteq \{i_k, \ldots, i_{k+s-1}\}$ for all $k \geq 0$. Such a control $\{i_k\}_{k=0}^\infty$ is called s-almost cyclic (cf. [11, Definition 3.4]). Then condition (iii) of Theorem 6.3 is satisfied. Consequently, u^k converges in norm to a unique solution of VIP(F, C). The convergence of the method was proved in [10, Theorem 21]. A special case of the above iteration with a cyclic control and T_k being nonexpansive was studied by Lions [30, Theorem 4], Bauschke [1, Theorem 3.1], Yamada in [38, Theorem 3.3], Xu and Kim in [37, Theorem 3.2] and Hirstoaga in [28, Corollary 3.1], where the convergence was proved under various assumptions on the sequence $\{\lambda_k\}_{k=0}^\infty$ and the family $\mathcal{F}$.

(b) (*maximal displacement control*) Let $i_k := \operatorname{argmax}\{\|U_i u^k - u^k\| \mid i \in I\}$. We show that the method is approximately shrinking with respect to C. Let $\eta > 0$. Because $\mathcal{F}$ is boundedly regular and $\{u^k\}_{k=0}^\infty$ is bounded (see Theorem

6.2), there are $\delta > 0$ and $k_0 \geq 0$ such that

$$\max_{i\in I} d(u^k, \operatorname{Fix} U_i) < \delta \Longrightarrow d(u^k, C) < \eta \tag{7.3}$$

for all $k \geq k_0$. Because U_i is approximately shrinking, there is $\gamma_i > 0$ and $k_i \geq 0$ such that for all $k \geq k_i$ it holds

$$\left\|U_i u^k - u^k\right\| < \gamma_i \Longrightarrow d(u^k, \operatorname{Fix} U_i) < \delta, \tag{7.4}$$

$i \in I$. Let $\gamma = \min_{i\in i} \gamma_i$ and $k' = \max_{i\in\{0,1,...,m\}} k_i$. Suppose that $k \geq k'$ and $\left\|U_{i_k} u^k - u^k\right\| < \gamma$. Then, $\left\|U_i u^k - u^k\right\| < \gamma_i$ and (7.3) and (7.4) imply $d(u^k, C) < \eta$. Therefore, the method is approximately shrinking. Theorem 6.2 yields that u^k converges to a unique solution of $\mathrm{VIP}(F, C)$.

Example 7.4. (*GHSD with simultaneous projections*) Consider GHSD method (6.2) combined with Procedure 7.1, where $t_k = m$, $m_k^p = 1$, $I_k^p = p$ and $a_k^p = \alpha_k$, $p = 1, 2, ..., m$, $k \geq 0$. The method has the form

$$T_k := \mathrm{Id} + \alpha_k (\sum_{p=1}^{m} \omega_k^p U_p - \mathrm{Id}). \tag{7.5}$$

By Proposition 2.2, $\operatorname{Fix} T_k = C$, $k \geq 0$, i.e., condition (iii) of Theorem 6.3 is satisfied. Condition (iv) follows from Corollary 5.7. Consequently, u^k converges in norm to a unique solution of $\mathrm{VIP}(F, C)$. The above construction of the operator T_k arises very often in the literature, not only related to the variational inequalities. For $\mathrm{VIP}(F, C)$ the method with U_i, $i \in I$, being firmly nonexpansive and with constant weight vectors $w_k = w$ and relaxation parameters $\alpha_k = \alpha$ was studied by Yamada in [38, Corollary 3.6], where $\alpha = 1$ and by Hirstoaga in [28, Remark 3.5], where $\alpha \in [0, 2]$.

Example 7.5. (*GHSD with sequential projections*) Consider GHSD method (6.2) combined with Procedure 7.1, where $t_k = 1$, $m_k := m_k^1 = m$ and $I_k := I_k^1$ consists of all elements of I, $k \geq 0$. Denote by $I_k := (i_{k1}, i_{k2}, ..., i_{km})$ the string in k-th iteration and by $a_k := (\alpha_{k1}, \alpha_{k2}, ..., \alpha_{km})$ the corresponding vector of relaxation parameters. Then,

$$T_k = U_{i_{km}, \alpha_{km}} U_{i_{k,m-1}, \alpha_{k,m-1}} ... U_{i_{k1}, \alpha_{k1}}$$

Similarly as in Example 7.4, condition (iii) of Theorem 6.3 is satisfied and condition (iv) follows from Corollary 5.8. Therefore, u^k converges in norm to a unique solution of $\mathrm{VIP}(F, C)$. Such sequential approach for VIP was studied by Yamada in [38, Theorem 3.5] and by Hirstoaga in [28, Corollary 3.4 and Remark 3.5], where U_i were supposed to be nonexpansive and $T_k = U_m U_{m-1} ... U_1$, where the convergence was proved under some additional assumptions on the sequence $\{\lambda_k\}_{k=0}^{\infty}$ and the family $\mathcal{F}$.

Although, I_k^p is an ordered subset in Procedure 7.1, by the same symbol we denote the corresponding subset of I, i.e., $I_k^p = \{i_{k1}^p, i_{k2}^p, ..., i_{k,m_k^p}^p\}$, $k \geq 0$, $p = 1, 2, ..., t_k$. Further, denote $I_k := I_k^1 \cup I_k^2 \cup ... \cup I_k^{t_k}$.

Let $\{T_k\}_{k=0}^{\infty}$ be a sequence of operators defined by Procedure 7.1 and suppose that the procedure is s-intermittent, where $s \geq m$, i.e., $I_k \cup I_{k+1} \cup ... \cup I_{k+s-1} = I$ for all $k \geq 0$ (cf. [3, Definition 3.18]). If we assume that all subfamilies of $\mathcal{F}$ are boundedly regular, then, by Theorems 5.4 and 5.6, the family $\mathcal{T} := \{T_k \mid k \geq 0\}$ is uniformly approximately shrinking. In this case the sequence $\{T_k\}_{k=0}^{\infty}$ fits into the frame of Theorem 6.3, which allows a more general construction of T_k than the string-averaging scheme. Consequently, the sequence $\{u^k\}_{k=0}^{\infty}$ generated by the GHSD method converges in norm to a unique solution of $\mathrm{VIP}(F, C)$. Unfortunately, the assumption that all subfamilies of $\mathcal{F}$ are boundedly regular is very restrictive. This assumption can be weakened, if we restrict the construction of T_k to the string-averaging scheme.

Theorem 7.6. *Let $U_i : \mathcal{H} \to \mathcal{H}$, $i \in I := \{1, 2, ..., m\}$, be approximately shrinking cutters with $C := \bigcap_{i \in I} \operatorname{Fix} U_i \neq \emptyset$, and u^k be generated by GHSD method* (6.2)*, where T_k, $k \geq 0$, is constructed by the string-averaging scheme (Procedure* 7.1*). If Procedure* 7.1 *is s-intermittent for some $s \geq m$ and the family $\mathcal{F}$ is boundedly regular, then the GHSD method is s-approximately shrinking with respect to C. Consequently u^k converges in norm to a unique solution of $VIP(F, C)$.*

Proof. Let Procedure 7.1 be s-intermittent for some $s \geq m$ and let $\eta > 0$. We have to show that there exist $\gamma > 0$ and $k_0 \geq 0$ such that for all $k \geq k_0$ it holds $d(u^k, C) < \eta$ whenever $\sum_{l=k-s+1}^{k} \rho_l \|T_l u^l - u^l\|^2 < \gamma$. By Theorem 6.2, the sequences $\{u^k\}_{k=0}^{\infty}$ and $\{F T_k u^k\}_{k=0}^{\infty}$ are bounded. We divide the proof into three steps.

Step 1. In this step we present three auxiliary properties of Procedure 7.1.

(a) Let $R > 0$ be such that $\left\|u^k - z\right\| \leq R$ for some $z \in \bigcap_{i \in I} \operatorname{Fix} U_i$ and for all $k \geq 0$. Let $\delta_0 > 0$ and $k_1 \geq 0$ be such that for any $k \geq k_1$ it holds

$$\max_{i \in I} d(u^k, \operatorname{Fix} U_i) < \delta_0 \Longrightarrow d(u^k, C) < \eta. \tag{7.6}$$

The existence of δ_0 and k_1 follows from the bounded regularity of $\mathcal{F}$.

(b) Let $i \in I$ be arbitrary. Let $k \geq s - 1$ and $l_k \in \{k - s + 1, k - s + 2, ..., k\}$, $p_k \in \{1, 2, ..., t_{l_k}\}$ and $j_k \in \{1, 2, ..., m_{l_k}^{p_k}\}$ be such that $U_i = U_{i_{l_k, j_k}^{p_k}}$. The existence of l_k, p_k and j_k follows from the assumption that Procedure 7.1 is s-intermittent. Because $S_{l_k, j_k - 1}^{p_k}$ is QNE as a composition of strongly quasi-nonexpansive operators having a common fixed point (see Proposition 2.2) and u^k is bounded, we have that $v^k := S_{l_k, j_k - 1}^{p_k} u^{l_k}$ is bounded. Note that

$$\begin{aligned} y^k &:= S_{l_k, j_k}^{p_k} u^{l_k} = V_{l_k, j_k}^{p_k}(v^k) = V_{l_k, j_k}^{p_k}(S_{l_k, j_k - 1}^{p_k} u^{l_k}) \\ &= U_{i, \alpha_{l_k, j_k}^{p_k}}(S_{l_k, j_k - 1}^{p_k} u^{l_k}) = U_{i, \alpha_{l_k, j_k}^{p_k}}(v^k), \end{aligned} \tag{7.7}$$

where $\alpha_{l_k, j_k}^{p_k} \geq \varepsilon$. Because U_i are approximately shrinking, the family $\{U_{i,\alpha} \mid \alpha \in [\varepsilon, 1], i \ \in I\}$ is UAS (see Lemma 5.3). Consequently, there are $\gamma_{\mathcal{U}} > 0$ and $k_{\mathcal{U}} \geq 0$ such that for all $k \geq k_{\mathcal{U}}$ it holds

$$\left\|U_{i,\alpha} v^k - v^k\right\| < \gamma_{\mathcal{U}} \Longrightarrow d(v^k, \operatorname{Fix} U_i) < \delta_0/4. \tag{7.8}$$

Let $\delta := \min\{\gamma_{\mathcal{U}}, \delta_0\}$. Recall that $V_{l_k, j_k}^{p_k} = U_{i, \alpha_{l_k, j_k}^{p_k}}$ and $\alpha_{l_k, j_k}^{p_k} \geq \varepsilon$. Therefore,

$$\left\|U_{i, \alpha_{l_k, j_k}^{p_k}} v^k - v^k\right\| < \delta \Longrightarrow d(v^k, \operatorname{Fix} U_i) < \delta_0/4. \tag{7.9}$$

(c) Let $k_2 \geq 0$ be such that for all $k \geq k_2$ it holds

$$\sum_{l=k-s+1}^{k} \lambda_l \|FT_l u^l\| < \delta/4. \tag{7.10}$$

The existence of k_2 follows from the assumption $\lim_k \lambda_k = 0$ and from the boundedness of $\{FT_k u^k\}_{k=0}^{\infty}$.

Step 2. We show the following inequality

$$\sum_{l=k-s+1}^{k} \left\|T_l u^l - u^l\right\| \geq \frac{\omega\varepsilon^3}{2^4 R^3 r} \sum_{l=k-s+1}^{k} \sum_{p=1}^{t_l} \sum_{j=1}^{m_l^p} \left\|V_{lj}^p(S_{l,j-1}^p u^l) - S_{l,j-1}^p u^l\right\|^4. \tag{7.11}$$

By Propositions 4.5, 4.6 and properties 1-4 above, we have

$$\begin{aligned}
\sum_{l=k-s+1}^{k} \left\|T_l u^l - u^l\right\| &\geq \sum_{l=k-s+1}^{k} \frac{1}{2R} \sum_{p=1}^{t_l} \omega_l^p \frac{\varepsilon}{2r} \left\|V_l^p u^l - u^l\right\|^2 \\
&\geq \sum_{l=k-s+1}^{k} \frac{1}{2R} \sum_{p=1}^{t_l} \omega_l^p \frac{\varepsilon}{2r} \left(\frac{1}{2R} \sum_{j=1}^{m_l^p} \frac{\varepsilon}{2} \left\|S_{lj}^p u^l - S_{l,j-1}^p u^l\right\|^2\right)^2 \\
&\geq \frac{\omega\varepsilon^3}{2^4 R^3 r} \sum_{l=k-s+1}^{k} \sum_{p=1}^{t_l} \sum_{j=1}^{m_l^p} \left\|V_{lj}^p(S_{l,j-1}^p u^l) - S_{l,j-1}^p u^l\right\|^4.
\end{aligned}$$

Step 3. Now we go the main part of the proof. Put

$$\gamma := \min\{\rho(\frac{\delta}{4s})^2, \rho(\frac{\delta\varepsilon}{8Rr})^8(\frac{\omega R}{\varepsilon rs})^2\} \tag{7.12}$$

and $k_0 := \max\{k_1, k_2, k_{\mathcal{U}}, s-1\}$. Assume that $k \geq k_0$ and

$$\sum_{l=k-s+1}^{k} \rho_l \|T_l u^l - u^l\|^2 < \gamma. \tag{7.13}$$

Then, for any $l \in \{k-s+1, k-s+2..., k\}$, we have

$$\sum_{l=k-s+1}^{k} \|T_l u^l - u^l\| < s(\gamma/\rho)^{\frac{1}{2}} \leq \frac{\delta}{4} \tag{7.14}$$

and by (7.11) and (7.12) it holds

$$\left\|V_{lj}^p(S_{l,j-1}^p u^l) - S_{l,j-1}^p u^l\right\| < \frac{\delta}{4r} \leq \delta, \tag{7.15}$$

for any $l \in \{k-s+1, k-s+2, ..., k\}$, $p \in \{1, 2, ..., t_l\}$ and $j \in \{1, 2, ..., m_l^p\}$. Now, (7.9), (7.7), the quasi nonexpansivity of $V_{l_k,j_k}^{p_k}$ and (7.8) yield

$$\begin{aligned}
\left\|y^k - P_{\operatorname{Fix} U_i} y^k\right\| &\leq \left\|y^k - P_{\operatorname{Fix} U_i} v^k\right\| = \left\|V_{l_k,j_k}^{p_k} v^k - P_{\operatorname{Fix} U_i} v^k\right\| \\
&\leq \left\|v^k - P_{\operatorname{Fix} U_i} v^k\right\| < \frac{\delta_0}{4}.
\end{aligned} \tag{7.16}$$

By the triangle inequality, (7.14), (7.10) and (7.15), we have

$$\begin{aligned}
\left\|u^k - y^k\right\| &= \left\|u^k - S^{p_k}_{l_k j_k} u^{l_k}\right\| \le \sum_{l=l_k-1}^{k-1} \left\|u^{l+1} - u^l\right\| + \left\|u^{l_k} - S^{p_k}_{l_k j_k} u^{l_k}\right\| \\
&\le \sum_{l=k-s+1}^{k-1} \left\|T_l u^l + \lambda_l F T_l u^l - u^l\right\| + \left\|u^{l_k} - S^{p_k}_{l_k j_k} u^{l_k}\right\| \\
&\le \sum_{l=k-s+1}^{k-1} \left\|T_l u^l - u^l\right\| + \sum_{l=k-s+1}^{k-1} \lambda_l \left\|F T_l u^l\right\| \\
&+ \sum_{j=0}^{j_k-1} \left\|S^{p_k}_{l_k,j+1} u^{l_k} - S^{p_k}_{l_k j} u^{l_k}\right\| \\
&= \underbrace{\sum_{l=k-s+1}^{k-1} \left\|T_l u^l - u^l\right\|}_{<\delta/4} + \underbrace{\sum_{l=k-s+1}^{k-1} \lambda_l \left\|F T_l u^l\right\|}_{<\delta/4} \\
&+ \underbrace{\sum_{j=0}^{j_k-1} \left\|V^{p_k}_{l_k,j+1}(S^{p_k}_{l_k j} u^{l_k}) - S^{p_k}_{l_k j} u^{l_k}\right\|}_{<\delta/4} \\
&< \frac{3}{4}\delta. \qquad (7.17)
\end{aligned}$$

Finally, the triangle inequality, (7.16) and (7.17) yield

$$\begin{aligned}
\left\|u^k - P_{\mathrm{Fix}\,U_i} u^k\right\| &= \left\|u^k - P_{\mathrm{Fix}\,U_i} y^k\right\| \\
&\le \left\|u^k - y^k\right\| + \left\|y^k - P_{\mathrm{Fix}\,U_i} y^k\right\| \\
&< \frac{3}{4}\delta + \frac{1}{4}\delta_0 \le \delta_0. \qquad (7.18)
\end{aligned}$$

Note that the choice of δ_0 and k_1 does not depend on i.
Therefore, $\max_{i\in I} d(u^k, \mathrm{Fix}\, U_i) \le \delta_0$ and, by (7.6), $d(u^k, C) < \eta$. We proved that the method is s-approximately shrinking with respect to C. By Theorem 6.2, $\{u^k\}_{k=0}^{\infty}$ converges to a unique solution of VIP(F, C). □

Note added in proof. We became aware of the paper [17] by Y. Censor and A.J. Zaslavski in which a string-averaging projected subgradient method is studied for constrained minimization problems. Although formulated in a different setting and employing different assumptions in its analysis, this paper is closely related to the results presented here.

Acknowledgment. Second author was financially supported by the Polish National Science Centre within the framework of Etiuda funding scheme under agreement No. DEC-2013/08/T/ST1/00177.

8. Appendix

Let $S \subseteq \mathcal{H}$ be a nonempty subset.

Lemma 8.1. *Let $f, g \colon \mathcal{H} \to [0, \infty)$ be given functions. Then the following conditions are equivalent:*

(i) *For any sequence $\{x^k\}_{k=0}^{\infty} \subseteq S$ it holds*

$$\lim_{k\to\infty} f(x^k) = 0 \Longrightarrow \lim_{k\to\infty} g(x^k) = 0.$$

(ii) *For any sequence $\{x^k\}_{k=0}^{\infty} \subseteq S$ and for any $\eta > 0$ there are $\gamma > 0$ and $k_0 \geq 0$ such that for all $k \geq k_0$ it holds*

$$f(x^k) < \gamma \Longrightarrow g(x^k) < \eta.$$

(iii) *For any $\eta > 0$ there is $\gamma > 0$ such that for any $x \in S$ it holds*

$$f(x) < \gamma \Longrightarrow g(x) < \eta.$$

Proof. By the definition of a limit, condition (i) is equivalent to the following one: For any sequence $\{x^k\}_{k=0}^{\infty} \subseteq S$ it holds:

$$\big(\forall_{\gamma>0}\exists_{k_1\geq 0}\forall_{k\geq k_1} f(x^k) < \gamma\big) \Longrightarrow \big(\forall_{\eta>0}\exists_{k_2\geq 0}\forall_{k\geq k_2} g(x^k) < \eta\big)$$

(iii)⇒(ii) The implication is obvious.
(ii)⇒(i) Let $\{x^k\}_{k=0}^{\infty} \subseteq S$. Suppose that

$$\forall_{\gamma>0}\exists_{k_1\geq 0}\forall_{k\geq k_1} f(x^k) < \gamma.$$

Let $\eta > 0$ be arbitrary. Let $\gamma > 0$ and $k_0 \geq 0$ be such that

$$\forall_{k\geq k_0}(f(x^k) < \gamma \Longrightarrow g(x^k) < \eta). \tag{8.1}$$

The existence of such γ and k_0 follows from (ii). Let $k_1 \geq 0$ be such that

$$\forall_{k\geq k_1} f(x^k) < \gamma. \tag{8.2}$$

Let $k_2 = \max\{k_0, k_1\}$. Then (8.2) yields that for all $k \geq k_2$ it holds $f(x^k) < \gamma$, consequently, (8.1) yields that $g(x^k) < \eta$.

(i)⇒(iii) Assume, to the contrary, that there is $\eta > 0$ such that

$$\forall_{\gamma>0}\exists_{x\in S}\left(f(x) < \gamma \text{ and } g(x) \geq \eta\right).$$

Put $\gamma_k := \frac{1}{k+1}$ and let $x^k \in S$ be such that

$$f(x^k) < \gamma_k \text{ and } g(x^k) \geq \eta.$$

Note that $\lim_{k\to\infty} f(x^k) = 0$ and, by (i), $\lim_{k\to\infty} g(x^k) = 0$, which is in contradiction with $\liminf_{k\to\infty} g(x^k) \geq \eta > 0$. □

References

[1] H.H. Bauschke, *The approximation of fixed points of composition of nonexpansive mappings in Hilbert space*, J. Math. Anal. Appl., **202**(1996), 150–159.

[2] H.H. Bauschke, J. Borwein, *On the convergence of von Neumann's alternating projection algorithm for two sets*, Set-Valued Analysis, **1**(1993), 185–212.

[3] H.H. Bauschke, J. Borwein, *On projection algorithms for solving convex feasibility problems*, SIAM Review, **38**(1996), 367–426.

[4] H.H. Bauschke, P.L. Combettes, *A weak-to-strong convergence principle for Fejér-monotone methods in Hilbert spaces*, Mathematics of Operation Research, **26**(2001), 248–264.

[5] C. Byrne, *A unified treatment of some iterative algorithms in signal processing and image reconstruction*, Inverse Problems, **20**(2004), 103–120.

[6] A. Cegielski, *A generalization of the Opial's theorem*, Control and Cybernetics, **36**(2007), 601–610.

[7] A. Cegielski, *Iterative Methods for Fixed Point Problems in Hilbert Spaces*, Springer, Heidelberg, 2012.

[8] A. Cegielski, Y. Censor, *Opial-type theorems and the common fixed point problem*, Fixed-Point Algorithms for Inverse Problems in Science and Engineering, (Eds. H. H. Bauschke, R. S. Burachik, P. L. Combettes, V. Elser, D. R. Luke, H. Wolkowicz), Springer Optimization and Its Applications **49**, New York, NY, USA, 2011, 155–183.

[9] A. Cegielski, A. Gibali, S. Reich, R. Zalas, *An algorithm for solving the variational inequality problem over the fixed point set of a quasi-nonexpansive operator in Euclidean space*, Numer. Funct. Anal. Optimiz., **34**(2013), 1067-1096.

[10] A. Cegielski, R. Zalas, *Methods for variational inequality problem over the intersection of fixed point sets of quasi-nonexpansive operators*, Numer. Funct. Anal. Optimiz., **34**(2013), 255-283.

[11] Y. Censor, *Row-action methods for huge and sparse systems and their applications*, SIAM Review, **23**(1981), 444–466.

[12] Y. Censor, T. Elfving, G.T. Herman, *Averaging strings of sequential iterations for convex feasibility problems*, Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications, (Eds. D. Butnariu, Y. Censor, S. Reich), Elsevier, Amsterdam, 2001, pp. 101-114.

[13] Y. Censor, A. Segal, *On the string averaging method for sparse common fixed points problems*, International Transactions in Operational Research, **16**(2009), 481-494.

[14] Y. Censor, A. Segal, *Sparse string-averaging and split common fixed points*, Contemporary Mathematics, **513**(2010), 125-142.

[15] Y. Censor, E. Tom, *Convergence of string-averaging projection schemes for inconsistent convex feasibility problems*, Optimization Methods and Software, **18**(2003), 543-554.

[16] Y. Censor, A.J. Zaslavski, *Convergence and perturbation resilience of dynamic string-averaging projection methods*, Computational Optimization and Applications, **54**(2013), 65-76.

[17] Y. Censor, A.J. Zaslavski, *String-averaging projected subgradient methods for constrained minimization*, Optimization Methods and Software, 29(2014), 658ń670.

[18] P.L. Combettes, *Hilbertian convex feasibility problem: Convergence of projection methods*, Appl. Math. Optim., **35**(1997), 311–330.

[19] P.L. Combettes, *Quasi-Fejérian analysis of some optimization algorithms*, Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications, (Eds. D. Butnariu, Y. Censor, S. Reich), Elsevier, Amsterdam, 2001, 115–152.

[20] F. Deutsch, *Best Approximation in Inner Product Spaces*, Springer, New York, 2001.

[21] I. Ekeland, R. Témam, *Convex Analysis and Variational Problems*, North-Holland Publishing Company, Amsterdam, 1976.

[22] F. Facchinei, J.S. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems,* Vol. I, II, Springer, New York, 2003.

[23] K. Goebel, W.A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge University Press, Cambridge, 1990.

[24] K. Goebel, S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, Marcel Dekker, New York and Basel, 1984.

[25] A.A. Goldstein, *Convex programming in Hilbert space*, Bull. Am. Math. Soc., **70**(1964), 709–710.

[26] L.G. Gurin, B.T. Polyak, E.V. Raik, *The method of projection for finding the common point in convex sets*, Zh. Vychisl. Mat. i Mat. Fiz., **7**(1967), 1211–1228 (in Russian). English translation in: USSR Comput. Math. Phys., **7**(1967), 1–24.

[27] B. Halpern, *Fixed points of nonexpanding maps*, Bull. Amer. Math. Soc., **73**(1967), 957–961.

[28] S.A. Hirstoaga, *Iterative selection methods for common fixed point problems*, J. Math. Anal. Appl., **324**(2006), 1020–1035.

[29] D. Kinderlehrer, G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, Academic Press, New York, 1980.

[30] P.-L. Lions, *Approximation de points fixes de contractions*, C. R. Acad. Sci. Paris Sér. A, **284**(1977), 1357–1359.

[31] St. Măruşter, C. Popîrlan, *On the Mann-type iteration and the convex feasibility problem*, J. Comput. Appl. Math., **212**(2008), 390–396.

[32] St. Măruşter, C. Popîrlan, *On the regularity condition in a convex feasibility problem*, Nonlinear Anal., **70**(2009), 1923–1928.

[33] N. Ogura, I. Yamada, *Non-strictly convex minimization over the fixed point set of an asymptotically shrinking nonexpansive mapping*, Numer. Funct. Anal. and Optimiz., **23**(2002), 113–137.

[34] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc., **73**(1967), 591–597.

[35] D. Schott, *A general iterative scheme with applications to convex optimization and related fields*, Optimization, **22**(1991), 885–902.

[36] R. Wittmann, *Approximation of fixed points of nonexpansive mappings*, Arch. Math., **58**(1992), 486–491.

[37] H.K. Xu, T.H. Kim, *Convergence of hybrid steepest-descent methods for variational inequalities*, JOTA, **119**(2003), 185–201.

[38] I. Yamada, *The hybrid steepest descent method for the variational inequality problem over the intersection of fixed point sets of nonexpansive mappings*, Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications, (Eds. D. Butnariu, Y. Censor, S. Reich), Elsevier, Amsterdam, 2001, 473–504.

[39] I. Yamada, N. Ogura, *Hybrid steepest descent method for variational inequality problem over the fixed point set of certain quasi-nonexpansive mapping*, Numer. Funct. Anal. Optimiz., **25**(2004), 619–655.

[40] E. Zeidler, *Nonlinear Functional Analysis and Its Applications, III – Variational Methods and Optimization*, Springer-Verlag, New York, 1985.

Received: March 03, 2013; Accepted: March 12, 2013