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TWO ORDER THEORETIC PROOFS FOR A FIXED POINT THEOREM IN PARTIALLY ORDERED METRIC SPACES AND ITS APPLICATION TO TRACE CLASS OPERATOR EQUATIONS

FAEZEH BAZAZI, KOUROSH NOUROUZI* AND DONAL O'REGAN**

* Faculty of Mathematics, K.N. Toosi University of Technology
P.O.Box 16315-1618, Tehran, Iran
E-mail: nourouzi@kntu.ac.ir

**School of Mathematics, Statistics and Applied Mathematics
National University of Ireland, Galway, University Road
Galway, Ireland

Abstract. In the present paper, we first give two order theoretic proofs for a known fixed point theorem in partially ordered metric spaces and then apply the results to find solutions of trace class operator equations.
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1. Introduction

Ran and Reurings [8] proved the following Banach type principle in ordered metric spaces.

Theorem 1.1. *Let X be a partially ordered set (poset) such that every pair $x, y \in X$ has a lower and an upper bound and let d be a complete metric on X. Let F be a continuous and increasing selfmap on X. Suppose further that:*

(1) *there exists $k \in [0, 1)$ such that $d(Fx, Fy) \leq kd(x, y)$ for each $x, y \in X$ with $x \preceq y$;*
(2) *there exists $b \in X$ such that $b \preceq Fb$.*

Then F has a (unique) fixed point.

The problem of existence of a fixed point for contractive type selfmaps on posets has been investigated by several authors. In [6] a stronger variant of Theorem 1.1 was given by removing the continuity of the selfmap F.

Theorem 1.2. ([6]; *Theorem 2.3) Let $(X, \preceq)$ be a poset and d be a complete metric on X. Let F be an increasing selfmap on X. Suppose that the following three assertions hold:*

(1) *there exists* $k \in [0,1)$ *such that* $d(Fx, Fy) \leq kd(x,y)$ *for each* $x, y \in X$ *with* $x \preceq y$*;*
(2) *there exists* $b \in X$ *such that* $b \preceq Fb$*;*
(3) *if an increasing net* $(x_\alpha)_{\alpha\in I}$ *converges to* x *in* X*, then* $x_\alpha \preceq x$ *for all* α*.*

Then F *has a (unique) fixed point.*

In this paper we aim to give an order theoretic proof of Theorem 1.2. We will also show that a slightly different version of Theorem 1.2 can be derived from the Knaster-Tarski theorem. Finally, in the last section, we will use the idea given in [8] for matrix equations to apply the result to trace class operator equations. This application is, in fact, a generalization of matrix equations to trace class operator equations.

2. Order theoretic approach

In this section we first reprove Theorem 1.2 via an order theoretic approach and then give another version of it through the Knaster-Tarski theorem. In the following by the symbol $\mathbb{R}_0^+$ we mean the set of all nonnegative real numbers.
Proof of Theorem 1.2. Define a partial order $\sqsubseteq$ in $X \times \mathbb{R}_0^+$ by

$$(x,\alpha) \sqsubseteq (y,\beta) \Leftrightarrow x \preceq y \text{ and } d(x,y) \leq \alpha - \beta, \tag{2.1}$$

where $x, y \in X$ and $\alpha, \beta \in \mathbb{R}_0^+$. Let Λ be the family of all chains M of $X \times \mathbb{R}_0^+$ such that

$$\exists\, (u,\gamma) \in M\,, \forall\, n \in \mathbb{N}: \quad (F^n u, k^n\gamma) \in M. \tag{2.2}$$

Now Λ is nonempty. In fact, we may choose $\alpha > 0$ such that $d(b, F^n b) \leq \alpha - k^n\alpha$, for each $n \geq 1$ (note $d(b, F^n b) \leq (1+k+...+k^{n-1})d(b, Fb)$). Then, because of assumptions (1) and (2) the set consisting of all $(F^n b, k^n\alpha)$, $n \geq 0$ belongs to Λ ((b,α) satisfies (2.2) and for $m > n \geq 0$ we have $F^n b \preceq F^m b$ and $d(F^n b, F^m b) \leq k^n\alpha - k^m\alpha$ which fulfils (2.1)). By Zorn's lemma the family Λ has a maximal element $\mathcal{P}$ with respect to the partial order $\subseteq$ in Λ. Note $(\mathcal{P}, \sqsubseteq)$ is a chain. Now using an argument similar to that of Lemma 1 in [5] we show that $\mathcal{P}$ has a maximum element. If $p \in \mathcal{P}$, then there exist $x_p \in X$ and $\alpha_p \in \mathbb{R}_0^+$ such that $p = (x_p, \alpha_p)$. Look at the net $\{\Gamma_p\}_{p\in\mathcal{P}}$, where $\Gamma_p = (x_p, \alpha_p)$. Note

$$p_1 \sqsubseteq p_2 \text{ means } (x_{p_1}, \alpha_{p_1}) \sqsubseteq (x_{p_2}, \alpha_{p_2}). \tag{2.3}$$

This along with (2.1) implies that $\{\alpha_p\}_{p\in\mathcal{P}}$ is decreasing and therefore is convergent, say $\alpha_p \to t$. From (2.3) it implies that the net $\{x_p\}_{p\in\mathcal{P}}$ is Cauchy. Thus there exists an element $a \in X$ to which $\{x_p\}_{p\in\mathcal{P}}$ is convergent. By assumption (3) we get $x_p \preceq a$, for each $p \in \mathcal{P}$. Again, from (2.3) it follows that

$$d(x_p, x_q) \leq \alpha_p - \alpha_q, \qquad (p, q \in \mathcal{P}).$$

Taking the limit with q we get $d(x_p, a) \leq \alpha_p - t$. Therefore $(x_p, \alpha_p) \sqsubseteq (a, t)$, for each p. This implies that (a,t) is the maximum of $\mathcal{P}$, since $\mathcal{P}$ is maximal and $\mathcal{P} \cup \{(a,t)\} \in \Lambda$, therefore $\mathcal{P} \cup \{(a,t)\} = \mathcal{P}$. Now, let (w,β) be an element of $\mathcal{P}$ such that $(F^n w, k^n\beta) \in \mathcal{P}$, for each $n \in \mathbb{N}$. Then, $(F^n w, k^n\beta) \sqsubseteq (a,t)$, so $0 \leq d(F^n w, a) \leq k^n\beta - t$, for each

$n \in \mathbb{N}$. This implies that $t = 0$, $d(F^n w, a) \to 0$, and therefore $d(F^{n+1}w, Fa) \to 0$, as $n \to \infty$. Hence, $Fa = a$.

The next theorem is due to Knaster-Tarski which can be found in many standard texts (see e.g. [3]).

Theorem 2.1. *Let $(P, \preceq)$ be a poset and $F : P \to P$ be increasing. Then F has a fixed point provided that*

(1) *there exists $a \in P$ such that $a \preceq Fa$;*
(2) *every chain in $\{x \in P : a \preceq x\}$ has a supremum.*

Theorem 2.2. *Suppose that all the assumptions of Theorem 1.2 are satisfied except we replace Condition (3) with*

3 ′. if the net $(x_\alpha)_{\alpha \in I}$ is increasing, convergent to x, then $\sup_{\alpha \in I} x_\alpha = x$.

Then F has a fixed point.

Proof. Define the partial order $\sqsubseteq$ on $P := X \times \mathbb{R}_0^+$ as in (2.1). Consider the selfmap T on P defined by

$$T(x, \alpha) = (Fx, k\alpha).$$

Note T is increasing (use F is increasing and assumption (1)). Now choose $\beta > 0$ such that $d(b, Fb) \le \beta - k\beta$ and we get $(b, \beta) \sqsubseteq (Fb, k\beta)$. As in the proof of Theorem 1.2, we see that every chain $C = \{(x_p, \alpha_p)\}_{p \in C}$ in

$$\Gamma = \{(x, \alpha) \in P : (b, \beta) \sqsubseteq (x, \alpha)\}$$

has an upper bound $(a, t) \in P$. That is, $(x_p, \alpha_p) \sqsubseteq (a, t)$, for each $p \in C$. In particular, (a, t) is the supremum of C. To see this let (y, β) be another upper bound of C. Then, $x_p \preceq y$ and $d(x_p, y) \le \alpha_p - \beta$, for each p. Now 3 ′ implies that $(a, t) \sqsubseteq (y, \beta)$. Finally, the desired result is an immediate consequence of the Knaster-Tarski theorem.

It should be mentioned that the partial order given in (2.1) was first introduced in [2]. Also, the selfmap T given in the proof of Theorem 2.2 was considered in [1].

3. Application to trace class operator equations

In [8] an application of Theorem 1.1 is given for finding the solutions of linear matrix equations. Using ideas and similar computations presented in the proof of Theorem 3.1 in [8], we generalize it to trace class operator equations.

Let $T_1, \cdots, T_n$ be bounded linear operators and P be a positive trace class operator acting on a separable Hilbert space H (denoted by $P \succeq 0$). We wish to find the solutions of equations

$$X = P + \sum_{i=1}^{n} T_i^* X T_i \tag{3.1}$$

and

$$X = P - \sum_{i=1}^{n} T_i^* X T_i. \tag{3.2}$$

If $\mathcal{L}_h$ denotes the algebra of all Hermitian operators on H, then the fixed points of the mappings $\varphi, \psi : \mathcal{L}_h \to \mathcal{L}_h$ defined by

$$\varphi(X) = P + \sum_{i=1}^{n} T_i^* X T_i$$

and

$$\psi(X) = P - \sum_{i=1}^{n} T_i^* X T_i$$

are the solutions of (3.1) and (3.2), respectively. For a compact linear operator T acting on a separable Hilbert space H, let

$$\lambda_1(T^*T) \geq \lambda_2(T^*T) \geq \lambda_3(T^*T) \geq \cdots \tag{3.3}$$

be the sequence of nonzero eigenvalues of the Hermitian compact operator T^*T, where multiplicity is taken into account. The number of eigenvalues of T^*T is finite if and only if T has finite rank and the sequence (3.3) can still be considered infinite if the sequence is extended by zero elements.

For $i = 1, 2, 3, \cdots$ the *i-th singular value* of T is defined as the number $s_i := \sqrt{\lambda_i(T^*T)}$. It should be mentioned that a compact operator T and its Hilbert-adjoint T^* have the same singular values (see [4], pp. 98, Corollary 1.2). The trace class operator space $\mathfrak{S}_1$ is defined as

$$\{T : H \to H : T \text{ compact}, \sum_{i=1}^{\infty} s_i(T) < \infty\}$$

which is a Banach space under the trace class norm

$$\|T\|_1 = \sum_{i=1}^{\infty} s_i(T).$$

Let F be a strictly positive bounded linear operator on H, that is F is positive (F is Hermitian and its spectral values are contained in the nonnegative real numbers) and that 0 is not a spectral value of F, denoted by $F \succ 0$. Then the set of all Hermitian operators in $\mathfrak{S}_1$ equipped with the norm

$$\|T\|_{1,F} = \|F^{\frac{1}{2}} T F^{\frac{1}{2}}\|_1, \tag{3.4}$$

constitutes a Banach space. To see this, suppose $T_n \to T$ in $\|\cdot\|_{1,F}$, where each T_n is Hermitian. Note

$$\begin{aligned}
\|T_n - T^*\|_{1,F} &= \|F^{\frac{1}{2}}(T_n - T)^* F^{\frac{1}{2}}\|_1 \\
&= \textstyle\sum_{i=1}^{\infty} s_i(F^{\frac{1}{2}}(T_n - T)^* F^{\frac{1}{2}}) \\
&= \textstyle\sum_{i=1}^{\infty} s_i(F^{\frac{1}{2}}(T_n - T) F^{\frac{1}{2}}) \\
&= \|F^{\frac{1}{2}}(T_n - T) F^{\frac{1}{2}}\|_1 \\
&= \|T_n - T\|_{1,F} \to 0.
\end{aligned}$$

This implies that $T = T^*$. Since the trace class operator space equipped with the norm $\| \cdot \|_{1,F}$ is a Banach space too, so the set of all Hermitian operators in $\mathfrak{S}_1$ equipped with the norm (3.4) is a Banach space.

If T is any bounded linear operator and F is positive trace class operator that acts on a separable Hilbert space H, then

$$|tr(TF)| \leq \|T\| \cdot tr(F),$$

where $tr(F) = \sum_i \lambda_i(F)$ and λ_i is the i-th eigenvalue of F, since,

$$\begin{aligned}
|tr(TF)| &\leq \|TF\|_1 && \text{([4], pp. 101, Corollary 2.4)} \\
&\leq \|T\| \cdot \|F\|_1 && \text{([4], pp. 106, Proposition 4.2)} \\
&= \|T\| \cdot \textstyle\sum_i s_i(F) \\
&= \|T\| \cdot \textstyle\sum_i \sqrt{\lambda_i(F^2)} \\
&= \|T\| \cdot \textstyle\sum_i \sqrt{(\lambda_i(F))^2} \\
&= \|T\| \cdot tr(F).
\end{aligned}$$

In what follows let $\mathfrak{S}_1^+$ be the set of all strictly positive trace class operators.

Theorem 3.1. *Let $P \in \mathfrak{S}_1^+$ and $\psi(R) \in \mathfrak{S}_1^+$ for some $R \in \mathfrak{S}_1^+$. Then φ and ψ have a unique fixed point in $\mathcal{L}_h$.*

Proof. Let $F, G \in \mathcal{L}_h$ and $F \preceq G$. Since $T_i^*(G-F)T_i \succeq 0$ for each i and the set of all positive operators is a cone, then $\varphi(F) \preceq \varphi(G)$ and we have

$$\begin{aligned}
\|\varphi(G) - \varphi(F)\|_{1,R} &= \|R^{\frac{1}{2}}(\varphi(G) - \varphi(F))R^{\frac{1}{2}}\|_1 \\
&= tr(R^{\frac{1}{2}}(\varphi(G) - \varphi(F))R^{\frac{1}{2}}) \\
&= tr(\textstyle\sum_{i=1}^n R^{\frac{1}{2}}T_i^*(G-F)T_i)R^{\frac{1}{2}}) \\
&= \textstyle\sum_{i=1}^n tr(R^{\frac{1}{2}}T_i^*(G-F)T_i)R^{\frac{1}{2}}) \\
&= \textstyle\sum_{i=1}^n tr(T_iPT_i^*(G-F)) \\
&= \textstyle\sum_{i=1}^n tr(T_iPT_i^*R^{-\frac{1}{2}}R^{\frac{1}{2}}(G-F)R^{\frac{1}{2}}R^{-\frac{1}{2}}) \\
&= \textstyle\sum_{i=1}^n tr(R^{-\frac{1}{2}}T_iPT_i^*R^{-\frac{1}{2}}R^{\frac{1}{2}}(G-F)R^{\frac{1}{2}}) \\
&= tr(\textstyle\sum_{i=1}^n R^{-\frac{1}{2}}T_iPT_i^*R^{-\frac{1}{2}}R^{\frac{1}{2}}(G-F)R^{\frac{1}{2}}) \\
&= tr((\textstyle\sum_{i=1}^n R^{-\frac{1}{2}}T_iPT_i^*R^{-\frac{1}{2}})(R^{\frac{1}{2}}(G-F)R^{\frac{1}{2}})) \\
&= |tr((\textstyle\sum_{i=1}^n R^{-\frac{1}{2}}T_iPT_i^*R^{-\frac{1}{2}})(R^{\frac{1}{2}}(G-F)R^{\frac{1}{2}}))| \\
&\leq \|\textstyle\sum_{i=1}^n R^{-\frac{1}{2}}T_iPT_i^*R^{-\frac{1}{2}}\| \cdot tr(R^{\frac{1}{2}}(G-F)R^{\frac{1}{2}}) \\
&= \|\textstyle\sum_{i=1}^n R^{-\frac{1}{2}}T_iPT_i^*R^{-\frac{1}{2}}\| \cdot \|G-F\|_{1,R}.
\end{aligned}$$

Putting $k = \|\sum_{i=1}^n R^{-\frac{1}{2}}T_iPT_i^*R^{-\frac{1}{2}}\|$, we get

$$\|\varphi(G) - \varphi(F)\|_{1,R} \leq k \cdot \|G-F\|_{1,R}.$$

The assumption that $\psi(R) \succ 0$ implies that

$$0 \preceq \sum_{i=1}^{n} R^{-\frac{1}{2}} T_i P T_i^* R^{-\frac{1}{2}} \prec I.$$

Since

$$I - \sum_{i=1}^{n} R^{-\frac{1}{2}} T_i P T_i^* R^{-\frac{1}{2}}$$

is invertible and k is a spectral value of

$$\sum_{i=1}^{n} R^{-\frac{1}{2}} T_i P T_i^* R^{-\frac{1}{2}},$$

it must be strictly less than 1. Now applying Theorem 1.2 or Theorem 2.2 a fixed point is obtained for φ which is certainly Hermitian. The same argument works for ψ to have a fixed point.

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