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ON NONLOCAL PROBLEMS FOR RETARDED FRACTIONAL DIFFERENTIAL EQUATIONS IN BANACH SPACES

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Abstract. We study the existence and asymptotic stability of solutions for a class of Cauchy problems involving retarded semilinear fractional differential equations subject to nonlocal conditions. The results are proved by means of fractional calculus and fixed point theory for condensing maps. They in particular extend and improve many recent existence results for fractional differential equations. An example is also given to illustrate the results.

Key Words and Phrases: Asymptotic stability; fractional differential equation; finite delay; nonlocal condition; condensing map; fixed point; measure of non-compactness; MNC-estimate.

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1. Introduction

The subject of fractional differential equations has received much attention in recent years due to its important applications in modeling phenomena of science and engineering. The employment of differential equations with fractional order allows to deal with many problems in numerous areas including fluid flows, rheology, electrical networks, viscoelasticity, electrochemistry, etc. For more details, we refer the reader to the monographs of Miller & Ross [22], Podlubny [25], and Kilbas et. al. [18]. In the last few years, the theory of fractional differential equations in Banach spaces has been studied extensively by several authors [3, 4, 8, 9, 14, 23, 28, 30, 31]. Notice that local and global existence results for the Cauchy problem for a similar semilinear fractional (with respect to the Riemann-Liouville derivative) functional differential equation in a Banach space were obtained by V. Obukhovskii and J.-C. Yao [24]. However, most of existing results are devoted to the existence and uniqueness of solutions.

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One of the most important and interesting problems in the theory of differential equations is to study the stability of solutions. While the stability theory for differential equations of integer order with/without delays has a long history of development and achieves many important principal results (see, e.g. [10, 11] and references therein), stability results for fractional differential equations seem to be less known, especially in infinite-dimensional spaces.

In this paper, by using the fixed point theory for condensing maps, we study the existence and asymptotic stability of solutions to the following problem

$$^{C}D_0^{\alpha}u(t) = Au(t) + f(t, u(t), u_t),\ t > 0, \tag{1.1}$$

$$u(s) + g(u)(s) = \varphi(s),\ s \in [-h, 0], \tag{1.2}$$

where the state function u takes values in a Banach space X, u_t stands for the history of the state function up to the time t, $^{C}D_0^{\alpha}, \alpha \in (0, 1]$, is the fractional derivative in the Caputo sense, A is a closed linear operator which generates a C_0-semigroup in X, f and g are the functions which will be specified in Section 3. The problem (1.1)-(1.2) is quite general and contains many important classes of Cauchy problems for differential equations.

The nonlocal problem for first order differential equations was first studied by Byszewski in [7]. This topic has been then studied extensively due to the fact that the nonlocal condition give a better description for Cauchy problems than the classical initial condition. Without being exhaustive with the references, let us quote some remarkable solvability results in [13, 15, 17, 19, 20, 21, 23, 30]. However, up to our knowledge, no attempt has been made to consider the stability of nonlocal problems for fractional differential equations. This is the main motivation of the present paper.

It is known that there are numerous technical difficulties in dealing with fractional differential equations with nonlocal conditions due to the nonlinearity of the initial conditions and the fractional derivatives involved. To overcome these difficulties, in this paper we exploit the fixed point method to prove the existence and asymptotic stability of the solutions. The idea of using the fixed point method to study the stability problem was initiated by Burton and Furumochi for ordinary/functional differential equations [5, 6] and developed later for functional partial differential equations (see e.g. [2, 14]). The main idea of this method is to construct a *stable subset*, in which the solution operator has a unique fixed point. To improve the existence conditions, we use the fixed point theorem for condensing maps. This is a quite general fixed point theorem, and in particular it covers the contraction mapping principle and the Krasnoselskii theorem. Thus, in particular, our existence results extend/improve many recent ones for fractional differential equations in [9, 23, 29, 30]. One technical difficulty we have to overcome here is to contruct some suitable measures of noncompactness and to establish the MNC-estimate for showing the condensivity of the solution semigroup generated by. Another new feature of our work is that we are able to establish stability results for problem (1.1)-(1.2) by means of fixed point theory. To do this we use the stability of the resolvent operators, which in particular is shown to hold when the operator A is exponentially stable. The approach used in the paper may be applied to investigate other differential equations, for example, the existence and stability of solutions to neutral/impulsive fractional differential equations.

The rest of the paper is organized as follows. In Section 2, for convenience of the reader, we recall some notions and results related to the resolvent operators for fractional differential equations and the fixed point theory for condensing maps. In Section 3, we prove existence results for problem (1.1)-(1.2) in the general case and in some special cases (of course with weaker conditions). Section 4 establishes the stability results for the problem. In the last section, we give an example to inlustrateillustrate the abstract results obtained in the paper.

2. Preliminaries

2.1. **Fractional calculus.** Let $L^1(0,T;X)$ be the space of integrable functions on $[0,T]$, in the Bochner sense.

Definition 2.1. *The fractional integral of order $\alpha > 0$ of a function $f \in L^1(0,T;X)$ is defined by*

$$I_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds,$$

where Γ is the Gamma function.

Definition 2.2. *For a function $f \in C^N([0,T];X)$, the Caputo fractional derivative of order $\alpha \in (N-1,N]$ is defined by*

$${}^C D_0^\alpha f(t) = \frac{1}{\Gamma(N-\alpha)} \int_0^t (t-s)^{N-\alpha-1} f^{(N)}(s)ds.$$

It should be noted that there are some notions of fractional derivatives, in which the Riemann-Liouville and Caputo definitions have been used widely. Many application problems, expressed by differential equations of fractional order, require initial conditions related to $u(0), u'(0)$, etc., and the Caputo fractional derivative satisfies these demands. For $u \in C^N([0,T];X)$, we have the following formulas

$$\begin{aligned} &{}^C D_0^\alpha I_0^\alpha u(t) = u(t), \\ &I_0^\alpha \, {}^C D_0^\alpha u(t) = u(t) - \sum_{k=0}^{N-1} \frac{u^{(k)}(0)}{k!} t^k. \end{aligned}$$

Consider the linear problem

$${}^C D_0^\alpha u(t) = Au(t) + f(t), t > 0, \tag{2.1}$$

$$u(0) = u_0, \tag{2.2}$$

where $\alpha \in (0,1], f \in L^1_{loc}(\mathbb{R}^+;X)$. By using the Laplace transform, the solution of (2.1)-(2.2) has the following presentation

$$u(t) = S_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1} P_\alpha(t-s) f(s)ds, \tag{2.3}$$

where S_α and P_α are so-called the resolvent operators for (2.1)-(2.2). Specifically,

$$\mathcal{L}(S_\alpha)(\lambda) = \lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1},$$
$$\mathcal{L}((\cdot)^{\alpha-1}P_\alpha)(\lambda) = (\lambda^\alpha I - A)^{-1},$$

here $\mathcal{L}$ denotes the Laplace transform for vector-valued functions. By the subordination principle (see [3]), S_α and $P_\alpha, \alpha \in (0,1]$, exist if A generates a C_0-semigroup $\{T(t)\}_{t\geq 0}$. The explicit formulation of S_α and P_α was given in [31]:

$$S_\alpha(t)x = \int_0^\infty \phi_\alpha(\theta)T(t^\alpha\theta)x d\theta,$$
$$P_\alpha(t)x = \alpha\int_0^\infty \theta\phi_\alpha(\theta)T(t^\alpha\theta)x d\theta,$$

where ϕ_α is a probability density function defined on $(0,\infty)$, that is, $\phi_\alpha(\theta) \geq 0$ and $\int_0^\infty \phi_\alpha(\theta)d\theta = 1$. Moreover, ϕ_α has the expression

$$\phi_\alpha(\theta) = \frac{1}{\alpha}\theta^{-1-\frac{1}{\alpha}}\psi_\alpha(\theta^{-\frac{1}{\alpha}}),$$
$$\psi_\alpha(\theta) = \frac{1}{\pi}\sum_{n=1}^\infty(-1)^{n-1}\theta^{-\alpha n-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha).$$

We now recall some basic results, which will be used in the sequel.

Lemma 2.3. *Let A generate a C_0-semigroup $\{T(t)\}_{t\geq 0}$ in X.*

i) *If $T(t)$ is compact for $t>0$, then $S_\alpha(t)$ and $P_\alpha(t)$ are compact for $t>0$;*
ii) *If $T(t)$ is norm continuous for $t>0$, then $S_\alpha(t)$ and $P_\alpha(t)$ are norm continuous for $t>0$.*

The proof of the first statement can be found in [31], while the second claim was proved in [28].

Let $\Phi(t,s)$ be a family of bounded linear operators on X for $t,s \in [0,T], s \leq t$. The following result was proved in [26, Lemma 1].

Lemma 2.4. *Assume that Φ satisfies the following conditions:*

(Φ1) *there exists a function $\rho \in L^q(J), q \geq 1$ such that $\|\Phi(t,s)\| \leq \rho(t-s)$ for all $t,s \in [0,T], s \leq t$;*
(Φ2) *$\|\Phi(t,s) - \Phi(r,s)\| \leq \epsilon$ for $0 \leq s \leq r-\epsilon, r < t = r+h \leq T$ with $\epsilon = \epsilon(h) \to 0$ as $h \to 0$.*

Then the operator $\mathbf{S} : L^{q'}(0,T;X) \to C([0,T];X)$ defined by

$$(\mathbf{S}g)(t) := \int_0^t \Phi(t,s)g(s)ds$$

sends any bounded set to an equicontinuous one, where q' is the conjugate of q ($q' = +\infty$ if $q = 1$). Denote

$$Q_\alpha : L^1([0,T];X) \to C([0,T];X),$$
$$Q_\alpha(f)(t) = \int_0^t (t-s)^{\alpha-1} P_\alpha(t-s) f(s) ds. \tag{2.4}$$

Now using the last two lemmas, we have the following result.

Proposition 2.5. *Let A generate a C_0-semigroup $\{T(t)\}_{t\geq 0}$ in X. Then for each bounded set $\Omega \subset L^1(0,T;X)$, $Q_\alpha(\Omega)$ is an equicontinuous set in $C([0,T];X)$ provided that the semigroup $\{T(t)\}_{t\geq 0}$ is norm continuous for $t > 0$.*

Proof. Since $T(t)$ is norm continuous for $t > 0$, so is $P_\alpha(t)$ thanks to Lemma 2.3. Then it deduces that $\Phi(t,s) = (t-s)^{\alpha-1} P_\alpha(t-s)$ satisfies $(\Phi1) - (\Phi2)$ in Lemma 2.4. Thus we have the conclusion as desired.

2.2. **Fixed point theory for condensing operators.** Let $\mathcal{E}$ be a Banach space. Denote by $\mathcal{B}(\mathcal{E})$ the collection of nonempty bounded subsets of $\mathcal{E}$. We will use the following definition of measure of noncompactness.

Definition 2.6. *A function $\beta : \mathcal{B}(\mathcal{E}) \to \mathbb{R}^+$ is called* a measure of noncompactness *(MNC) in $\mathcal{E}$ if*

$$\beta(\overline{co}\,\Omega) = \beta(\Omega) \ \textit{for every} \ \Omega \in \mathcal{B}(\mathcal{E}),$$

where $\overline{co}\,\Omega$ is the closure of the convex hull of Ω. An MNC β is called

i) *monotone if $\Omega_0, \Omega_1 \in \mathcal{B}(\mathcal{E})$, $\Omega_0 \subset \Omega_1$ implies $\beta(\Omega_0) \leq \beta(\Omega_1)$;*
ii) *nonsingular if $\beta(\{a\} \cup \Omega) = \beta(\Omega)$ for any $a \in \mathcal{E}, \Omega \in \mathcal{B}(\mathcal{E})$;*
iii) *invariant with respect to union with compact set if $\beta(K \cup \Omega) = \beta(\Omega)$ for every relatively compact set $K \subset \mathcal{E}$ and $\Omega \in \mathcal{B}(\mathcal{E})$;*
iv) *algebraically semi-additive if $\beta(\Omega_0 + \Omega_1) \leq \beta(\Omega_0) + \beta(\Omega_1)$ for any $\Omega_0, \Omega_1 \in \mathcal{B}(\mathcal{E})$;*
v) *regular if $\beta(\Omega) = 0$ is equivalent to the relative compactness of Ω.*

An important example of MNC is *the Hausdorff* MNC $\chi(\cdot)$, which is defined as follows

$$\chi(\Omega) = \inf\{\varepsilon : \ \Omega \text{ has a finite } \varepsilon\text{-net}\}.$$

It should be mentioned that the Hausdorff MNC has also the following additional properties:

- semi-homogeneity: $\chi(t\Omega) \leq |t|\chi(\Omega)$ for any $\Omega \in \mathcal{B}(\mathcal{E})$ and $t \in \mathbb{R}$;
- in a separable Banach space $\mathcal{E}$, $\chi(\Omega) = \lim\limits_{m\to\infty} \sup\limits_{x\in\Omega} d(x, \mathcal{E}_m)$, where $\{\mathcal{E}_m\}$ is a sequence of finite dimensional subspaces of $\mathcal{E}$ such that $\mathcal{E}_m \subset \mathcal{E}_{m+1}, m = 1,2,...$ and $\overline{\bigcup\limits_{m=1}^{\infty} \mathcal{E}_m} = \mathcal{E}$.

Based on the Hausdorff MNC χ in $\mathcal{E}$, one can define *the sequential MNC* χ_0 as follows:

$$\chi_0(\Omega) = \sup\{\chi(D) : D \in \Delta(\Omega)\}, \tag{2.5}$$

where $\Delta(\Omega)$ is the collection of all at-most-countable subsets of Ω (see [1]). We know that

$$\frac{1}{2}\chi(\Omega) \leq \chi_0(\Omega) \leq \chi(\Omega), \tag{2.6}$$

for all bounded set $\Omega \subset \mathcal{E}$. Then the following property is evident.

Proposition 2.7. *Let χ be the Hausdorff MNC in $\mathcal{E}$ and $\Omega \subset \mathcal{E}$ be a bounded set. Then for every $\epsilon > 0$, there exists a sequence $\{x_n\} \subset \Omega$ such that*

$$\chi(\Omega) \leq 2\chi(\{x_n\}) + \epsilon.$$

We need the following assertion, whose proof can be found in [16].

Proposition 2.8. *If $\{w_n\} \subset L^1(0,T;\mathcal{E})$ such that*

$$||w_n(t)||_{\mathcal{E}} \leq \nu(t), \text{ for a.e. } t \in [0,T],$$

for some $\nu \in L^1(0,T)$, then we have

$$\chi(\{\int_0^t w_n(s)ds\}) \leq 2\int_0^t \chi(\{w_n(s)\})ds$$

for $t \in [0,T]$.

Let J be a compact interval of $\mathbb{R}$ and χ_C be the Hausdorff MNC in $C(J;\mathcal{E})$. We recall the following facts (see [1]), which will be used later: for each bounded set $D \subset C(J;\mathcal{E})$, one has

- $\chi(D(t)) \leq \chi_C(D)$, for all $t \in J$, where $D(t) := \{x(t) : x \in D\}$.
- If D is an equicontinuous set, then

$$\chi_C(D) = \sup_{t\in J} \chi(D(t)).$$

Let $\mathcal{T} \in L(\mathcal{E})$, i.e., $\mathcal{T}$ is a bounded linear operator from $\mathcal{E}$ into $\mathcal{E}$. We recall the notion of χ-norm (see, e.g. [1]) as follows:

$$\|\mathcal{T}\|_\chi := \inf\{M : \chi(\mathcal{T}\Omega) \leq M\chi(\Omega),\ \Omega \subset \mathcal{E} \text{ is a bounded set}\}. \tag{2.7}$$

The χ-norm of $\mathcal{T}$ can be evaluated as

$$\|\mathcal{T}\|_\chi = \chi(\mathcal{T}\mathbf{S}_1) = \chi(\mathcal{T}\mathbf{B}_1),$$

where $\mathbf{S}_1$ and $\mathbf{B}_1$ are the unit sphere and the unit ball in $\mathcal{E}$, respectively. It is easy to see that

$$\|\mathcal{T}\|_\chi \leq \|\mathcal{T}\|_{L(\mathcal{E})}. \tag{2.8}$$

Definition 2.9. *A continuous map $\mathcal{F} : Z \subseteq \mathcal{E} \to \mathcal{E}$ is said to be condensing with respect to an MNC β (β-condensing) if for any bounded set $\Omega \subset Z$, the relation*

$$\beta(\Omega) \leq \beta(\mathcal{F}(\Omega))$$

implies the relative compactness of Ω.

Let β be a monotone nonsingular MNC in $\mathcal{E}$. The application of the topological degree theory for condensing maps (see, e.g., [1, 16]) yields the following fixed point principle.

Theorem 2.10. [16, Corollary 3.3.1] *Let $\mathcal{M}$ be a bounded convex closed subset of $\mathcal{E}$ and let $\mathcal{F} : \mathcal{M} \to \mathcal{M}$ be a β-condensing map. Then* $\text{Fix}\mathcal{F} := \{x = \mathcal{F}(x)\}$ *is a non-empty compact set.*

3. Existence results

Given $T > 0$, we denote $\mathcal{C}^T = C([-h,T];X), \mathcal{C}_h = C([-h,0];X)$.

3.1. **The general case.** Concerning problem (1.1)-(1.2), we give the following assumptions.

(**A**) *The semigroup $\{T(t)\}_{t\geq 0}$ generated by A is norm continuous for $t > 0$.*

(**F**) *The nonlinear function $f : \mathbb{R}^+ \times X \times \mathcal{C}_h \to X$ satisfies:*

(1) *$f(\cdot, v, w)$ is measurable for each $(v,w) \in X \times \mathcal{C}_h$, $f(t,\cdot,\cdot)$ is continuous for a.e. $t \in [0,T]$ and*

$$||f(t,v,w)||_X \leq m(t)\Psi_f(||v||_X + ||w||_{\mathcal{C}_h}),$$

for all $(v,w) \in X \times \mathcal{C}_h$, where $m \in L^p_{loc}(\mathbb{R}^+), p > \frac{1}{\alpha}$ and Ψ_f is a real-valued, continuous and nondecreasing function;

(2) *there exists a function $k : \mathbb{R}^2_+ \to \mathbb{R}^+$ such that $k(t,\cdot) \in L^p(0,t), t > 0$, and for all bounded subsets $V \subset X, W \subset \mathcal{C}_h$,*

$$\chi(P_\alpha(t-s)f(s,V,W)) \leq k(t,s)[\chi(V) + \vartheta(W)],$$

for a.e. $t,s \in [0,T], s \leq t$, where χ and ϑ are the Hausdorff MNC in X and $\mathcal{C}_h$, respectively.

(**G**) *The nonlocal function $g : \mathcal{C}^T \to \mathcal{C}_h$ obeys the following conditions:*

(1) *g is continuous and*

$$||g(u)||_{\mathcal{C}_h} \leq \Psi_g(||u||_{\mathcal{C}^T}),$$

for all $u \in \mathcal{C}^T$, where Ψ_g is a continuous and nondecreasing function on $\mathbb{R}^+$;

(2) *there exists $\eta \geq 0$ such that for any bounded set $D \subset \mathcal{C}^T$,*

$$\vartheta(g(D)) \leq \eta\chi_{\mathcal{C}}(D),$$

here $\chi_{\mathcal{C}}$ stands for the Hausdorff MNC in $\mathcal{C}^T$.

Remark 3.1. *Let us give some comments on assumptions (**F**)(2) and (**G**)(2).*

(1) *If $f(t,\cdot,\cdot)$ satisfies the Lipschitz condition, i.e.,*

$$||f(t,v_1,w_1) - f(t,v_2,w_2)||_X \leq k_f(t)(||v_1 - v_2||_X + ||w_1 - w_2||_{\mathcal{C}_h}),$$

*for some $k_f \in L^p_{loc}(\mathbb{R}^+)$, then (**F**)(2) holds for $k(t,s) = ||P_\alpha(t-s)||k_f(s)$. On the other hand, if $P_\alpha(t), t > 0$, is compact or $f(t,\cdot,\cdot)$ is completely continuous (for each fixed t) then* (**F**)(2) *is obviously fulfilled with $k = 0$.*

(2) *Similarly for (**G**)(2), if g is Lipschitzian, that is,*

$$||g(u) - g(v)||_{\mathcal{C}_h} \leq \eta||u - v||_{\mathcal{C}^T},$$

*then (**G**)(2) takes place. This condition is also satisfied with $\eta = 0$ if g is completely continuous.*

In accordance with formula (2.3), we have the following definition.

Definition 3.2. *A function $u \in \mathcal{C}^T$ is said to be an integral solution of problem* (1.1)-(1.2) *on the interval* $[-h,T]$ *if and only if* $u(t) = \varphi(t) - g(u)(t)$ *for* $t \in [-h,0]$*, and*

$$u(t) = S_\alpha(t)[\varphi(0) - g(u)(0)] + \int_0^t (t-s)^{\alpha-1} P_\alpha(t-s) f(s,u(s),u_s)ds$$

for any $t \in [0,T]$*.*

Let

$$\mathcal{F}(u)(t) = \begin{cases} S_\alpha(t)[\varphi(0) - g(u)(0)] + \int_0^t (t-s)^{\alpha-1} P_\alpha(t-s) f(s,u(s),u_s)ds, \ t > 0, \\ \varphi(t) - g(u)(t), \ t \in [-h,0]. \end{cases} \tag{3.1}$$

Then u is an integral solution of (1.1)-(1.2) iff it is a fixed point of *the solution operator* $\mathcal{F}$. From the assumptions imposed on f and g, $\mathcal{F}$ is a continuous map on $\mathcal{C}^T$.

It should be mentioned that, since f and g may be not Lipschitzian, the existence of solutions of (1.1)-(1.2) cannot be obtained by the Banach contraction principle. In this paper, we deploy the fixed point theory for condensing maps by establishing the so-called MNC-estimate (i.e. estimate via MNC) to prove the condensivity of $\mathcal{F}$.

The following lemma is the key in this section.

Lemma 3.3. *Let the hypotheses* **(A)**, **(F)** *and* **(G)** *hold. Then the solution operator* $\mathcal{F}$ *given by* (3.1) *satisfies*

$$\chi_{\mathcal{C}}(\mathcal{F}(D)) \le \Big[\eta \sup_{t\in[0,T]} ||S_\alpha(t)|| + 8 \sup_{t\in[0,T]} \int_0^t (t-s)^{\alpha-1} k(t,s)ds\Big] \chi_{\mathcal{C}}(D),$$

for all bounded set $D \subset \mathcal{C}^T$*.*

Proof. Let $D \subset \mathcal{C}^T$ be a bounded set. Then we have

$$\mathcal{F}(D) = \mathcal{F}_1(D) + \mathcal{F}_2(D),$$

where

$$\mathcal{F}_1(u)(t) = \begin{cases} S_\alpha(t)[\varphi(0) - g(u)(0)], \ t > 0, \\ \varphi(t) - g(u)(t), \ t \in [-h,0]; \end{cases}$$

$$\mathcal{F}_2(u)(t) = \begin{cases} \int_0^t (t-s)^{\alpha-1} P_\alpha(t-s) f(s,u(s),u_s)ds, \ t > 0, \\ 0, \ t \in [-h,0]. \end{cases}$$

Then

$$\chi_{\mathcal{C}}(\mathcal{F}(D)) \le \chi_{\mathcal{C}}(\mathcal{F}_1(D)) + \chi_{\mathcal{C}}(\mathcal{F}_2(D)).$$

For $z_1, z_2 \in \mathcal{F}_1(D)$, there exist $u_1, u_2 \in D$ such that

$$z_1(t) = S_\alpha(t)[\varphi(0) - g(u_1)(0)], \ z_2(t) = S_\alpha(t)[\varphi(0) - g(u_2)(0)] \text{ if } t > 0,$$
$$z_1(t) = \varphi(t) - g(u_1)(t), \ z_2(t) = \varphi(t) - g(u_2)(t) \text{ if } t \in [-h,0].$$

Then

$$||z_1(t) - z_2(t)||_X \le ||S_\alpha(t)||||g(u_1)(0) - g(u_2)(0)||_X \le ||S_\alpha(t)||||g(u_1) - g(u_2)||_{\mathcal{C}_h}$$
$$\text{if } t > 0,$$
$$||z_1(t) - z_2(t)||_X \le ||g(u_1)(t) - g(u_2)(t)||_X \le ||g(u_1) - g(u_2)||_{\mathcal{C}_h} \text{ if } t \in [-h, 0].$$

Since $||S_\alpha(t)|| \ge 1$, we have

$$||z_1 - z_2||_{\mathcal{C}^T} \le \sup_{t\in[0,T]} ||S_\alpha(t)||||g(u_1) - g(u_2)||_{\mathcal{C}_h}.$$

Thus

$$\chi_{\mathcal{C}}(\mathcal{F}_1(D)) \le \sup_{t\in[0,T]} ||S_\alpha(t)||\vartheta(g(D)).$$

Employing $(\mathbf{G})(2)$, we have

$$\chi_{\mathcal{C}}(\mathcal{F}_1(D)) \le \eta \sup_{t\in[0,T]} ||S_\alpha(t)||\chi_{\mathcal{C}}(D). \tag{3.2}$$

For $\epsilon > 0$, choosing a sequence $\{u_n\} \subset \mathcal{C}^T$ such that

$$\chi_{\mathcal{C}}(\mathcal{F}_2(D)) \le 2\chi_{\mathcal{C}}(\mathcal{F}_2(\{u_n\})) + \epsilon, \tag{3.3}$$

thanks to Proposition 2.7. Taking into account hypothesis $(\mathbf{A})$ and Proposition 2.5, we see that $\mathcal{F}_2(\{u_n\})$ is an equicontinuous set in $C([0,T];X)$. Then

$$\begin{aligned}
\chi_{\mathcal{C}}(\mathcal{F}_2(\{u_n\})) &= \sup_{t\in[0,T]} \chi(\mathcal{F}_2(\{u_n\})(t)) \\
&\le 2 \sup_{t\in[0,T]} \int_0^t (t-s)^{\alpha-1}\chi\big(P_\alpha(t-s)f(s,\{u_n(s)\},\{(u_n)_s\})\big)ds \\
&\le 2 \sup_{t\in[0,T]} \int_0^t (t-s)^{\alpha-1}k(t,s)\big(\chi(\{u_n(s)\}) + \vartheta(\{(u_n)_s\})\big)ds \\
&\le 4\chi_{\mathcal{C}}(\{u_n\}) \sup_{t\in[0,T]} \int_0^t (t-s)^{\alpha-1}k(t,s)ds \\
&\le 4\chi_{\mathcal{C}}(D) \sup_{t\in[0,T]} \int_0^t (t-s)^{\alpha-1}k(t,s)ds,
\end{aligned}$$

here we have used Proposition 2.8. In view of (3.3), one has

$$\chi_{\mathcal{C}}(\mathcal{F}_2(D)) \le 8\chi_{\mathcal{C}}(D) \sup_{t\in[0,T]} \int_0^t (t-s)^{\alpha-1}k(t,s)ds,$$

since $\epsilon > 0$ can be chosen arbitrarily.

Combining the last inequality with (3.2), we arrive at

$$\chi_{\mathcal{C}}(\mathcal{F}(D)) \le \Big[\eta \sup_{t\in[0,T]} ||S_\alpha(t)|| + 8 \sup_{t\in[0,T]} \int_0^t (t-s)^{\alpha-1}k(t,s)ds\Big]\chi_{\mathcal{C}}(D).$$

The proof is complete.

Theorem 3.4. *Assume that the hypotheses of Lemma 3.3 hold. Then problem* (1.1)-(1.2) *has at least one integral solution in* $\mathcal{C}^T$ *provided that*

$$\eta \sup_{t\in[0,T]} ||S_\alpha(t)|| + 8 \sup_{t\in[0,T]} \int_0^t (t-s)^{\alpha-1} k(t,s) ds < 1, \tag{3.4}$$

$$\liminf_{r\to\infty} \frac{1}{r}\big[\Psi_g(r) \sup_{t\in[0,T]} ||S_\alpha(t)|| + \Psi_f(2r) \sup_{t\in[0,T]} \int_0^t (t-s)^{\alpha-1} ||P_\alpha(t-s)|| m(s) ds\big] < 1. \tag{3.5}$$

Proof. By (3.4), we obtain the χ_C-condensing property for $\mathcal{F}$ thanks to Lemma 3.3. In order to apply Theorem 2.10, it remains to show that $\mathcal{F}(B_R) \subset B_R$ for some $R > 0$, where B_R is the closed ball in $\mathcal{C}^T$ centered at 0 with radius R. Assume to the contrary that there exists a sequence $\{u_n\} \subset \mathcal{C}^T$ such that $||u_n||_{\mathcal{C}^T} \le n$ but $||\mathcal{F}(u_n)||_{\mathcal{C}^T} > n$. From the formulation of $\mathcal{F}$, one has

$$||\mathcal{F}(u_n)(t)||_X \le ||\varphi||_{\mathcal{C}_h} + \Psi_g(||u_n||_{\mathcal{BC}}) \le ||\varphi||_{\mathcal{C}_h} + \Psi_g(n) \text{ for } t \in [-h, 0],$$

and, for $t > 0$,

$$\begin{aligned} ||\mathcal{F}(u_n)(t)||_X \le (||\varphi||_{\mathcal{C}_h} &+ \Psi(||u_n||_{\mathcal{C}^T})) \sup_{t\in[0,T]} ||S_\alpha(t)|| \\ &+ \int_0^t (t-s)^{\alpha-1} ||P_\alpha(t-s)|| m(s) \Psi_f(||u_n(s)||_X + ||(u_n)_s||_{\mathcal{C}_h}) ds \\ &\le (||\varphi||_{\mathcal{C}_h} + \Psi(n)) \sup_{t\in[0,T]} ||S_\alpha(t)|| \\ &+ \Psi_f(2n) \int_0^t (t-s)^{\alpha-1} ||P_\alpha(t-s)|| m(s) ds. \end{aligned}$$

Therefore,

$$\begin{aligned} 1 \le &\frac{1}{n} ||\mathcal{F}(u_n)||_{\mathcal{C}^T} \le \frac{1}{n}\Big[||\varphi||_{\mathcal{C}_h} + \Psi(n) \sup_{t\in[0,T]} ||S_\alpha(t)||\Big] \\ &+ \frac{1}{n}\Big[\Psi_f(2n) \sup_{t\in[0,T]} \int_0^t (t-s)^{\alpha-1} ||P_\alpha(t-s)|| m(s) ds\Big], \end{aligned}$$

thanks to the fact that $\sup_{t\in[0,T]} ||S_\alpha(t)|| \ge 1$. Passing to the limits in the last inequality, one gets a contradiction. The proof is just complete.

Remark 3.5. *Assume that* $\Psi_f(r) = C_f(1 + r^\beta), \Psi_g(r) = C_g(1 + r^\gamma)$ *for some* $\beta, \gamma \in [0, 1]$. *If* $\beta, \gamma < 1$ *(the sublinear case), then condition* (3.5) *is fulfilled evidently. If* $\beta = \gamma = 1$, *then* (3.5) *is relaxed to*

$$C_g \sup_{t\in[0,T]} ||S_\alpha(t)|| + 2C_f \sup_{t\in[0,T]} \int_0^t (t-s)^{\alpha-1} ||P_\alpha(t-s)|| m(s) ds < 1.$$

3.2. **The non-retarded case.** It is worth noting that, if the delay is absent (i.e., $h = 0$) and g is completely continuous, condition (3.4) is no longer necessary since one can take a suitable MNC instead of χ_C. Indeed, by ω_C we denote the modulus of equicontinuity in $\mathcal{C}^T$, that is,

$$\omega_C(D) = \lim_{\delta\to 0}\sup_{y\in D}\max_{|t_1-t_2|<\delta}\|y(t_1)-y(t_2)\|,\ D\in\mathcal{C}^T. \tag{3.6}$$

Then, as mentioned in [16], the MNC given by

$$\chi^*(D) = \sup_{t\in[0,T]} e^{-Lt}\chi(D(t)) + \omega_C(D), \tag{3.7}$$

where L is a nonnegative number, satisfies all properties stated in Definition 2.6.

In the non-retarded case, problem (1.1)-(1.2) reads

$$^C D_0^\alpha u(t) = Au(t) + f(t,u(t)), t > 0, \tag{3.8}$$

$$u(0) + g(u) = \varphi, \tag{3.9}$$

where $\varphi\in X$ given. The assumptions (F) and (G) should be changed as the follows:

(**Fa**) *The nonlinear function $f:\mathbb{R}^+\times X\to X$ satisfies:*

(1) *$f(\cdot,v)$ is measurable for each $v\in X$, $f(t,\cdot)$ is continuous for a.e. $t\in[0,T]$ and*

$$||f(t,v)||_X \le m(t)\Psi_f(||v||_X),$$

for all $v\in X$, where $m\in L^p_{loc}(\mathbb{R}^+), p>\frac{1}{\alpha}$, Ψ_f is continuous and decreasing;

(2) *there exists a function $k:\mathbb{R}^2_+\to\mathbb{R}^+$ such that $k(t,\cdot)\in L^p(0,t), t>0$, and for any bounded set $V\subset X$,*

$$\chi(P_\alpha(t-s)f(s,V)) \le k(t,s)\chi(V),$$

for a.e. $t,s\in[0,T], s\le t$.

(**Ga**) *The nonlocal function $g:\mathcal{C}^T\to X$ obeys the following conditions:*

(1) *g is continuous and*

$$||g(u)||_X \le \Psi_g(||u||_{\mathcal{C}^T}),$$

for all $u\in\mathcal{C}^T$. Here Ψ_g is a continuous and decreasing function;

(2) *there exists a nonnegative number η such that*

$$\chi(g(D)) \le \eta\chi_C(D),$$

for any bounded set $D\subset\mathcal{C}^T$.

Choosing L in (3.7) such that

$$4\sup_{t\in[0,T]}\int_0^t e^{-L(t-s)}(t-s)^{\alpha-1}k(t,s)ds < 1,$$

we prove the condensivity of the solution operator with respect to MNC χ^* in the next proposition.

Proposition 3.6. *Let* (**A**), (**Fa**) *and* (**Ga**) *hold. If the nonlocal function g is completely continuous, then $\mathcal{F}$ is χ^*-condensing.*

Proof. Let D be a bounded set in $\mathcal{C}^T$. We have

$$\mathcal{F}(D)(t) = S_\alpha(t)[\varphi - g(D)] + Q_\alpha(D)(t),$$

where Q_α is defined in (2.4). Since $g(D)$ is a relatively compact set in X, we have

$$\omega_C(S_\alpha(\cdot)[\varphi - g(D)]) = 0. \tag{3.10}$$

On the other hand, $Q_\alpha(D)$ is an equicontinuous set in $\mathcal{C}^T$ thanks to Proposition 2.5. Thus

$$\omega_C(Q_\alpha(D)) = 0. \tag{3.11}$$

It is obvious that

$$\chi(\mathcal{F}(D)(t)) \le \chi(S_\alpha(t)g(D)) + \chi(Q_\alpha(D)(t)) = \chi(Q_\alpha(D)(t)),\ t \ge 0. \tag{3.12}$$

For a given $\epsilon > 0$, one can take $\{u_n\} \subset D$ such that

$$\begin{aligned}
\chi(Q_\alpha(D)(t)) &\le 2\chi(Q_\alpha(\{u_n\})(t)) + \epsilon \\
&\le 2\chi\Big(\{\int_0^t (t-s)^{\alpha-1} P_\alpha(t-s) f(s, u_n(s))\}\Big) + \epsilon \\
&\le 4\int_0^t (t-s)^{\alpha-1} k(t,s)\chi(\{u_n(s)\})ds + \epsilon \\
&\le \Big(4\int_0^t e^{Ls}(t-s)^{\alpha-1}k(t,s)ds\Big) \sup_{t\in[0,T]} e^{-Lt}\chi(D(t)) + \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we get

$$e^{-Lt}\chi(Q_\alpha(D)(t)) \le \Big(4\int_0^t e^{-L(t-s)}(t-s)^{\alpha-1}k(t,s)ds\Big) \sup_{t\in[0,T]} e^{-Lt}\chi(D(t)). \tag{3.13}$$

Combining (3.10)-(3.13), we arrive at

$$\chi^*(\mathcal{F}(D)) \le \Big(\sup_{t\in[0,T]} 4\int_0^t e^{-L(t-s)}(t-s)^{\alpha-1}k(t,s)ds\Big)\chi^*(D).$$

The proof is complete.

We also mention that, in the non-retarded case, condition (3.5) can be removed if the nonlocal function g is uniformly bounded and the nonlinearity f has a linear growth, namely, $\Psi_g(r) = C_g$ being a constant and $\Psi_f(r) = C_f(1+r)$.

Let $\mathcal{M}_\psi = \{u \in \mathcal{C}^T : ||u(t)||^p \le \psi(t), t \in [0,T]\}$, where ψ is the solution of the integral equation

$$\psi(t) = (||\varphi|| + C_g)^p \sup_{t\in[0,T]} ||S_\alpha(t)||^p + C_P\int_0^t |m(s)|^p(1+\psi(s))ds,\ t\in[0,T],$$

$$C_P = 2^p C_f^p \sup_{t\in[0,T]} ||P_\alpha(t)||^p \Big(\frac{p-1}{p\alpha-1}\Big)^{p-1} T^{p\alpha-1}.$$

It is obvious that $\mathcal{M}_\psi$ is a closed, bounded and convex subset of $\mathcal{C}^T$. We now prove that $\mathcal{M}_\psi$ is invariant under the solution operator of problem (3.8)-(3.9).

Proposition 3.7. *Assume that g is uniformly bounded and f has a linear growth, that is, $\Psi_g(r) = C_g, \Psi_f(r) = C_f(1+r)$ for all $r \in \mathbb{R}^+$. Let $\mathcal{F}$ be the solution operator for the non-retarded problem* (3.8)-(3.9)*. Then $\mathcal{F}(\mathcal{M}_\psi) \subset \mathcal{M}_\psi$.*

Proof. The solution operator for problem (3.8)-(3.9) is given by

$$\mathcal{F}(u)(t) = S_\alpha(t)[\varphi - g(u)] + \int_0^t (t-s)^{\alpha-1} P_\alpha(t-s) f(s, u(s)) ds.$$

Then we have

$$\begin{aligned}
||\mathcal{F}(u)(t)|| &\le (||\varphi|| + C_g) \sup_{t\in[0,T]} ||S_\alpha(t)|| \\
&\quad + C_f \int_0^t (t-s)^{\alpha-1} ||P_\alpha(t-s)|| m(s)(1 + ||u(s)||) ds \\
&\le (||\varphi|| + C_g) \sup_{t\in[0,T]} ||S_\alpha(t)|| \\
&\quad + C_f \sup_{t\in[0,T]} ||P_\alpha(t)|| \int_0^t (t-s)^{\alpha-1} m(s)(1 + ||u(s)||) ds \\
&\le (||\varphi|| + C_g) \sup_{t\in[0,T]} ||S_\alpha(t)|| \\
&\quad + 2C_f \sup_{t\in[0,T]} ||P_\alpha(t)|| \Big(\frac{p-1}{p\alpha-1}\Big)^{\frac{p-1}{p}} T^{\frac{p\alpha-1}{p}} \Big[\int_0^t |m(s)|^p (1 + ||u(s)||^p) ds\Big]^{\frac{1}{p}},
\end{aligned}$$

thanks to the Hölder inequality. Thus

$$\begin{aligned}
||\mathcal{F}(u)(t)||^p &\le (||\varphi|| + C_g)^p \sup_{t\in[0,T]} ||S_\alpha(t)||^p \\
&\quad + 2^p C_f^p \sup_{t\in[0,T]} ||P_\alpha(t)||^p \Big(\frac{p-1}{p\alpha-1}\Big)^{p-1} T^{p\alpha-1} \int_0^t |m(s)|^p (1 + ||u(s)||^p) ds.
\end{aligned}$$

The last inequality implies that $||\mathcal{F}(u)(t)||^p \le \psi(t)$, provided $||u(t)||^p \le \psi(t)$ for all $t \in [0, T]$. The proof is complete.

The following result is a consequence of Propositions 3.6, 3.7 and Theorem 2.10.

Theorem 3.8. *Assume that* (**A**), (**Fa**) *and* (**Ga**) *hold. Assume further that the nonlocal function g is completely continuous and uniformly bounded, the nonlinearity f has a linear growth. Then the solution set of problem* (3.8)-(3.9) *is nonempty and compact.*

4. Stability results

In order to establish stability results for problem (1.1)-(1.2), we consider this problem in the space of continuous and uniformly bounded functions on $[-h, +\infty)$:

$$\mathcal{BC} = \{u \in C([-h, +\infty); X) : \sup_{t\ge -h} ||u(t)|| < +\infty\},$$

endowed with the norm

$$||u||_{\mathcal{BC}} = \sup_{t\ge -h} ||u(t)||.$$

Let $\pi_T, T > 0$, be the truncate function on $\mathcal{BC}$, i.e., for $D \subset \mathcal{BC}$, $\pi_T(D)$ is the restriction of D on interval $[-h, T]$. Then one can see that the MNC $\chi_{\mathcal{BC}}$ in $\mathcal{BC}$ defined by

$$\chi_{\mathcal{BC}}(D) = \sup_{T>0} \chi_{\mathcal{C}}(\pi_T(D))$$

satisfies all specifications in Definition 2.6. Using Lemma 3.3, we get the condensing property for the solution operator $\mathcal{F}$ on $\mathcal{BC}$.

Lemma 4.1. *Assume that* (**A**), (**F**) *and* (**G**) *take place for any* $T > 0$. *Then the solution operator* $\mathcal{F}$ *acting on* $\mathcal{BC}$ *is* $\chi_{\mathcal{BC}}$*-condensing provided that*

$$\eta \sup_{t\geq 0} ||S_\alpha(t)|| + 8 \sup_{t\geq 0} \int_0^t (t-s)^{\alpha-1} k(t,s) ds < 1. \tag{4.1}$$

In order to study the stability of solutions to problem (1.1)-(1.2), we need the following assumption on the resolvent operators.

(**R**) *The resolvent operators* $\{S_\alpha(t), P_\alpha(t)\}_{t\geq 0}$ *are stable, i.e.,*

$$\lim_{t\to\infty} ||S_\alpha(t)|| = 0, \ \lim_{t\to\infty} ||P_\alpha(t)|| = 0.$$

Now we show a particular case, in which (**R**) is satisfied.

Proposition 4.2. *If the semigroup* $\{T(t)\}_{t\geq 0}$ *generated by* A *is exponentially stable, i.e., there are positive numbers* a, M *such that*

$$||T(t)|| \leq Me^{-at},$$

then (**R**) *is testified.*

Proof. Let $E_{\alpha,\beta}$ be the Mittag-Leffler function, that is,

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \ \alpha, \beta > 0, \ z \in \mathbb{C}.$$

By the fact that (see, e.g., [28])

$$\int_0^\infty \phi_\alpha(\theta) e^{-z\theta} d\theta = E_{\alpha,1}(-z),$$
$$\int_0^\infty \alpha\theta\phi_\alpha(\theta) e^{-z\theta} d\theta = E_{\alpha,\alpha}(-z),$$

we have

$$\begin{aligned}
||S_\alpha(t)|| &\le \int_0^\infty \phi_\alpha(\theta)||T(\theta t^\alpha)||d\theta \\
&\le M\int_0^\infty \phi_\alpha(\theta)e^{-at^\alpha\theta}d\theta = ME_{\alpha,1}(-at^\alpha), \\
||P_\alpha(t)|| &\le \int_0^\infty \alpha\theta\phi_\alpha(\theta)||T(\theta t^\alpha)||d\theta \\
&\le M\int_0^\infty \alpha\theta\phi_\alpha(\theta)e^{-at^\alpha\theta}d\theta = ME_{\alpha,\alpha}(-at^\alpha).
\end{aligned}$$

On the other hand, we have the following asymptotic expansion for $E_{\alpha,\beta}$ as $z \to \infty$ (see, e.g., [12]):

$$E_{\alpha,\beta}(z) = \begin{cases} \frac{1}{\alpha}z^{(1-\beta)/\alpha}\exp z^{1/\alpha} + \varepsilon_{\alpha,\beta}(z) & \text{if } |\arg z| \le \frac{1}{2}\pi\alpha, \\ \varepsilon_{\alpha,\beta}(z) & \text{if } |\arg(-z)| \le (1-\frac{1}{2}\alpha)\pi, \end{cases}$$

where

$$\varepsilon_{\alpha,\beta}(z) = -\sum_{n=1}^{N-1}\frac{z^{-n}}{\Gamma(\beta-\alpha n)} + O(|z|^{-N}), \text{ as } z \to \infty.$$

Thus, in our case

$$\begin{aligned}
||S_\alpha(t)|| &\le ME_{\alpha,1}(-at^\alpha) = M\varepsilon_{\alpha,1}(-at^\alpha), \\
||P_\alpha(t)|| &\le ME_{\alpha,\alpha}(-at^\alpha) = M\varepsilon_{\alpha,\alpha}(-at^\alpha).
\end{aligned}$$

Two last inequalities ensure that $||S_\alpha(t)||$ and $||P_\alpha(t)||$ tend to zero as $t \to +\infty$. The proposition is proved.

The following theorem is devoted to the stability result.

Theorem 4.3. *Let the hypotheses* $(\mathbf{A}), (\mathbf{F}), (\mathbf{G}), (\mathbf{R})$ *and relation* (4.1) *hold. Then there exists an integral solution u of problem* (1.1)-(1.2) *such that* $\lim_{t\to+\infty} u(t) = 0$, *provided that* $\Psi_f(0) = 0$ *and*

$$\liminf_{r\to\infty}\frac{1}{r}\Big[\Psi_g(r)\sup_{t\ge 0}||S_\alpha(t)|| + \Psi_f(2r)\sup_{t\ge 0}\int_0^t (t-s)^{\alpha-1}||P_\alpha(t-s)||m(s)ds\Big] < 1. \quad (4.2)$$

Proof. For each $\varphi \in \mathcal{C}_h$, by the same arguments as in the proof of Theorem 3.4, condition (4.2) ensures the existence of a number $R > 0$ such that $\mathcal{F}(B_R) \subset B_R$. Now we denote

$$\mathcal{M}_R = \{u \in B_R : u(t) \to 0 \text{ as } t \to +\infty\}.$$

Since $\mathcal{F}$ is $\chi_{\mathcal{BC}}$-condensing due to relation (4.1), it remains to show that $\mathcal{F}(\mathcal{M}_R) \subset \mathcal{M}_R$. Let $u \in \mathcal{M}_R$, we prove that $\mathcal{F}(u)(t) \to 0$ as $t \to +\infty$. Let $\epsilon > 0$ be given. Then there is $t_1 > 0$ such that

$$||u(t)||_X < \epsilon, \text{ for all } t \ge t_1, \quad (4.3)$$

$$||u(t+\tau)||_X < \epsilon, \text{ for all } t \ge t_1 + h, \tau \in [-h, 0]. \quad (4.4)$$

Taking into account assumption (**R**), there exist $t_2, t_3 > 0$ such that

$$||S_\alpha(t)|| < \epsilon, \text{ for all } t \geq t_2, \tag{4.5}$$

$$\int_0^{t_1+h} (t_1 + h - s)^{\alpha-1} ||P_\alpha(t-s)|| m(s) ds < \epsilon, \text{ for all } t \geq t_3. \tag{4.6}$$

One has, for $t > t_1 + h$,

$$\begin{aligned}
||\mathcal{F}(u)(t)|| &\leq ||S_\alpha(t)||(||\varphi(0)||_X + ||g(u)(0)||_X) \\
&\quad + \int_0^t (t-s)^{\alpha-1} ||P_\alpha(t-s)|| m(s) \Psi_f(||u(s)||_X + \sup_{\tau \in [-h,0]} ||u(s+\tau)||_X) ds \\
&\leq ||S_\alpha(t)||(R + \Psi_g(R)) + \Psi_f(2R) \int_0^{t_1+h} (t-s)^{\alpha-1} ||P_\alpha(t-s)|| m(s) ds \\
&\quad + \Psi_f(2\epsilon) \int_{t_1+h}^t (t-s)^{\alpha-1} ||P_\alpha(t-s)|| m(s) ds, \\
&\leq ||S_\alpha(t)||(R + \Psi_g(R)) + \Psi_f(2R) \int_0^{t_1+h} (t_1 + h - s)^{\alpha-1} ||P_\alpha(t-s)|| m(s) ds \\
&\quad + \Psi_f(2\epsilon) \int_{t_1+h}^t (t-s)^{\alpha-1} ||P_\alpha(t-s)|| m(s) ds.
\end{aligned}$$

Now for $t > \max\{t_1 + h, t_2, t_3\}$, we have

$$||\mathcal{F}(u)(t)|| \leq [R + \Psi_g(R) + \Psi_f(2R)]\epsilon + C_0 \Psi_f(2\epsilon),$$

where

$$C_0 = \sup_{t \geq 0} \int_0^t (t-s)^{\alpha-1} ||P_\alpha(t-s)|| m(s) ds$$

which is finite due to (3.5). Since Ψ_f is continuous and $\Psi_f(0) = 0$, it follows that $\Psi_f(2\epsilon) \to 0$ as $\epsilon \to 0$.

Thus $\mathcal{F}(\mathcal{M}_R) \subset \mathcal{M}_R$ and Theorem 2.10 ensures the existence of an integral solution $u(\cdot, \varphi)$ for (1.1)-(1.2) in $\mathcal{M}_R$. The proof is complete.

Corollary 4.4. *Let the hypotheses of Theorem 4.3 hold. Assume that* $\Psi_g(0) = \Psi_f(0) = 0$. *If the solution of problem* (1.1)-(1.2) *is unique for each given initial datum* φ*, then the zero solution is asymptotically stable.*

Proof. Since $\Psi_g(0) = \Psi_f(0) = 0$, $u = 0$ is a solution of (1.1)-(1.2) with respect to the initial datum $\varphi = 0$. The conclusion follows from Theorem 4.3.

Remark 4.5. *1. The uniqueness of solution to* (1.1)-(1.2) *is fulfilled if* (**F**) *and* (**G**) *are replaced by stronger ones, e.g.,*

($\mathbf{F}^*$) *The nonlinear function* f *is such that* $f(t, 0, 0) = 0$ *and*

$$||f(t, u, \xi) - f(t, v, \eta)||_X \leq m(t)(||u - v||_X + ||\xi - \eta||_{\mathcal{C}_h}),$$

where $m \in L^p_{loc}(\mathbb{R}^+), p > \frac{1}{\alpha}$.

($\mathbf{G}^*$) *The nonlocal function g satisfies $g(0) = 0$ and it is Lipschitzian, i.e.,*

$$||g(u) - g(v)||_{\mathcal{C}_h} \le \eta ||u - v||_{\mathcal{BC}},$$

for all $u, v \in \mathcal{BC}$.

2. In the case when $\alpha = 1$, one knows that $S_\alpha(t) = P_\alpha(t) = T(t)$, thanks to the fact that

$$\mathcal{L}(S_1)(\lambda) = \mathcal{L}(P_1)(\lambda) = (\lambda I - A)^{-1} = \mathcal{L}(T)(\lambda).$$

Moreover, if the nonlocal condition is removed and the delay is absent, that is $g = 0, h = 0$ and $f = f(t, u)$, then condition (3.5) is reduced to

$$\liminf_{r\to\infty} \frac{1}{r}\Psi_f(r) \sup_{t\ge 0} \int_0^t ||T(t-s)|| m(s) ds < 1. \tag{4.7}$$

In a more specific situation, if $\Psi_f(r) = r$, m is a constant and $||T(t)|| \le e^{-at}$, then (4.7) is satisfied for $m < a$. The latter condition is exactly the stability condition given in the work of Travis and Webb [27].

However, in the case when $0 < \alpha < 1$, the problem is much more complicated in the sense that, the resolvent operators $S_\alpha(t)$ and $P_\alpha(t)$ have no exponential decay like $T(t)$ when $t \to +\infty$.

5. An example

Consider the following fractional partial differential equation

$$\begin{aligned} \partial_t^\alpha u(x,t) = \partial_x^2 u(x,t) + \mu(t)\ln(1 + u^2(x,t)) \\ + \int_0^\pi dy \int_{t-h}^t \xi(t,y) K(x, y, u(y,s)) ds,\ \alpha \in (0,1], \end{aligned} \tag{5.1}$$

for $x \in (0,\pi), t > 0$, subject to the boundary condition

$$u(0,t) = u(\pi,t) = 0, \tag{5.2}$$

and the nonlocal condition

$$u(x,s) + \sum_{j=1}^p \beta_j u(x, t_j + s) = \varphi(x,s),\ s \in [-h, 0],\ x \in [0,\pi], \tag{5.3}$$

where $\beta_j \in \mathbb{R}, t_j > 0, j = 1, ..., p$, are given. In the above model, ∂_t^α stands for the Caputo derivative of order α with respect to the time variable, ∂_x denotes the generalized derivative in variable x.

Let $A = \partial_x^2$ with the domain $D(A) = H^2(0,\pi) \cap H_0^1(0,\pi)$ and $X = L^2(0,\pi)$ with the norm

$$||v|| = \Big(\int_0^\pi |v(x)|^2 dx\Big)^{\frac{1}{2}}.$$

It is known that A generates a compact (and hence norm continuous) semigroup $\{T(t)\}_{t\ge 0}$ such that

$$||T(t)|| \le Me^{-t},\ t \ge 0.$$

This guarantees that the corresponding resolvent operators $\{S_\alpha(t), P_\alpha(t)\}_{t\geq 0}$ are compact and stable, thanks to Lemma 2.3 and Proposition 4.2. Thus the hypotheses (**A**) and (**R**) are verified.

Regarding the nonlinearity in equation (5.1), we assume that

(1) $\mu \in L^p_{loc}(\mathbb{R}^+), p > \frac{1}{\alpha}$, is a nonnegative function, ξ is continuous and $|\xi(t,y)| \leq \nu(t)$ for all $y \in [0,\pi], t \geq 0$, where $\nu \in L^p_{loc}(\mathbb{R}^+)$;

(2) K is defined in $[0,\pi] \times [0,\pi] \times \mathbb{R}$ such that K is continuous, and

$$K(x,y,0) = 0,$$
$$|K(x,y,z_1) - K(x,y,z_2)| \leq w(x)|z_1 - z_2|,\ \forall x,y \in [0,\pi], z_1, z_2 \in \mathbb{R},$$

where $w \in L^2(0,\pi)$.

Let $f_2(t,u_t) = \int_0^\pi dy \int_{t-h}^t \xi(t,y)K(x,y,u(y,s))ds$. Then

$$||f_2(t,\phi)|| \leq \nu(t)||w||\sqrt{\pi}\int_{-h}^0 ||\phi(\cdot,s)||ds \leq \nu(t)||w||h\sqrt{\pi}\|\phi\|_{\mathcal{C}_h}.$$

Let $f_1(t,z) = \mu(t)\ln(1+z^2)$. Then

$$f_1(t,0) = 0,$$
$$|f_1(t,z_1) - f(t,z_2)| \leq \mu(t)|z_1 - z_2|,\ \forall t \geq 0, z_1, z_2 \in \mathbb{R}.$$

Hence, the nonlinearity

$$f(t,v,\phi) = f_1(t,v) + f_2(t,\phi)$$

satisfies ($\mathbf{F}^*$) with $m(t) = \mu(t) + \nu(t)||w||h\sqrt{\pi}$, $\Psi_f(r) = r$.

As far as the nonlocal function is concerned, $g(u)(s) = \sum_{j=1}^N \beta_j u(t_j + s)$ satisfies the Lipschitz condition with the Lipschitz constant $\eta = \sum_{j=1}^N \beta_j$. Thus, ($\mathbf{G}^*$) takes place.

Since $\Psi_g(r) = \Psi_f(r) = r$, one sees that (3.5) is equivalent to

$$\eta \sup_{t\geq 0} ||S_\alpha(t)|| + 2\sup_{t\geq 0}\int_0^t (t-s)^{\alpha-1}||P_\alpha(t-s)||m(s)ds < 1. \tag{5.4}$$

Noting that, since $P_\alpha(t)$ is compact for $t > 0$, one can take $k(t,s) = 0$, then (5.4) implies (4.1). Therefore, if (5.4) is satisfied, the zero solution of problem (5.1)-(5.3) is asymptotically stable.

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