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FIXED POINT THEORY FOR QUASI-CONTRACTION MAPS IN b-METRIC SPACES

A. AMINI-HARANDI

Department of Mathematics, University of Isfahan,
Isfahan, 81745-163, Iran

School of Mathematics, Institute for Research in Fundamental Sciences (IPM)
Tehran, Iran
E-mail: aminih_a@yahoo.com

Abstract. In this paper, we first give a new fixed point theorem for quasi-contraction maps in b-metric spaces which gives a partial answer to a question raised in [S. L. Singh, S. Czerwik, K. Król, A. Singh, *Coincidences and fixed points of hybrid contractions*, Tamsui Oxf. J. Math. Sci., 24 (2008), 401-416]. Then we derive some fixed point results for contractive type maps. An example is also given to support our main result. Our results extend and improve some fixed point theorems in the literature.
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1. Introduction

Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be quasi-contraction if there exists $c < 1$ such that

$$d(Tx, Ty) \leq c \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},$$

for any $x, y \in X$. In 1974, Ćirić [3] introduced these maps and proved an existence and uniqueness fixed point theorem for quasi-contraction maps. Our aim in this paper is to prove a fixed point result for quasi-contraction maps in the setting of b-metric spaces.

The concept of a b-metric space was introduced by Czerwik in [5], see also [6, 7].

Definition 1.1. *Let X be a nonempty set. Let $D : X \times X \to [0, \infty)$ be a function satisfies the following conditions:*

(1) $D(x, y) = 0$ *if and only if* $x = y$*;*
(2) $D(x, y) = D(y, x)$*, for each* $x, y \in X$*;*
(3) $D(x, y) \leq K(D(x, z) + D(z, y))$*, for each* $x, y, z \in X$*, where* $K > 0$ *is a constant.*

Then the pair (X, D) is called a b-metric space or a metric type space.

There are many examples where the triangle inequality fails but property (3) holds, see [5, 8]. Every metric space (X, d) satisfies (1)-(3) with $K = 1$. Without loss of generality, we may always assume that $K \geq 1$. In this case, it is easy to show that for each $x_1, ..., x_n \in X$, we have

$$D(x_1, x_n) \leq KD(x_1, x_2) + K^2 D(x_2, x_3) + ... + K^{n-1} D(x_{n-1}, x_n).$$

For more on fixed point theory in b-metric spaces see [5, 6, 7, 1, 2] and references therein.

In this paper, we first obtain a generalization of the above mentioned Ćirić's result to b-metric spaces which gives a partial answer to the question raised in [9]. Then we derive a fixed point theorem for Lipschitzian mappings which improves Theorem 3.3 in [6]. We also give an example to support our main result.

2. Main results

We first introduce the concept of a *quasi-contraction* map in b-metric spaces.

Definition 2.1. *Let (X, D) be a b-metric space. The self-map $T : X \to X$ is said to be quasi-contraction if there exists a $0 \leq c < 1$ such that*

$$D(Tx, Ty) \leq c \max\{D(x, y), D(x, Tx), D(y, Ty), D(x, Ty), D(y, Tx)\},$$

for any $x, y \in X$.

Remark 2.2. *In the reasoning of some recent fixed point results in the setting of b-metric spaces* [9]*, the authors used the following implication*

$$x_n \to x \Rightarrow D(x_n, y) \to D(x, y) \text{ for each } y \in X,$$

which, as the following example shows, does not hold in b-metric spaces, in general.

Example 2.3. *Let $X = \mathbb{N} \cup \{\infty\}$ and let $D : X \times X \to \mathbb{R}$ defined by*

$$D(m, n) = \begin{cases} 0, & \text{if } m = n \\ \frac{1}{2}, & \text{if } m \neq n \in \mathbb{N} \text{ and } m + n \text{ is odd} \\ |\frac{1}{m} - \frac{1}{n}|, & \text{otherwise} \end{cases}$$

Then it is easy to see that for each $m, n, p \in X$, we have

$$D(m, p) \leq 10(D(m, n) + D(n, p)).$$

Thus (X, D) is a b-metric space. Let $x_n = 2n$ for each $n \in \mathbb{N}$, then

$$D(2n, \infty) = \frac{1}{2n} \to 0 \text{ as } n \to \infty,$$

that is, $x_n \to \infty$ but $D(x_n, 1) = \frac{1}{2} \not\to D(\infty, 1) = 1$ as $n \to \infty$.

Motivated by the above example, we introduce the following property which is crucial in proving our main result.

Definition 2.4. *Let (X, D) be a b-metric space. We say that (X, D) has the Fatou property if*

$$D(x, y) \leq \liminf_{n \to \infty} D(x_n, y),$$

whenever $\lim_{n\to\infty} D(x_n, x) = 0$ *and* $y \in X$.

It is clear that every metric space enjoys Fatou property. Other examples are given below:

Example 2.5. *Let* X *be the set of Lebesgue measurable functions on* $[0,1]$ *such that*

$$\int_0^1 |f(x)|^2 dx < \infty.$$

Define $D : X \times X \to [0, \infty$ *by*

$$D(f,g) = \int_0^1 |f(x) - g(x)|^2 dx.$$

Then (X, D) *is a b-metric space* [6, 7]. *It is straightforward to show that,* (X, D) *has the Fatou property.*

Example 2.6. *Let* X *be a nonempty set and* A, B *are two nonempty disjoint subsets of* X *such that* $X \setminus (A \cup B) \neq \emptyset$. *Let* $D : X \times X \to [0, \infty)$ *defined by* $D(x, y) = 0$ *for* $x = y$, *and for each* $x \neq y$

$$D(x,y) = \begin{cases} 3 & (x,y) \in A \times B \text{ or } (x,y) \in B \times A, \\ 1 & \text{otherwise.} \end{cases}$$

Let $x \in A$, $y \in B$ *and* $z \notin A \cup B$. *Then*

$$3 = D(x,y) > D(x,z) + D(z,y) = 2,$$

and so (X, D) *is not a metric space. It is easy to show that for each* $x, y, z \in X$, *we have*

$$D(x,y) \leq 2(D(x,z) + D(z,y)).$$

Hence (X, D) *is a b-metric space. To show that* (X, D) *enjoys Fatou property, let* $x_n \in X$ *be a sequence which converges to* $x \in X$, *that is,* $D(x_n, x) \to 0$. *Then there exists* $N \in \mathbb{N}$ *such that* $x_n = x$ *for* $n \geq N$. *Thus for each* $y \in X$, $D(x_n, y) = D(x, y)$ *for* $n \geq N$. *Hence*

$$\liminf_{n\to\infty} D(x_n, y) = D(x, y),$$

and this completes the proof.

Remark 2.7. *Let* (X, D) *and the sequence* $\{x_n\}$ *be as in the Example 2. Then*

$$1 = D(\infty, 1) \not\leq \liminf_{n\to\infty} D(x_n, 1) = \frac{1}{2},$$

and so the b-metric space (X, D) *does not enjoy the Fatou property.*

Now, we are ready to state our main result which extend the main result of [1].

Theorem 2.8. *Let* (X, D) *be a complete b-metric space with the Fatou property and let* $T : X \to X$ *be a quasi-contraction map with* $c < \frac{1}{K}$. *Then,* T *has a unique fixed point* $\overline{x} \in X$ *and for each* $x \in X$, $\lim_{n\to\infty} T^n x = \overline{x}$.

Proof. Since T is quasi-contraction then for each $x, y \in X$, we have

$$D(Tx, Ty) \leq c \max\{D(x,y), D(x,Tx), D(y,Ty), D(x,Ty), D(y,Tx)\},$$

Let $x \in X$ be arbitrary. If for some $n_0 \in \mathbb{N}$, $T^{n_0-1}x = T^{n_0}x = T(T^{n_0-1}x)$ then $T^n x = T^{n_0-1}x$ for $n \geq n_0$. Thus $T^{n_0-1}x$ is a fixed point of T, the sequence $\{T^n x\}$ is convergent to $T^{n_0-1}x$ and we are finished. So, we may assume that $T^{n-1}x \neq T^n x$ for each $n \in \mathbb{N}$. Now, we show that $\{T^n x\}$ is a Cauchy sequence. To prove the claim, we first show by induction that for each $n \geq 2$ there exists $1 \leq m \leq n$ such that

$$D(T^{n-1}x, T^n x) \leq c^{n-1} D(x, T^m x). \tag{2.1}$$

If $n = 2$ then, we get

$$\begin{aligned} D(Tx, T^2x) &\leq c \max\{D(x, Tx), D(Tx, T^2x), D(x, T^2x)\}) \\ &= c \max\{D(x, Tx), D(x, T^2x)\} = cD(x, T^m x), \end{aligned}$$

for some $1 \leq m \leq 2$. Thus (2.1) holds for $n = 2$. Suppose that (2.1) holds for each $k < n$ and we show that it holds for $k = n$. Since T is quasi-contraction then, we have $D(T^{n-1}x, T^n x) \leq cu$, where

$$u \in \{D(T^{n-2}x, T^{n-1}x), D(T^{n-2}x, T^n x)\}.$$

It is trivial that (2.1) holds if $u = D(T^{n-2}x, T^{n-1}x)$. Now suppose that

$$u = D(T^{n-2}x, T^n x).$$

In this case we have

$$D(T^{n-2}x, T^n x) \leq cu_1,$$

where

$$\begin{aligned} u_1 \in \{&D(T^{n-3}x, T^{n-1}x), D(T^{n-2}x, T^{n-1}x), \\ &D(T^{n-3}x, T^{n-2}x), D(T^{n-3}x, T^n x), D(T^{n-1}x, T^n x)\}. \end{aligned}$$

Again, it is trivial that (2.1) holds if $u_1 = D(T^{n-1}x, T^n x)$ or $u_1 = D(T^{n-3}x, T^{n-2}x)$. If $u_1 = D(T^{n-2}x, T^{n-1}x)$, then

$$D(T^{n-1}x, T^n x) \leq c^2 D(T^{n-2}x, T^{n-1}x).$$

By assumption of induction, there exists $1 \leq m \leq n-1$ such that

$$D(T^{n-2}x, T^{n-1}x) \leq c^{n-2} D(x, T^m x).$$

Hence

$$D(T^{n-1}x, T^n x) \leq c^n D(x, T^m x) \leq c^{n-1} D(x, T^m x).$$

If $u_1 = D(T^{n-3}x, T^{n-1}x)$, then

$$D(T^{n-1}x, T^n x) \leq c^2 D(T^{n-3}x, T^{n-1}x).$$

If $u_1 = D(T^{n-3}x, T^n x)$, then

$$D(T^{n-1}x, T^n x) \leq c^2 D(T^{n-3}x, T^n x).$$

Therefore by continuing this process, we see that (2.1) holds for each $n \geq 2$. From (2.1) we get

$$D(T^{n-1}x, T^n x) \leq c^{n-1}(KD(x, Tx) + K^2 D(Tx, T^2x) + ... + K^n D(T^{n-1}x, T^n x)).$$

Let $a_n = K^n D(T^{n-1}x, T^n x)$ and let $s_n = \sum_{i=1}^n a_i$. Then from the above, we have

$$a_n \leq K^n c^{n-1} s_n,$$

and so

$$\frac{a_n}{s_n} \leq K(Kc)^{n-1}.$$

Since $Kc < 1$ then the series $\sum_{n=1}^{\infty} K(Kc)^{n-1}$ is convergent. Thus

$$\sum_{n=1}^{\infty} \frac{a_n}{s_n} < \infty. \tag{2.2}$$

Now, we show that

$$\sum_{n=1}^{\infty} a_n < \infty$$

or equivalently

$$\lim_{n\to\infty} s_n < \infty.$$

On the contrary, assume that $\lim\limits_{n\to\infty} s_n = \infty$. From (2.2), we get there exists $n \in \mathbb{N}$ such that for each $m \in \mathbb{N}$,

$$\frac{1}{2} > \sum_{j=1}^{m} \frac{a_{n+j}}{s_{n+j}} \geq \frac{a_{n+1} + ... + a_{n+m}}{s_{n+m}} = \frac{s_{n+m} - s_n}{s_{n+m}} = 1 - \frac{s_n}{s_{n+m}}.$$

Therefore

$$\frac{1}{2} > 1 - \frac{s_n}{s_{n+m}}, \text{ for each } m \in \mathbb{N}.$$

Letting $m \to \infty$ in the above inequality, we get $\frac{1}{2} \geq 1$, a contradiction. Since

$$\sum_{n=1}^{\infty} K^{n+1} D(T^n x, T^{n+1} x) < \infty$$

then

$$\begin{aligned} D(T^n x, T^m x) &\leq \sum_{j=1}^{m-n} K^j D(T^{n+j-1} x, T^{n+j} x) \\ &\leq \sum_{j=1}^{m-n} K^{n+j} D(T^{n+j-1} x, T^{n+j} x) < \epsilon, \end{aligned}$$

for sufficiently large $m > n \in \mathbb{N}$. Then $\{T^n x\}$ is a Cauchy sequence and since (X, D) is a complete b-metric space, then there exists a $\overline{x} \in X$ such that $\lim_{n\to\infty} T^n(x) = \overline{x}$, that is, $\lim_{n\to\infty} D(T^n x, \overline{x}) = 0$. Since

$$D(T\overline{x}, T^n x) \leq K(D(T\overline{x}, \overline{x}) + D(\overline{x}, T^n x)),$$

then

$$\limsup_{n\to\infty} D(T\overline{x}, T^n x) \leq K D(T\overline{x}, \overline{x}). \tag{2.3}$$

Since

$$D(T\overline{x}, T^{n+1} x) \leq$$
$$c \max\{D(\overline{x}, T^n x), D(\overline{x}, T\overline{x}), D(T^n x, T^{n+1} x), D(\overline{x}, T^{n+1} x), D(T^n x, T\overline{x})\},$$

then, we have

$$\limsup_{n\to\infty} D(T\overline{x}, T^{n+1}x) \le c \limsup_{n\to\infty} D(T\overline{x}, T^n x) \le (cK)D(\overline{x}, T\overline{x}).$$

Since (X, D) has the Fatou property, letting $n \to \infty$ from the above, we get

$$D(T\overline{x}, \overline{x}) \le \liminf_{n\to\infty} D(T\overline{x}, T^n x) \le$$

$$\limsup_{n\to\infty} D(T\overline{x}, T^n x) = \limsup_{n\to\infty} D(T\overline{x}, T^{n+1}x)$$

$$\le (cK)D(T\overline{x}, \overline{x}),$$

which yields $D(T\overline{x}, \overline{x}) = 0$, and so $\overline{x} = T\overline{x}$ (note that $Kc < 1$). To prove the uniqueness, let us assume that $\overline{x}$ and $\overline{y}$ are fixed points of T. Then

$$D(\overline{x}, \overline{y}) = D(T\overline{x}, T\overline{y})$$

$$\le c \max\{D(\overline{x}, \overline{y}), D(\overline{x}, T\overline{x}), D(\overline{y}, T\overline{y}), D(\overline{x}, T\overline{y}), D(\overline{y}, T\overline{x})\} = c \max D(\overline{x}, \overline{y}),$$

and so $\overline{x} = \overline{y}$.

Let $T : X \to X$ be a map. T is called Lipschitzian if there exists a constant $\lambda \ge 0$ such that

$$D(Tx, Ty) \le \lambda D(x, y),$$

for each $x, y \in X$. The smallest constant λ will be denoted $Lip(T)$.

In the following, we give a fixed point theorem for Lipschitzian mappings in b-metric spaces without assuming Fatou property.

Corollary 2.9. *Let (X, D) be a b-metric space and let $T : X \to X$ be a map satisfies*

$$D(Tx, Ty) \le cD(x, y),$$

for each $x, y \in X$ where $c < \frac{1}{K}$. Then, T has a unique fixed point $\overline{x}$ and for each $x \in X$,

$$\lim_{n\to\infty} T^n x = \overline{x}.$$

Proof. Since T is a quasi-contraction map with $c < \frac{1}{K}$, then by the proof of Theorem 2.6,

$$\lim_{n\to\infty} T^n x = \overline{x},$$

for each $x \in X$. Since

$$D(T^{n+1}x, T\overline{x}) \le cD(T^n x, \overline{x}),$$

and $\lim_{n\to\infty} D(T^n x, \overline{x}) = 0$ then $\lim_{n\to\infty} T^{n+1}x = T\overline{x}$. Since the limit of the sequences in b-metric spaces is unique then, we get $T\overline{x} = \overline{x}$, and we are finished.

The following Corollary improves Theorem 3.3 in [6].

Corollary 2.10. *Let (X, D) be a b-metric space and let $T : X \to X$ be a map such that T^n is Lipschitzian for any $n \in \mathbb{N}$ and $\lim_{n\to\infty} Lip(T^n) = 0$. Then, T has a unique fixed point $\overline{x}$ and for each $x \in X$, $\lim_{n\to\infty} T^n x = \overline{x}$.*

Proof. Since $\lim_{n\to\infty} Lip(T^n) = 0$, then there exists $N \in \mathbb{N}$ such that

$$Lip(T^N) < \frac{1}{K}.$$

Then by Corollary 2.9, T^N has a unique fixed point $\overline{x}$ and for each $x \in X$, $\lim_{n\to\infty}(T^N)^n x = \lim_{n\to\infty} T^{Nn}x = \overline{x}$. Since $T^N\overline{x} = \overline{x}$ then, we have $T^N(T\overline{x}) = T\overline{x}$

and so $T\overline{x}$ is a fixed point of T^N. Since T^N has a unique fixed point, then, we get $T\overline{x} = \overline{x}$. Now we show that $\lim_{n\to\infty} T^n x = \overline{x}$. Since $\lim_{k\to\infty} T^{Nk}x = \overline{x}$ and $\lim_{k\to\infty} Lip(T^k) = 0$ then for each $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that

$$D(T^{Nk}, \overline{x}) < \frac{\epsilon}{2K} \text{ and } \mathrm{Lip}(T^{Nk}) < \frac{\epsilon}{2K \max(D(x,Tx),...,D(x,T^N x))}.$$

for each $k \geq k_0$. Let $n > Nk_0$ and $k \in \mathbb{N}$ be such that $Nk \leq n < N(k+1)$. Then

$$D(T^n x, \overline{x}) \leq K(D(T^n x, T^{Nk}x) + D(T^{Nk}x, \overline{x})) \leq$$

$$K(\mathrm{Lip}(T^{Nk}) \max(D(x,Tx),...,D(x,T^N x)) + \frac{\epsilon}{2K}) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Now, we give an example illustrating our main result.

Example 2.11. *Let $X = [0,1]$ and let $D(x,y) = |x-y| + |x-y|^2$ for each $x, y \in X$. It is easy to see that (X, D) is a complete b-metric space with Fatou property with $K = 3$. Let $T : X \to X$ be defined by*

$$Tx = \begin{cases} \frac{1}{4}, & x = 1 \\ \frac{1}{2}. & x \neq 1 \end{cases}$$

It is straightforward to see that for each $x, y \in X$, we have

$$D(Tx,Ty) \leq \frac{1}{4} \max\{D(x,y), D(x,Tx), D(y,Ty), D(x,Ty), D(y,Tx)\}.$$

Since $\frac{1}{4} \leq \frac{1}{K}$, then by Theorem 2.8, T has a unique fixed point ($x = \frac{1}{2}$ is the unique fixed point of T).

Remark 2.12. *Fixed and common fixed point theorems for quasi-contraction type mappings have some applications in the existence theory of solution of variational inequalities, see* [4] *and references therein.*

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