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LERAY-SCHAUDER ALTERNATIVES IN BANACH ALGEBRA INVOLVING THREE OPERATORS WITH APPLICATIONS

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Abstract. In this paper, we establish some nonlinear alternatives of Leray-Schauder type in Banach algebra satisfying certain sequential condition $(\mathcal{P})$ for the sum and the product of nonlinear weakly sequentially continuous operators and we give an example of application to a functional nonlinear integral equation.

Key Words and Phrases: Banach algebra, sequentially weakly continuous, fixed point theorems, relatively weakly compact, integral equations.

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1. Introduction

One of the main object of research on nonlinear functional analysis is the study of functional integral equations. In particular, we cite nonlinear integral equations of mixed type that have been discussed for a long time in Banach spaces and their study has been developed in various directions. Some of these equations can be formulated into nonlinear operator equations:

$$x = AxBx + Cx \tag{1.1}$$

The resolution of equation (1.1) was the main interest of many scientists and their results were very interesting (see for example [3, 4, 5, 9, 10, 11, 12, 13]). Recently, Dhage in his papers [9, 10] initiated the study of integral equation in Banach algebra via fixed point techniques and he developed his researches to prove Leray-Schauder type hybrid fixed point theorems [11, 12] . These studies were mainly based on Lipschitz and complete continuity conditions on operators A, B and C.
Since the weak topology is the practice setting and natural to investigate the problems of existence of solutions of different types of nonlinear integral equations and nonlinear differential equations in Banach algebras, it turns out that the results mentioned

above cannot be easily applied. This is because a bounded linear functional φ acting on Banach algebra does not necessarily satisfy the following inequality:

$$|\varphi(x, y)| \leq c|\varphi(x)||\varphi(y)|$$

with $c > 0$ and $x, y \in X$.

In 2010, Ben Amar, Chouayekh, and Jeribi [2] introduced a new class of Banach algebra satisfying certain sequential condition called condition $(\mathcal{P})$ (see Definition 3.3). They focused on the main conditions which were formulated in term of weak sequential continuity to the three nonlinear operators A, B and C involved in equation (1.1) and proved some new fixed point theorems in a nonempty closed convex subset of any Banach algebras or Banach algebras satisfying the condition $(\mathcal{P})$.

In the present paper, we prove a Leray-Schauder type fixed point theorems in Banach algebra and Banach algebra satisfying condition $(\mathcal{P})$ under weak topology setting. Our results are applied to study the existence of solutions for the following nonlinear functional integral equation (in short FIE):

$$x(t) = a(\nu(t)) + (Tx)(t)\Big[\Big(q(t) + \int_0^{\sigma(t)} p(t, s, g(x(\xi(s))), h(x(\eta(s))))ds\Big).u\Big] \qquad (1.2)$$

for all $t \in J$, where $u \neq 0$ is a fixed vector of X and the functions $a, \nu, q, \sigma, \xi, \eta, p, g, h$ and T are given while $x = x(t)$ is an unknown function.

The content of this paper in organized in four sections. In section 2, we give some definitions and preliminaries that will be needed in the sequel. In section 3, we present new fixed point theorems of Leray-Schauder type in Banach algebras and Banach algebra satisfying condition $(\mathcal{P})$, and in the last section we apply one of the obtained result in precedent section to investigate the FIE (1.2).

2. Preliminaries

In this section, we recall some definitions and we give some results that we will need in the sequel.

Definition 2.1. An algebra $\mathcal{E}$ is a vector space endowed with an internal composition law noted by (.) *i, e.*,

$$\begin{cases} (.) : \mathcal{E} \times \mathcal{E} \longrightarrow \mathcal{E} \\ (x, y) \longrightarrow x.y \end{cases}$$

which is associative and bilinear.

A normed algebra is an algebra endowed with a norm satisfying the following property

$$\text{for all } x, y \in \mathcal{E};\ \|x.y\| \leq \|x\|\|y\|.$$

A complete normed algebra is called a Banach algebra.

Definition 2.2. Let X be a Banach space. An operator $A : X \to X$ is said to be weakly compact if $A(B)$ is relatively weakly compact for every bounded subset $B \subset X$.

Definition 2.3. Let X be a Banach space. An operator $A : X \to X$ is said to be sequentially weakly continuous on X if for every sequence $(x_n)_n$ with $x_n \rightharpoonup x$, we have $Ax_n \rightharpoonup Ax$; here $\rightharpoonup$ denotes weak convergence.

Definition 2.4. Let X be a Banach space with norm $\|.\|$. A mapping $T : X \to X$ is called D-Lipschitz if there exists a continuous nondecreasing function $\Phi_T : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\|Tx - Ty\| \leq \Phi_T(\|x - y\|) \tag{2.1}$$

for all $x, y \in X$ with $\Phi_T(0) = 0$. Sometimes we call the function Φ_T a D-function of T on X. In the special case when $\Phi_T(r) = \alpha r$ for some $\alpha > 0$, T is called a Lipschitz with a Lipschitz constant α. In particular if $\alpha < 1$, T is called a contraction on X with a contraction constant α.

Remark 2.5. Every Lipschitz mapping is D-Lipschitz, but the converse may not be true. If Φ_T is not necessarily nondecreasing and satisfies $\Phi_T(r) < r$ for $r > 0$, then T is called a nonlinear contraction on X.

An important fixed point theorem that has been commonly used in the theory of nonlinear differential and integral equations is the following result proved by Boyd and Wong in [6].

Theorem 2.6. [6] Let $A : X \to X$ be a nonlinear contraction. Then A has a unique fixed point x^* and the sequence $(A^n x)_n$ of successive iterations of A converges to x^* for each $x \in X$.

Now, we state a nonlinear alternative of Leray-Schauder type given in [1] that will be needed in the sequel.

Theorem 2.7. [1, Theorem 3.1] Let X be a Banach space, $\Omega \subset X$ a closed convex subset, $U \subset \Omega$ a weakly open set (with respect to the weak topology of Ω) and such that $0 \in U$. Assume that $\overline{U^w}$ is a weakly compact subset of Ω and $F : \overline{U^w} \to \Omega$ is weakly sequentially continuous mapping. Then, either
$(A1)$ F has a fixed point on U, or
$(A2)$ there is a point $u \in \partial_\Omega U$ (the weak boundary of U in Ω) and $\lambda \in]0, 1[$ with $u = \lambda F u$.

Remark 2.8. The condition that $\overline{U^w}$ is weakly compact in the statement of the last Theorem can be removed if we assume that $F(\overline{U^w})$ is relatively weakly compact.

3. Fixed Point Theorems

In what follows, we are going to give our first nonlinear alternative of Leray-Schauder type in a Banach algebra for three operators.

Theorem 3.1. Let Ω be a closed convex subset in a Banach algebra $\mathcal{E}$, $U \subset \Omega$ a weakly open set (with respect to the weak topology of Ω) such that $0 \in U$ and $\overline{U^w}$

is a weakly compact subset of Ω. Let $A, C : \mathcal{E} \to \mathcal{E}$ $B : \overline{U^w} \to \mathcal{E}$ be three operators satisfying
(i) A and C are D-Lipschitz with D-functions Φ_A and Φ_C respectively,
(ii) A is regular on $\mathcal{E}$, $i, e.$, A maps $\mathcal{E}$ into the set of all invertible elements of $\mathcal{E}$,
(iii) B is weakly sequentially continuous on $\overline{U^w}$,
(iv) $M\Phi_A(r) + \Phi_C(r) < r$ for $r > 0$, with $M = \|B(\overline{U^w})\|$,
(v) $x = AxBy + Cx \Rightarrow x \in \Omega$ for all $y \in \overline{U^w}$,
(vi) $(\frac{I-C}{A})^{-1}$ is weakly sequentially continuous on $B(\overline{U^w})$.
Then either
$(A1)$ the equation $\lambda A(\frac{x}{\lambda})Bx + \lambda C(\frac{x}{\lambda}) = x$ has a solution for $\lambda = 1$, or
$(A2)$ there is an element $u \in \partial_\Omega(U)$ such that $\lambda A(\frac{u}{\lambda})Bu + \lambda C(\frac{u}{\lambda}) = u$ for some $0 < \lambda < 1$.

Proof. Let $y \in \overline{U^w}$ be fixed and define the mapping $\phi_y : \mathcal{E} \to \mathcal{E}$ by

$$\phi_y(x) = AxBy + Cx \tag{3.1}$$

for $x \in \mathcal{E}$. Then for any $x_1, x_2 \in \mathcal{E}$, we have

$$\begin{aligned} \|\phi_y(x_1) - \phi_y(x_2)\| &\le \|Ax_1By - Ax_2By\| + \|Cx_1 - Cx_2\| \\ &\le \|Ax_1 - Ax_2\|\|By\| + \|Cx_1 - Cx_2\| \\ &\le M\Phi_A(\|x_1 - x_2\|) + \Phi_C(\|x_1 - x_2\|). \end{aligned}$$

In view of the hypothesis (iv), we deduce that ϕ_y is a nonlinear contraction on $\mathcal{E}$. Therefore an application of Theorem 2.6 yields that ϕ_y has a unique fixed point, say x in $\mathcal{E}$.
This means $\exists\ !\ x \in X \setminus AxBy + Cx = x$.
By hypothesis (e), it's clear that $x \in \Omega$, so $\exists\ !\ x \in \Omega \setminus AxBy + Cx = x$.
By virtue of the hypothesis (b),

$$\exists\ !\ x \in \Omega \setminus (\frac{I-C}{A})x = By \text{ and so } x = (\frac{I-C}{A})^{-1}By$$

Hence $(\frac{I-C}{A})^{-1}B : \overline{U^w} :\to \Omega$ is well defined.
Since B is weakly sequentially continuous on $\overline{U^w}$ and $(\frac{I-C}{A})^{-1}$ is weakly sequentially continuous on $B(\overline{U^w})$, so by composition we have $(\frac{I-C}{A})^{-1}B$ is weakly sequentially continuous on $\overline{U^w}$.

Now an application of Theorem 2.7 implies that either

$(A1)$ $(\frac{I-C}{A})^{-1}B$ has a fixed point, or

$(A2)$ there is a point $u \in \partial_\Omega U$ and $\lambda \in]0, 1[$ with $u = \lambda(\frac{I-C}{A})^{-1}Bu$.

Assume first that $x \in U$ is a fixed point of the operator $(\frac{I-C}{A})^{-1}B$. Then

$x = (\frac{I-C}{A})^{-1}Bx$ which implies that

$$AxBx + Cx = x.$$

Suppose next that there is an element $u \in \partial_\Omega(U)$ and a real number $\lambda \in]0,1[$ such that $u = \lambda(\frac{I-C}{A})^{-1}Bu$. Then

$$(\frac{I-C}{A})^{-1}Bu = \frac{u}{\lambda},$$

so that

$$\lambda A(\frac{u}{\lambda})Bu + \lambda C(\frac{u}{\lambda}) = u.$$

This completes the proof.

Remark that this result remains true even when $C \equiv 0$, and we get a nonlinear alternative of Leray-Schauder type in a Banach algebra for the product of two operators:

Corollary 3.2. Let Ω be a closed convex subset in a Banach algebra $\mathcal{E}$, $U \subset \Omega$ a weakly open set (with respect to the weak topology of Ω) such that $0 \in U$ and $\overline{U^w}$ is a weakly compact subset of Ω. Let $A : \mathcal{E} \to \mathcal{E}$ $B : \overline{U^w} \to \mathcal{E}$ be two operators satisfying
(i) A is D-Lipschitz with D-function Φ_A ,
(ii) A is regular on $\mathcal{E}$,
(iii) B is weakly sequentially continuous on $\overline{U^w}$,
(iv) $M\Phi_A(r) < r$ for $r > 0$, with $M = \|B(\overline{U^w})\|$,
(v) $x = AxBy \Rightarrow x \in \Omega$ for all $y \in \overline{U^w}$,
(vi) $(\frac{I}{A})^{-1}$ is weakly sequentially continuous on $B(\overline{U^w})$.
Then either
$(A1)$ the equation $\lambda A(\frac{x}{\lambda})Bx = x$ has a solution for $\lambda = 1$, or
$(A2)$ there is an element $u \in \partial_\Omega(U)$ such that $\lambda A(\frac{u}{\lambda})Bu = u$ for some $0 < \lambda < 1$.

Because it lacks the stability of convergence for the product sequences under the weak topology, A. Ben Amar, S. Chouayekh and A. Jeribi have introduced a new class of Banach Algebra:

Definition 3.3. We will say that the Banach algebra $\mathcal{E}$ satisfies condition $(\mathcal{P})$ if

$$(\mathcal{P}) \begin{cases} \text{For any sequences} \{x_n\} \text{ and } \{y_n\} \text{ in } \mathcal{E} \text{ such that } x_n \rightharpoonup x \text{ and } y_n \rightharpoonup y, \\ \text{then } x_n y_n \rightharpoonup xy; \text{ here } \rightharpoonup \text{ denotes weak convergence.} \end{cases}$$

Note that every finite dimensional Banach algebra satisfies condition $(\mathcal{P})$. Even if X satisfies condition $(\mathcal{P})$ then $\mathcal{C}(K, X)$ is also Banach algebra satisfying condition $(\mathcal{P})$, where K is a compact Hausdorff space. The proof is based on Dobrakov's Theorem:

Theorem 3.4. [8, Dobrakov, p.36] Let K be a compact Hausdorff space and X be a Banach space. Let $(f_n)_n$ be a bounded sequence in $\mathcal{C}(K, X)$, and $f \in \mathcal{C}(K, X)$. Then

$(f_n)_n$ is weakly convergent to f if and only if $(f_n(t))_n$ is weakly convergent to $f(t)$ for each $t \in K$.

Now, we will state our results in Banach algebra satisfying condition $(\mathcal{P})$.

Theorem 3.5. Let Ω be a closed convex subset in a Banach algebra $\mathcal{E}$ satisfying the condition $(\mathcal{P})$ and $U \subset \Omega$ a weakly open set (with respect to the weak topology of Ω) such that $0 \in U$. Let $A, C : \mathcal{E} \to \mathcal{E}$ $B : \overline{U^w} \to \mathcal{E}$ be three operators satisfying
(i) A and C are D-Lipschitz with D-functions Φ_A and Φ_C respectively,
(ii) A is regular on $\mathcal{E}$,
(iii) $B(\overline{U^w})$ is bounded with bound M,
(iv) $M\Phi_A(r) + \Phi_C(r) < r$ for $r > 0$,
(v) $x = AxBy + Cx \Rightarrow x \in \Omega$ for all $y \in \overline{U^w}$,
(vi) A, C are weakly sequentially continuous on Ω and B is weakly sequentially continuous $\overline{U^w}$,
(vii) $(\frac{I-C}{A})^{-1}B(\overline{U^w})$ is relatively weakly compact.
Then either
$(A1)$ the equation $\lambda A(\frac{x}{\lambda})Bx + \lambda C(\frac{x}{\lambda}) = x$ has a solution for $\lambda = 1$, or
$(A2)$ there is an element $u \in \partial_\Omega(U)$ such that $\lambda A(\frac{u}{\lambda})Bu + \lambda C(\frac{u}{\lambda}) = u$ for some $0 < \lambda < 1$.

Proof. Similarly to the proof of Theorem 3.1, we obtain $(\frac{I-C}{A})^{-1}B$ is well defined from $\overline{U^w}$ to Ω, it suffices to establish that $(\frac{I-C}{A})^{-1}B$ is weakly sequentially continuous on $\overline{U^w}$. To see this, Let $\{u_n\}$ be a weakly convergent sequence of $\overline{U^w}$ to a point u in $\overline{U^w}$. Now, define the sequence $\{v_n\}$ of the subset Ω by

$$v_n = (\frac{I-C}{A})^{-1}Bu_n.$$

Since $(\frac{I-C}{A})^{-1}B(\overline{U^w})$ is relatively weakly compact, so, there is a renamed subsequence such that

$$v_n = (\frac{I-C}{A})^{-1}Bu_n \rightharpoonup v.$$

But on the other hand, the subsequence $\{v_n\}$ verifies $v_n - Cv_n = Av_nBu_n$. Therefore, from assumption (f) and in view of condition $(\mathcal{P})$, we deduce that v verifies the following equation

$$v - Cv = AvBu,$$

or equivalently

$$v = (\frac{I-C}{A})^{-1}Bu.$$

Next we claim that the whole sequence $\{u_n\}$ verifies

$$(\frac{I-C}{A})^{-1}Bu_n = v_n \rightharpoonup v.$$

Indeed, suppose that this is not the case, so, there is V^w a weakly neighborhood of v satisfying for all $n \in \mathbb{N}$, there exists an $N \geq n$ such that $v_N \notin V^w$. Hence, there is a renamed subsequence $\{v_n\}$ verifying the property

$$\text{for all } n \in \mathbb{N}, \{ v_n\} \notin V^w. \tag{3.2}$$

However for all $n \in \mathbb{N}, v_n \in (\frac{I-C}{A})^{-1}B(\overline{U^w})$.
Again, there is a renamed subsequence such that $v_n \rightharpoonup v'$.
According to the preceding, we have $v' = (\frac{I-C}{A})^{-1}Bu$, and consequently $v' = v$, which is in contradiction with the property (3.2). This yields that $(\frac{I-C}{A})^{-1}B$ is weakly sequentially continuous.
In view of Remark 2.8, an application of Theorem 2.7 implies that either

$(A1)$ $(\frac{I-C}{A})^{-1}B$ has a fixed point, or

$(A2)$ there is a point $u \in \partial_\Omega U$ and $\lambda \in]0,1[$ with $u = \lambda(\frac{I-C}{A})^{-1}Bu$.

Assume first that $x \in \overline{U^w}$ is a fixed point of the operator $(\frac{I-C}{A})^{-1}B$. Then $x = (\frac{I-C}{A})^{-1}Bx$ which implies that

$$AxBx + Cx = x.$$

Suppose next that there is an element $u \in \partial_\Omega(U)$ and a real number $\lambda \in]0,1[$ such that $u = \lambda(\frac{I-C}{A})^{-1}Bu$. Then

$$(\frac{I-C}{A})^{-1}Bu = \frac{u}{\lambda},$$

so that

$$\lambda A(\frac{u}{\lambda})Bu + \lambda C(\frac{u}{\lambda}) = u.$$

This completes the proof.

Also, this result remains true when $C \equiv 0$ and we get the following results:

Corollary 3.6. Let Ω be a closed convex subset in a Banach algebra $\mathcal{E}$ satisfying the condition $(\mathcal{P})$ and $U \subset \Omega$ a weakly open set (with respect to the weak topology of Ω) such that $0 \in U$. Let $A : \mathcal{E} \to \mathcal{E}$ $B : \overline{U^w} \to \mathcal{E}$ be two operators satisfying
(i) A is D-Lipschitz with D-functions Φ_A,
(ii) A is regular on $\mathcal{E}$,
(iii) $B(\overline{U^w})$ is bounded with bound M,
(iv) $M\Phi_A(r) < r$ for $r > 0$,
(v) $x = AxBy \Rightarrow x \in \Omega$ for all $y \in \overline{U^w}$,
(vi) A is weakly sequentially continuous on Ω and B is weakly sequentially continuous $\overline{U^w}$,
(vii) $(\frac{I}{A})^{-1}B(\overline{U^w})$ is relatively weakly compact.
Then either

$(A1)$ the equation $\lambda A(\frac{x}{\lambda})Bx = x$ has a solution for $\lambda = 1$, or
$(A2)$ there is an element $u \in \partial_\Omega(U)$ such that $\lambda A(\frac{u}{\lambda})Bu = u$ for some $0 < \lambda < 1$.

Theorem 3.7. Let Ω be a closed convex subset in a Banach algebra $\mathcal{E}$ satisfying the condition $(\mathcal{P})$ and $U \subset \Omega$ a weakly open set (with respect to the weak topology of Ω) such that $0 \in U$. Let $A, C : \mathcal{E} \to \mathcal{E}$ $B : \overline{U^w} \to \mathcal{E}$ be three operators satisfying
(i) A and C are D-Lipschitz with D-functions Φ_A and Φ_C respectively,
(ii) A is regular on $\mathcal{E}$,
(iii) $M\Phi_A(r) + \Phi_C(r) < r$ for $r > 0$,
(iv) $x = AxBy + Cx \Rightarrow x \in \Omega$ for all $y \in \overline{U^w}$,
(v) A, C are weakly sequentially continuous on Ω and B is weakly sequentially continuous $\overline{U^w}$,
(vi) $A(\Omega)$, $B(\overline{U^w})$ and $C(\Omega)$ are relatively weakly compacts.
Then either
$(A1)$ the equation $\lambda A(\frac{x}{\lambda})Bx + \lambda C(\frac{x}{\lambda}) = x$ has a solution for $\lambda = 1$, or
$(A2)$ there is an element $u \in \partial_\Omega(U)$ such that $\lambda A(\frac{u}{\lambda})Bu + \lambda C(\frac{u}{\lambda}) = u$ for some $0 < \lambda < 1$.

Proof. In view of the last Theorem, it's enough to prove that $(\frac{I-C}{A})^{-1}B(\overline{U^w})$ is relatively weakly compact. To do this, let $\{u_n\}$ be any sequence in $(\overline{U^w})$ and let

$$v_n = (\frac{I-C}{A})^{-1}Bu_n. \tag{3.3}$$

Since $B(\overline{U^w})$ is relatively weakly compact, there is a renamed subsequence $\{Bu_n\}$ weakly converging to an element w. On the other hand, by the equation (3.3), we obtain

$$v_n = Av_nBu_n + Cv_n. \tag{3.4}$$

Since $\{v_n\}$ is a sequence in $(\frac{I-C}{A})^{-1}B(\overline{U^w})$, so by assumption (f), there is a renamed subsequence such that $Av_n \rightharpoonup x$ and $Cv_n \rightharpoonup y$. Hence, in view of condition $(\mathcal{P})$ and the last equation, we obtain

$$v_n \rightharpoonup xw + y.$$

This shows that $(\frac{I-C}{A})^{-1}B(\overline{U^w})$ is relatively weakly sequentially compact. An application of the Eberlein-Smulian theorem yields that $(\frac{I-C}{A})^{-1}B(\overline{U^w})$ is relatively weakly compact.

When $C \equiv 0$ we get the following results:

Corollary 3.8. Let Ω be a closed convex subset in a Banach algebra $\mathcal{E}$ satisfying the condition $(\mathcal{P})$ and $U \subset \Omega$ a weakly open set (with respect to the weak topology of Ω) such that $0 \in U$. Let $A : \mathcal{E} \to \mathcal{E}$ $B : \overline{U^w} \to \mathcal{E}$ be two operators satisfying
(i) A is D-Lipschitz with D-functions Φ_A ,
(ii) A is regular on $\mathcal{E}$,
(iii) $M\Phi_A(r) < r$ for $r > 0$,

(iv) $x = AxBy \Rightarrow x \in \Omega$ for all $y \in \overline{U^w}$,
(v) A is weakly sequentially continuous on Ω and B is weakly sequentially continuous $\overline{U^w}$,
(vi) $A(\Omega)$ and $B(\overline{U^w})$ are relatively weakly compacts.
Then either
$(A1)$ the equation $\lambda A(\frac{x}{\lambda})Bx = x$ has a solution for $\lambda = 1$, or
$(A2)$ there is an element $u \in \partial_\Omega(U)$ such that $\lambda A(\frac{u}{\lambda})Bu = u$ for some $0 < \lambda < 1$.

4. Functional integral equation

In this section we will apply Theorem 3.7 to a nonlinear functional integral equation. Let $(X, \|\ \|)$ be a Banach algebra satisfying condition $(\mathcal{P})$. Let $J = [0, 1]$ the closed and bounded interval in $\mathbb{R}$, the set of all real numbers. Let $\mathcal{E} = \mathcal{C}(J, X)$ the Banach algebra of all continuous functions from J to X, endowed with the sup-norm $\|\ \|_\infty$, defined by $\|f\|_\infty = \sup\{\|f(t)\|_\infty; t \in [0, 1]\}$, for each $f \in \mathcal{C}(J, X)$.
Consider the nonlinear functional integral equation (1.2):

$$x(t) = a(\nu(t)) + (Tx)(t)\Big[\Big(q(t) + \int_0^{\sigma(t)} p(t, s, g(x(\xi(s))), h(x(\eta(s))))ds\Big).u\Big]$$

$(H1)$ $a : J \to X$ is a continuous function.
$(H2)$ $\nu, \sigma, \xi, \eta : J \to J$ are continuous.
$(H3)$ $q : J \to \mathbb{R}$ is a continuous function.
$(H4)$ The operator $T : \mathcal{C}(J, X) \to \mathcal{C}(J, X)$ is such that:
 (a) T is Lipschitzian with a Lipschitz constant α,
 (b) T is regular on $\mathcal{C}(J, X)$,
 (c) T is sequentially weakly continuous on $\mathcal{C}(J, X)$,
 (d) T is weakly compact.
$(H5)$ The functions $g, h : X \to X$ are weakly sequentially continuous on X such that for each $r > 0$, g and h map the bounded subset $(r\mathcal{B}_\mathcal{X})$ into itself,
$(H6)$ The function $p : J \times J \times X \times X \to \mathbb{R}$ is weakly sequentially continuous such that for arbitrary fixed $s \in J$ and $x, y \in X$, the partial function $t \to p(t, s, x, y)$ is continuous uniformly for $(s, x, y) \in J \times X \times X$, and
$(H7)$ The exists $r_0 > 0$ such that:
 (a) $|p(t, s, x, y)| \leq M$ for each $t, s \in J; x, y \in X$ such that $\|x\| \leq r_0$ and $\|y\| \leq r_0$,
 (b) $\|u\|\|(Tx)\|_\infty \leq 1$ for each $x \in \mathcal{C}(J, X)$ such that $\|x\|_\infty \leq r_0$,
 (c) $\|a\|_\infty + \|q\|_\infty + M \leq r_0$,
 (d) $\alpha.r_0 < 1$.
Let us define the subset Ω of $\mathcal{C}(J, X)$ by

$$\Omega := \{x \in \mathcal{C}(J, X), \|x\|_\infty \leq r_0\} = B_{r_0}.$$

Obviously, Ω is nonempty, convex and closed. Let U be a weakly open subset of Ω such that $0 \in U$.
To make lecture of the FIE (1.2) easier, let us consider two operators A, C defined on

Ω and B defined on $\overline{U^w}$ as following:
$(Ax)(t) = (Tx)(t)$,
$(Bx)(t) = \Big[q(t) + \int_0^{\sigma(t)} p(t, s, g(x(\xi(s))), h(x(\eta(s))))ds\Big].u$,
$(Cx)(t) = a(\nu(t))$.
This means that equation (1.2) in equivalent to:

$$x = AxBx + Cx.$$

Theorem 4.1. Assume that $(H1) - (H7)$ hold. Let $\Omega = B_{r_0}$ and U be a weakly open subset of Ω such that $0 \in U$. In addition, suppose that for any solution x to the equation $x = \lambda A(\frac{x}{\lambda})Bx + \lambda C(\frac{x}{\lambda})$ for some $0 < \lambda < 1$, we have $x \notin \partial_\Omega(U)$, then the Equation (1.2) has a solution in U.

Proof. First, we begin by showing that the space $\mathcal{C}(J, X)$ verifies condition $(\mathcal{P})$. To see this, let $\{x_n\}, \{y_n\}$ any sequences in $\mathcal{C}(J, X)$, such that $x_n \rightharpoonup x$ and $y_n \rightharpoonup y$. So, for each $t \in J$, we have by Theorem 3.4 $x_n(t) \rightharpoonup x(t)$ and $y_n(t) \rightharpoonup y(t)$ in X. Since X verifies condition $(\mathcal{P})$, then

$$x_n(t)y_n(t) \rightharpoonup x(t)y(t),$$

because $(x_ny_n)_n$ is bounded [15], this, further, implies that

$$x_ny_n \rightharpoonup xy,$$

which shows that the space $\mathcal{C}(J, X)$ verifies the condition $(\mathcal{P})$.

We shall prove that operators A, B and C satisfy all the conditions of Theorem 3.7.
(i) From assumption $(H4)(a)$, it follows that A is Lipschitzian with a Lipschitz constant α. It is clear that C is Lipschitzian with a Lipschitz constant 0.
(ii) From assumption $(H4)(b)$, it follows that A is regular on $\mathcal{C}(J, X)$.
(iii) This condition is satisfied by $(H7)(d)$.
(iv) We shall show that the hypothesis (iv) of Theorem 3.7 is satisfied. In fact, we fix arbitrarily $x \in \mathcal{C}(J, X)$ and $y \in \overline{U^w}$ such that

$$x = AxBy + Cx,$$

or equivalently for all $t \in J$,

$$x(t) = (Tx)(t)(By)(t) + a(\nu(t)).$$

In view of hypothesis (H_7), for all $t \in J$ one has

$$\begin{aligned}\|(Ax)(t)(By)(t) + (Cx)(t)\| &= \|(Tx)(t)(Bx)(t) + a(\nu(t))\|\\ &\le \|a\|_\infty + \|(By)(t)\|\|(Tx)\|_\infty\\ &\le \|a\|_\infty + (M + \|q\|_\infty)\|u\|\|(Tx)\|_\infty\end{aligned}$$

$$\leq r_0.$$

From the last inequality and taking the supremum over t, we obtain

$$\|(Ax)(By) + (Cx)\|_\infty \leq r_0,$$

and consequently $(Ax)(By) + (Cx) \in \Omega$.

(v) In view of hypothesis $(H4)(c)$, A is sequentially weakly continuous on $\mathcal{E}$. Since C is constant, so, C is sequentially weakly continuous on Ω. Now, we show that B is sequentially weakly continuous on $\overline{U^w}$. Firstly, we verify that if $x \in \overline{U^w}$, then $Bx \in \mathcal{C}(J, X)$. For that, let $\{t_n\}$ be any sequence in J converging to a point in J. Then

$$\begin{aligned}
&\|(Bx)(t_n) - (Bx)(t)\| \leq |q(t_n) - q(t)|\|u\| \\
&\quad + \Big[\int_0^{\sigma(t_n)} |p(t_n, s, g(x(\xi(s))), h(x(\eta(s)))) - p(t, s, g(x(\xi(s))), h(x(\eta(s))))|ds\Big]\|u\| \\
&\quad + \Big[\int_{\sigma(t_n)}^{\sigma(t)} |p(t, s, g(x(\xi(s))), h(x(\eta(s))))|ds\Big]\|u\| \\
&\quad \leq |q(t_n) - q(t)|\|u\| \\
&\quad + \Big[\int_0^1 |p(t_n, s, g(x(\xi(s))), h(x(\eta(s)))) - p(t, s, g(x(\xi(s))), h(x(\eta(s))))|ds\Big]\|u\| \\
&\quad + M|\sigma(t) - \sigma(t_n)|\|u\|.
\end{aligned}$$

Since $t_n \to t$ then, for all $s \in J$ we have:

$$(t_n, s, g(x(\xi(s))), h(x(\eta(s)))) \to (t, s, g(x(\xi(s))), h(x(\eta(s)))).$$

Taking into account the hypothesis (H_6), we obtain

$$p(t_n, s, g(x(\xi(s))), h(x(\eta(s)))) \to p(t, s, g(x(\xi(s))), h(x(\eta(s)))) \text{ in } \mathbb{R}.$$

Moreover, the use of assumption $(H7)$ leads to

$$|p(t_n, s, g(x(\xi(s))), h(x(\eta(s)))) - p(t, s, g(x(\xi(s))), h(x(\eta(s))))| \leq 2M$$

for all $t, s \in J$. We can now apply the Dominated Converge Theorem and since assumption (H_3) holds, we get

$$(Bx)(t_n) \to (Bx)(t) \text{ in } X.$$

It follows that

$$Bx \in \mathcal{C}(J, X).$$

Next, we prove that B is weakly sequentially continuous on $\overline{U^w}$. Let $\{x_n\}$ be any sequence in $\overline{U^w}$ weakly converging to a point $x \in \overline{U^w}$. Then, $\{x_n\}$ is bounded. Applying Theorem 3.4, we get

$$x_n(t) \to x(t), \ \forall t \in J.$$

So, by assumptions $(H_5)-(H_6)-(H_7)$ and the Dominated convergence Theorem, we obtain:

$$\lim_{n\to\infty}\int_0^{\sigma(t)} p(t,s,g(x_n(\xi(s))),h(x_n(\eta(s))))ds = \int_0^{\sigma(t)} p(t,s,g(x(\xi(s))),h(x(\eta(s))))ds$$

which implies

$$\lim_{n\to\infty}\Big(q(t)+\int_0^{\sigma(t)} p(t,s,g(x_n(\xi(s))),h(x_n(\eta(s))))ds\Big).u$$
$$= \Big(q(t)+\int_0^{\sigma(t)} p(t,s,g(x(\xi(s))),h(x(\eta(s))))ds\Big).u$$

Hence

$$(Bx_n)(t)\to (Bx)(t) \text{ in } X$$

and so

$$(Bx_n)(t)\rightharpoonup (Bx)(t) \text{ in } X.$$

It's clear that the sequence $\{Bx_n\}$ is bounded by $(\|q\|_\infty+M)\|u\|$, then by the Theorem 3.4, we get

$$(Bx_n)\rightharpoonup (Bx)$$

Thus, we conclude that B is weakly sequentially continuous on $\overline{U^w}$.

(vi) Using the fact that Ω is bounded by r_0 and in view of assumption $(H4)(d)$, it follows that $A(\Omega)$ is relatively weakly compact. As $C(\Omega)=\{a\}$, hence $C(\Omega)$ is relatively weakly compact.

It remains to prove that $B(\overline{U^w})$ is relatively weakly compact.

By definition,

$$B(\overline{U^w}) := \{Bx : \|x\|_\infty \le r_0\}.$$

For all $t\in J$, we have

$$B(\overline{U^w})(t) := \{(Bx)(t) : \|x\|_\infty \le r_0\}.$$

We claim that $B(\overline{U^w})(t)$ is weakly sequentially relatively compact in X. To see this, let $\{x_n\}$ be any sequence in $\overline{U^w}$, we have $(Bx_n)(t)=r_n(t).u$, where

$$r_n(t)=q(t)+\int_0^{\sigma(t)} p(t,s,g(x_n(\xi(s))),h(x_n(\eta(s))))ds.$$

It's clear that $|r_n(t)|\le(\|q\|_\infty+M)$ and $\{r_n(t)\}$ is a real sequence, so, by the Bolzano Weirstrass Theorem [15], there is a renamed subsequence such that

$$r_n(t)\to r(t) \text{ in } \mathbb{R},$$

which implies

$$r_n(t).u\to r(t).u \text{ in } X,$$

and consequently

$$(Bx_n)(t)\to r(t).u \text{ in } X.$$

Hence, we conclude that $B(\overline{U^w})(t)$ is sequentially relatively compact in X, then $B(\overline{U^w})(t)$ is sequentially relatively weakly compact in X.
Now, we have to prove that $B(\overline{U^w})$ is weakly equicontinuous on J. Let $\varepsilon > 0$; $x \in \overline{U^w}$; $x^* \in X^*$; $t, t' \in J$ such that $t \leq t'$ and $t' - t \leq \varepsilon$. Then

$$\|x^*((Bx)(t) - (Bx)(t'))\| \leq |q(t) - q(t')||x^*(u)| + |x^*(u)| \times$$
$$\Big[\int_0^{\sigma(t)} |p(t, s, g(x(\xi(s))), h(x(\eta(s))))ds - \int_0^{\sigma(t')} p(t', s, g(x(\xi(s))), h(x(\eta(s))))|ds\Big]$$
$$\leq |q(t) - q(t')||x^*(u)|$$
$$+\Big[\int_0^{\sigma(t)} \Big|p(t, s, g(x(\xi(s))), h(x(\eta(s)))) - p(t', s, g(x(\xi(s))), h(x(\eta(s))))\Big|ds\Big]|x^*(u)|$$
$$+\Big|\int_{\sigma(t)}^{\sigma(t')} p(t', s, g(x(\xi(s))), h(x(\eta(s))))ds\Big||x^*(u)|$$
$$\leq [w(q, \varepsilon) + w(p, \varepsilon) + Mw(\sigma, \varepsilon)]|x^*(u)|.$$

where:
$w(q, \varepsilon) := \sup\{|q(t) - q(t')| : t, t' \in J; |t - t'| \leq \varepsilon\}$
$w(p, \varepsilon) := \sup\{|p(t, s, x, y)) - p(t, s, x, y)| : t, t', s \in J; |t - t'| \leq \varepsilon; x, y \in \Omega\}$
$w(\sigma, \varepsilon) := \sup\{|\sigma(t) - \sigma(t')| : t, t' \in J; |t - t'| \leq \varepsilon\}$
Taking into account the hypothesis $(H6)$ and in view of the uniform continuity of the functions q, σ, it follows that $w(q, \varepsilon) \to 0$, $w(p, \varepsilon) \to 0$, and $w(\sigma, \varepsilon) \to 0$ when $\varepsilon \to 0$. Hence, the application of the Arzel-Ascoli Theorem [14] leads to have that $B(\overline{U^w})$ is sequentially relatively weakly compact in $\mathcal{E}$. Now, the Eberlein-Smulian Theorem [7] yields that $B(\overline{U^w})$ is relatively weakly compact.
Thus all the conditions of Theorem 3.7 are satisfied and hence an application of it yields that either the conclusion $(A1)$ or $(A2)$ holds. By the fact that for any solution x to the equation $x = \lambda A(\frac{x}{\lambda})Bx + \lambda C(\frac{x}{\lambda})$ for some $0 < \lambda < 1$, $x \notin \partial_\Omega(U)$, we conclude that conclusion $(A2)$ is eliminated and hence the Equation (1.2) has a solution in U.

References

[1] A. Ben Amar, M. Mnif, *Leray-Schauder alternatives for weakly sequentially continuous mappings and application to transport equation*, Math. Methods Appl. Sci., **33**(2010), no. 1, 80-90.

[2] A. Ben Amar, S. Chouayekh, A. Jeribi, *New fixed point theorems in Banach algebra under weak topology features and applications to nonlinear integral equations*, J. Funct. Anal., **259**(2010), no. 9, 2215-2237.

[3] J. Banas, L. Lecko, *Fixed points of the product of operators in Banach algebra*, Panamer. Math. J., **12**(2002), no. 2, 101-109.

[4] J. Banas, K. Sadarangui, *Solutions of some functional-integral equations in Banach algebras*, Math. Comput. Modelling, **38**(2003), no. 3-4, 245-250.

[5] J. Banas, B. Rzepka, *Monotonic solutions of a quadratic integral equation of fractional order*, J. Math. Anal. Appl., **332**(2007), no. 2, 1371-1379.

[6] D.W. Boyd, J.S.W. Wong, *On nonlinear contractions,* Proc. Amer. Math. Soc., **20**(1969), 458-464.

[7] J.B. Conway, *A Course in Functionnal Analysis*, Springer-Verlag, Berlin, 1990.

[8] I. Dobrakov, *On representation of linear operators on* $\mathcal{C}_0(T, X)$, Czechoslovak Math. J., **21**(1971), 13-30.

[9] B.C. Dhage, *On some variants of Schauder's fixed point principle and applications to nonlinear integral equations*, J. Math. Phys. Sci., **22**(1988), no. 5, 603-611.

[10] B.C. Dhage, *On* α*-condensing mappings in Banach algebras*, Math. Student, **63**(1994), no. 1-4, 146-152.

[11] B.C. Dhage, *Some nonlinear alternatives in Banach algebras with application II*, Kyungpook Math. J., **45**(2005), no. 2, 281-292.

[12] B.C. Dhage, *On some nonlinear alternatives of Leray Schauder type and functional integral equations*, Arch. Math., Brno, **42**(2006), no. 1, 11-23.

[13] D.R. Smart, *Fixed Point Theorems*, Camb. Univ. Press, 1980.

[14] I.I. Vrabie, A.I. Cusa, O. Mayer, C_0*-Semigroups and Applications*, Elsevier, New-York, 2003.

[15] C. Wagschal, *Topologie et analyse fonctionnelle*, Hermann, Paris, 2003.

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