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STRONG CONVERGENCE THEOREM FOR INFINITE FAMILY OF TOTAL QUASI-ϕ-ASYMPTOTICALLY NONEXPANSIVE MULTI-VALUED MAPPINGS

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Abstract. We prove strong convergence theorem for infinite family of uniformly $L-$Lipschitzian total quasi-ϕ-asymptotically nonexpansive multi-valued mappings using a generalized $f-$projection operator in a real uniformly convex and uniformly smooth Banach space. The result presented in this paper improve and unify important recent results announced by many authors.

Key Words and Phrases: total quasi-ϕ-asymptotically nonexpansive multi-valued mapping, fixed point, strong convergence, f-projection operator.
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1. Introduction

Let E be a real Banach space and C a nonempty closed convex subset of E. We denote by J the normalized duality map from E to 2^{E^*} defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}$$

for all $x \in E$, where E^* is the dual space of E and $\langle ., .\rangle$ denotes the duality pairing between elements of E and those of E^*. The space E is said to be *uniformly convex* if given $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that for all $x, y \in E$ with $||x|| \leq 1, ||y|| \leq 1$ and $||x - y|| \geq \epsilon$, we have $\left\|\frac{x+y}{2}\right\| \leq 1 - \delta$. E is *strictly convex* if $\left\|\frac{x+y}{2}\right\| < 1$ for all $x, y \in E$ with $||x|| = ||y|| = 1$ and $x \neq y$.
The space E is smooth if for any $x \in E$, with $\|x\| = 1$ there exists a unique $x^* \in E^*$, with $\|x^*\| = 1$, such that $x^*(x) = 1$. If dimE ≥ 2, then the *modulus of smoothness* of E is a function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(t) := \sup\left\{\frac{\|x + y\| \ + \|x - y\|}{2} - 1 : \|x\| = 1; \|y\| = t\right\}.$$

The space E is called *uniformly smooth* if and only if $\lim\limits_{t\to 0^+} \frac{\rho_E(t)}{t} = 0$.

It is known that if E is strictly convex, smooth and reflexive then the duality map J is one-to-one, single-valued and onto. Also if E is uniformly smooth, then J is norm-to-norm uniformly continuous on bounded subsets of E.
Let E be a smooth Banach space. We always use $\phi : E \times E \to \mathbb{R}^+$ to denote the Lyapunov functional defined by

$$\phi(x, y) = ||x||^2 - 2\langle x, Jy\rangle + ||y||^2, \ \forall x, y \in E. \tag{1.1}$$

It is obvious from the definition that

$$(||x|| - ||y||)^2 \leq \phi(x, y) \leq (||x|| + ||y||)^2, \ \forall x, y \in E. \tag{1.2}$$

Let E be a smooth Banach space, C be a nonempty, closed and convex subset of E. Let $N(C)$ be a family of nonempty subsets of C and $T : C \to N(C)$ be a mutivalued mapping. A point $p \in C$ is a fixed point of T if $p \in Tp$. The point $p \in C$ is said to be an asymptotic fixed point of T if $Tx \neq \emptyset \ \forall x \in C$, there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset C$ such that $x_n \rightharpoonup p$ and $d(x_n, Tx_n) \to 0$ as $n \to \infty$. Also the point $p \in C$ is said to be a strong asymptotic fixed point of T (see [28]) if there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset C$ such that $x_n \to p$ and $d(x_n, Tx_n) \to 0$ as $n \to \infty$ where $d(x_n, Tx_n) = \underset{u \in Tx_n}{inf} \ \|x_n - u\|$
We denote the set of all fixed point of T, the set of all asymptotic fixed point of T and the set of all strong asymptotic fixed point of T by $F(T)$, $\widehat{F(T)}$ and $\widetilde{F(T)}$, respectively.

Definition 1.1. (1) A mapping $T : C \to C$ is said to be *relatively* nonexpansive (see e.g. [14]) if $F(T) \neq \emptyset, \ F(T) = \widehat{F(T)}$ and

$$\phi(p, Tx) \leq \phi(p, x), \ \forall x \in C, \ p \in F(T).$$

(2) A mapping $T : C \to C$ is said to be quasi-ϕ-asymptotically nonexpansive if $F(T) \neq \emptyset$ and there exists a real sequence $\{k_n\}$ satisfying $k_n \to 1$ as $n \to \infty$ such that

$$\phi(p, T^n x) \leq k_n \phi(p, x), \ \forall n \geq 1, \ x \in C, \ p \in F(T). \tag{1.3}$$

(3) A mapping $T : C \to C$ is said to be total quasi-ϕ-asymptotically nonexpansive if $F(T) \neq \emptyset$ and there exist nonnegative real sequences $\{v_n\}$ and $\{u_n\}$ with $v_n \to 0, \ u_n \to 0$ (as $n \to \infty$) and a strictly increasing continuous function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ with $\zeta(0) = 0$ such that for all $x \in C, \ p \in F(T)$

$$\phi(p, T^n x) \leq \phi(p, x) + u_n \zeta(\phi(p, x)) + v_n, \ \forall n \geq 1. \tag{1.4}$$

Definition 1.2. (1) A multi-valued mapping $T : C \to N(C)$ is said to be *relatively nonexpansive* if $F(T) \neq \emptyset, F(T) = \widehat{F(T)}$ and

$$\phi(p, w) \leq \phi(p, x), \ \forall x \in C, \ w \in Tx, \ p \in F(T).$$

(2) A multi-valued mapping $T : C \to N(C)$ is said to be $quasi - \phi - asymptotically$ *nonexpansive* if $F(T) \neq \emptyset$ and there exists a real sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ (as $n \to \infty$) such that

$$\phi(p, w_n) \leq k_n \phi(p, x), \ \forall n \geq 1, \ x \in C, \ w_n \in T^n x, \ p \in F(T).$$

(3) A multi-valued mapping $T : C \to N(C)$ is said to be $total \ quasi - \phi -$ *asymptotically nonexpansive* if $F(T) \neq \emptyset$ and there exist nonnegative real sequences

$\{u_n\}$ and $\{v_n\}$ with $u_n \to 0, v_n \to 0$ (as $n \to \infty$) and a strictly increasing continuous function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ with $\zeta(0) = 0$ such that for all $x \in C,\ p \in F(T)$

$$\phi(p, w_n) \leq \phi(p, x) + u_n\zeta(\phi(p, x)) + v_n, ,\ \forall n \geq 1,\ w_n \in T^n x. \qquad (1.5)$$

(4) A multi-valued mapping $T : C \to N(C)$ is said to be *closed* if, for any sequence $\{x_n\} \in C$ with $x_n \to x$ and $w_n \in T(x_n)$ with $w_n \to y$, then $y \in Tx$.
(5) A multi-valued mapping $T : C \to N(C)$ is said to be $uniformly\ L-Lipschitzian$ if there exists a constant $L > 0$ such that for all $x, y \in C$,

$$||w_n - s_n|| \leq L||x - y||,\ \forall\ w_n \in T^n x,\ s_n \in T^n y,\ n \geq 1.$$

The class of total quasi$-\phi-$asymptotically nonexpansive mappings is important as a generalisation of the class of relatively nonexpansive mappings. Approximating fixed points of multivalued nonlinear mappings if they exist are of enomous importance due to their applications in various fields such as in game theory and market economy (see for example [15, 16]), nonsmooth differential equations (see for example [5, 6, 7]) to mention just a few.

In 2005, Matsushita and Takahashi [14] proved weak and strong convergence theorems for fixed point of a single relative nonexpansive mapping in a uniformly convex and uniformly smooth Banach space. Various generalizations of relatively nonexpansive mappings were introduced and studied by numerous authors see for example [2, 11, 13],[17]-[20], [21, 23, 24],[26]-[31] (just to mention but a few) and the references contained in them. In 2010, Chang et al. [3] obtained a strong convergence theorem for an infinite family of quasi-ϕ-asymptotically nonexpansive mappings in a uniformly smooth and strictly convex Banach space with Kadec-Klee property.

Still in 2010, Li et al. [12] introduced the following hybrid iterative scheme for approximation of fixed point of a relatively nonexpansive mapping T using properties of generalized f-projection operator in a uniformly smooth real Banach space which is also uniformly convex: $x_0 \in C$,

$$\begin{cases} y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JTx_n), \\ C_{n+1} = \{w \in C_n : G(w, Jy_n) \leq G(w, Jx_n)\}, \\ x_{n+1} = \Pi^f_{C_{n+1}} x_0,\ n \geq 0. \end{cases}$$

They proved a strong convergence of the scheme to an element in the fixed point set of T.
Recently, Homaeipour and Razani [9] proved weak and strong convergence theorems for a single relatively nonexpansive multi-valued mapping in a uniformly convex and uniformly smooth Banach space.
Quite recently, Tang and Chang [22] introduced a new hybrid iterative scheme for approximation of fixed point of a total quasi-ϕ-asymptotically nonexpansive multi-valued mapping in a uniformly smooth and strictly convex Banach space with Kadec-Klee property. They actually proved the following theorem.
Theorem 1.3. [22] Let E be a real uniformly smooth and strictly convex Banach space with Kadec-Klee property, and C be a nonempty closed and convex subset of E. Let $T : C \to N(C)$ be a closed and total quasi-ϕ-asymptotically nonexpansive multi-valued mapping with nonnegative real sequences $\{v_n\}$ and $\{\mu_n\}$ and strictly

increasing continuous function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\mu_1 = 0, v_n \to 0, \mu_n \to 0$ (as $n \to \infty$) and $\zeta(0) = 0$. Let $x_0 \in C, C_0 = C$ and $\{x_n\}$ be a sequence generated by

$$\begin{cases} y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)Jz_n) \\ z_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)Jw_n) \\ C_{n+1} = \{v \in C_n : \phi(v, Jy_n) \leq \phi(v, Jx_n) + \xi_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \ n \geq 0 \end{cases} \tag{1.6}$$

where $w_n \in T^n x_n, \ \forall n \geq 1, \ \xi_n = v_n \sup_{p \in F(T)} \zeta(\phi(p, x_n)) + \mu_n, \ \Pi_{C_n+1}$ is the generalized projection of E onto $C_{n+1}, \ \{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ satisfying the following conditions:
(a) $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$;
(b) $0 \leq \alpha_n \leq \alpha < 1$ for some $\alpha \in (0,1)$. If $F(T)$ is a nonempty and bounded subset of C, then the sequence $\{x_n\}$ converges strongly to $\Pi_{F(T)} x_0$.

More recently, Zhang et al. [28] introduced the notion of weak relatively nonexpansive mappings. They studied a new hybrid algorithm for fixed point of multivalued weak relatively nonexpansive mappings. They proved the following Theorem.

Theorem 1.4. [28] Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, let $T : C \to C$ be a weak relatively multivalued nonexpansive mapping. Assume that $\{\alpha_n\}$,$\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$ such that $\alpha_n + \beta_n + \gamma_n = 1, \ \underset{n\to\infty}{lim} \alpha_n = 0$ and $0 < \gamma \leq \gamma_n \leq 1 \ \forall n \geq 1$ for some constant $\gamma \in (0,1)$. Define a sequence $\{x_n\}$ in C by the following algorithm:

$$\begin{cases} x_0 \in C \\ y_n = J^{-1}(\alpha_n Jx_0 + \beta_n Jx_n + \gamma_n Jz_n), z_n \in Tx_n \\ C_n = \{z \in C_{n-1} : \phi(z, y_n) \leq (1-\alpha_n)\phi(z, x_n) + \alpha_n \phi(z, x_0), n \geq 1, \\ C_0 = C \\ x_{n+1} = \Pi_{C_n} x_0. \end{cases}$$

Then $\{x_n\}$ converges to $q = \Pi_{F(T)} x_0$.

Motivated by the above results, it is our purpose in this paper to study a new modified iterative scheme and prove strong convergence theorem for infinite family of total quasi-ϕ-asymptotically nonexpansive multi-valued mappings in a real uniformly convex and uniformly smooth Banach space using a generalized $f-$projection operator. Our result improve and unify several recent important results announced by numerous authors.

2. Preliminaries

Let E be a smooth, strictly convex and reflexive real Banach space and let C be a nonempty, closed and convex subset of E. Following Alber [1], the generalized projection Π_C from E onto C is defined by

$$\Pi_C(x) := \underset{y \in C}{argmin} \phi(y, x), \ \forall x \in E.$$

The existence and uniqueness of Π_C follows from the property of the functional $\phi(x, y)$ and strict monotonicity of the mapping J (see, for example,[1]). If E is a Hilbert space, then Π_C become the metric projection of H onto C.

Let $G : C \times E^* \to \mathbb{R} \cup \{+\infty\}$ be a functional defined as follows:

$$G(\xi, \varphi) = ||\xi||^2 - 2\langle \xi, \varphi \rangle + ||\varphi||^2 + 2\rho f(\xi), \tag{2.1}$$

where $\xi \in C, \varphi \in E^*, \rho$ is a positive number and $f : C \to \mathbb{R} \cup \{+\infty\}$ is proper, convex and lower semi-continuous. From the definition of G and f, it is easy to observe the following properties:
(i) $G(\xi, \varphi)$ is convex and continuous with respect to φ when ξ is fixed;
(ii) $G(\xi, \varphi)$ is convex and lower semi-continuous with respect to ξ when φ is fixed.
Definition 2.1. (Wu and Huang [25]) Let E be a real Banach space with its dual E^*. Let C be a nonempty closed and convex subset of E. A mapping $\Pi_C^f : E^* \to 2^C$ defined by

$$\Pi_C^f \varphi = \{u \in C : G(u, \varphi) = \inf_{\xi \in C} G(\xi, \varphi)\}, \ \forall \varphi \in E^*.$$

is called a generalized f-projection operator.
For the generalized f-projection operator, Wu and Hung [25] proved the following basic properties:
Lemma 2.2. (Wu and Huang [25]) Let E be a real reflexive Banach space with its dual space E^*. Let C be a nonempty closed and convex subset of E. Then the following statement hold:
(i) $\Pi_C^f \varphi$ is a nonempty closed convex subset of C for all $\varphi \in E^*$,
(ii) If E is smooth, then for all $\varphi \in E^*, x \in \Pi_C^f \varphi$ if and only if

$$\langle x - y, \varphi - Jx \rangle + \rho f(y) - \rho f(x) \geq 0, \forall y \in C;$$

(iii) If E is strictly convex and $f : C \to \mathbb{R} \cup \{+\infty\}$ is positive homogeneous (i.e., $f(tx) = tf(x)$ for all $t > 0$ with $tx \in C$ where $x \in C$), then $\Pi_C^f \varphi$ is a single valued mapping.
Fan et al. [8] showed that the condition f is positive homogeneous which appeared in Lemma 2 can be dropped.
Lemma 2.3. (Fan et al. [8]) Let E be a real reflexive Banach space with its dual space E^* and C a nonempty closed and convex subset of E. If E is strictly convex, then Π_C^f is a single valued mapping.
Recall that J is single valued in a real smooth Banach space E. This implies that for each $x \in E$ there exists a unique element $\varphi \in E^*$ such that $\varphi = Jx$. Using this in (2.1), we obtain, in a real smooth Banach space,

$$G(\xi, Jx) = ||\xi||^2 - 2\langle \xi, Jx \rangle + ||x||^2 + 2\rho f(\xi). \tag{2.2}$$

With this, the generalized f-projection operator in a real smooth Banach space can be defined as follows.
Definition 2.4. Let E be a real Banach space and C a nonempty closed and convex subset of E. We say that $\Pi_C^f : E \to 2^C$ is a generalized f-projection operator if

$$\Pi_C^f x = \{u \in C : G(u, Jx) = \inf_{\xi \in C} G(\xi, Jx)\}, \ \forall x \in E.$$

The following definition follows from the relations between ϕ and G.

Definition 2.5. A multi-valued mapping $T : C \to N(C)$ is
(i) relatively nonexpansive if $F(T) \neq \emptyset,\ F(T) = \widehat{F(T)}$ and

$$G(p, Jw) \leq G(p, Jx),\ \forall x \in C,\ p \in F(T),\ x \in C,\ \text{and}\ w \in Tx.$$

(ii) weak relatively nonexpansive if $F(T) \neq \emptyset,\ F(T) = \widetilde{F(T)}$ and

$$G(p, Jw) \leq G(p, Jx),\ \forall x \in C,\ p \in F(T),\ x \in C,\ \text{and}\ w \in Tx.$$

(iii) going to be call *total quasi* $-\,\phi\,-$ *asymptotically nonexpansive* with respect to f if $F(T) \neq \emptyset$ and there exist nonnegative real sequences $\{u_n\}$ and $\{v_n\}$ with $u_n \to 0, v_n \to 0$ (as $n \to \infty$) and a strictly increasing continuous function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ with $\zeta(0) = 0$ such that for all $x \in C,\ p \in F(T)$

$$G(p, Jw_n) \leq G(p, Jx) + u_n \zeta(G(p, Jx)) + v_n,,\ \forall n \geq 1,\ w_n \in T^n x. \tag{2.3}$$

Lemma 2.6. (Li et al. [12]) Let E be a Banach space and $f : E \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous convex functional. Then there exist $x^* \in E^*$ and $\alpha \in \mathbb{R}$ such that

$$f(x) \geq \langle x, x^* \rangle + \alpha,\ \forall x \in E.$$

Lemma 2.7. (Li et al. [12]) Let C be a nonempty, closed and convex subset of a smooth and reflexive Banach space E. Then the following statements hold:
(i) $\Pi_C^f x$ is a nonempty, closed and convex subset of C for all $x \in E$;
(ii) for all $x \in E,\ \hat{x} \in \Pi_C^f x$ if and only if

$$\langle \hat{x} - y, Jx - J\hat{x} \rangle + \rho f(y) - \rho f(\hat{x}) \geq 0,\ \forall y \in C;$$

(iii) if E is strictly convex, then $\Pi_C^f x$ is a single valued mapping.
Lemma 2.8. (Li et al. [12]) Let C be a nonempty, closed and convex subset of a smooth and reflexive Banach space E. Let $x \in E$ and $\hat{x} \in \Pi_C^f x$, then

$$\phi(y, \hat{x}) + G(\hat{x}, Jx) \leq G(y, Jx),\ \forall y \in C.$$

Lemma 2.9. (Kamimura and Takahashi [10]) Let C be a nonempty closed and convex subset of a smooth uniformly convex Banach space E. Let $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ be sequences in E such that either $\{x_n\}_{n=1}^\infty$ or $\{y_n\}_{n=1}^\infty$ is bounded. If $\lim_{n\to\infty} \phi(x_n, y_n) = 0$, then $\lim_{n\to\infty} ||x_n - y_n|| = 0$.
Lemma 2.10. (Chang et al. [4]) Let E be a uniformly convex real Banach space. For arbitrary $r > 0$, let

$$B_r(0) := \{x \in E : ||x|| \leq r\}.$$

Then, for any given sequence $\{x_n\}_{n=1}^\infty \subset B_r(0)$ and for any given sequence $\{\lambda_n\}_{n=1}^\infty$ of positive numbers such that $\sum_{i=1}^\infty \lambda_i = 1$, there exists a continuous strictly increasing convex function

$$g : [0, 2r] \to \mathbb{R},\ g(0) = 0$$

such that for any positive integers i, j with $i < j$, the following inequality holds:

$$\Big\| \sum_{n=1}^\infty \lambda_n x_n \Big\|^2 \leq \sum_{n=1}^\infty \lambda_n ||x_n||^2 - \lambda_i \lambda_j g(||x_i - x_j||).$$

3. Main results

Lemma 3.1. Let E be a real uniformly convex and uniformly smooth Banach space, and C be a nonempty closed and convex subset of E. Let $T : C \to N(C)$ be a closed and total quasi-ϕ-asymptotically nonexpansive multi-valued mapping with nonnegative real sequences $\{u_n\}$ and $\{v_n\}$ and a strictly increasing continuous function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ such that $u_n \to 0, v_n \to 0$ as $n \to \infty$ and $\zeta(0) = 0$. Let $f : E \to \mathbb{R}$ be a convex, and lower semi continuous mapping with $C \subset int(D(f))$. Then $F(T)$ is closed and convex.

Proof. First, we show that $F(T)$ is closed. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $F(T)$ such that $x_n \to p$ (as $n \to \infty$). Since $x_n \in Tx_n$, by the closure of T, we have $p \in Tp$, that is $p \in F(T)$. Thus, $F(T)$ is closed.

Next we show that $F(T)$ is convex. Let $x, y \in F(T)$, put $z = tx + (1-t)y$ for any $t \in (0,1)$, we prove that $z \in F(T)$. Indeed, let $\{w_n\}$ be a sequence generated by $w_1 \in Tz, w_2 \in Tw_1 \subset T^2z, \ldots, w_n \in Tw_{n-1} \subset T^nz, \ldots$ by the convexity of $||.||^2$, letting $\mu_n := t[u_n\zeta(G(x,Jz)) + v_n] + (1-t)[u_n\zeta(G(y,Jz)) + v_n]$, we have

$$G(z, Jw_n) = ||z||^2 - 2\langle z, Jw_n\rangle + ||w_n||^2 + 2\rho f(z)$$

$$= ||z||^2 - 2t\langle x, Jw_n\rangle - 2(1-t)\langle y, Jw_n\rangle + t||w_n||^2 + (1-t)||w_n||^2 + 2\rho f(z)$$

$$= ||z||^2 + tG(x, Jw_n) + (1-t)G(y, Jw_n) + 2\rho f(z)$$

$$-t||x||^2 - (1-t)||y||^2 - 2t\rho f(x) - 2(1-t)\rho f(y)$$

$$\leq ||z||^2 + t[G(x, Jz) + u_n\zeta(G(x, Jz)) + v_n]$$

$$+(1-t)[G(y, Jz) + u_n\zeta(G(y, Jz)) + v_n] + 2\rho f(z)$$

$$-t||x||^2 - (1-t)||y||^2 - 2t\rho f(x) - 2(1-t)\rho f(y)$$

$$= ||z||^2 - 2\langle z, Jz\rangle + ||z||^2 + 2\rho f(z) + \mu_n = G(z, Jz) + \mu_n.$$

Since $G(z, Jz) \leq G(z, Jw_n)$ and $\mu_n \to 0$ as $n \to \infty$, we have $G(z, Jw_n) \to G(z, Jz)$, which implies that $\phi(z, w_n) \to 0$ as $n \to \infty$, by Lemma 2.9 we have $w_n \to z$ (as $n \to \infty$). Since T is closed, we have $z \in Tz$ that is $z \in F(T)$. Therefore $F(T)$ is convex.

Theorem 3.2. Let E be a real uniformly convex and uniformly smooth Banach space, and C be a nonempty closed and convex subset of E. Let $T_i : C \to N(C)$, $i \in \mathbb{N}$ be an infinite family of uniformly L_i-Lipschitzian, closed and total quasi-ϕ-asymptotically nonexpansive multi-valued mappings with nonnegative real sequences $\{u_{ni}\}$ and $\{v_{ni}\}$ and sequences of strictly increasing continuous functions $\zeta_i : \mathbb{R}^+ \to \mathbb{R}^+$ such that $u_{ni} \to 0, v_{ni} \to 0$ as $n \to \infty$, and $\zeta_i(0) = 0$, $i \in \mathbb{N}$. Let $F = \cap_{i=1}^{\infty} F(T_i) \neq \emptyset$, bounded and $f : E \to \mathbb{R}$ be convex and lower semi continuous mapping with $C \subset int(D(f))$. Suppose $\{x_n\}_{n=0}^{\infty}$ is a sequence iteratively generated by $x_0 \in C, C_0 = C$,

$$\begin{cases} z_n = J^{-1}(\alpha_{n0}Jx_n + \sum_{i=1}^{\infty} \alpha_{ni}Jw_n^{(i)}); \\ y_n = J^{-1}(\beta_n Jz_n + (1-\beta_n)Jx_n); \\ C_{n+1} = \{v \in C_n : G(v, Jy_n) \leq G(v, Jx_n) + \sigma_n\}; \\ x_{n+1} = \Pi_{C_{n+1}}^f x_0, \ n \geq 0, \end{cases} \tag{3.1}$$

where $\{\alpha_{ni}\}_{n=1}^{\infty}$ are sequences of positive real numbers, $w_n^{(i)} \in T_i^n x_n,\ i \in \mathbb{N}$,

$$\sigma_n := \sum_{i=1}^{\infty} \alpha_{ni}\Big(v_{ni}\sup_{p\in F}\zeta_i(G(p, Jx_n)) + u_{ni}\Big),$$

and

(i) $\sum_{i=0}^{\infty} \alpha_{ni} = 1$; $\liminf_{n\to\infty} \alpha_{n0}\alpha_{ni} > 0\ \forall i \in \mathbb{N}$;
(ii) $0 < \beta < \beta_n \leq 1$ for some $\beta \in (0, 1)$.

Then, $\{x_n\}_{n=1}^{\infty}$ converges strongly to $\Pi_F^f x_0$.

Proof. We first show that $C_n, \forall n \geq 0$ is closed and convex. It is obvious that $C_0 = C$ is closed and convex. Thus, we only need to show that C_n is closed and convex for each $n \geq 1$. Since $G(v, Jy_n) \leq G(v, Jx_n) + \sigma_n$ is equivalent to $2\langle v, Jx_n - Jy_n\rangle \leq ||Jy_n||^2 - ||Jx_n||^2 + \sigma_n,\ n \in \mathbb{N}$, we obtain that C_{n+1} is closed and convex for all $n \geq 0$.

We now show that $\{x_n\}$ is bounded and the limit $\lim_{n\to\infty} G(x_n, Jx_0)$ exists. Since $f : E \to \mathbb{R}$ is convex and lower semi-continuous, by Lemma 2.6, there exists $u^* \in E^*$ and $\alpha \in \mathbb{R}$ such that.

$$f(y) \geq \langle y, u^*\rangle + \alpha,\ \forall y \in E.$$

It follows that

$$\begin{aligned} G(x_n, Jx_0) &= ||x_n||^2 - 2\langle x_n, Jx_0\rangle + ||x_0||^2 + 2\rho f(x_n) \\ &\geq ||x_n||^2 - 2\langle x_n, Jx_0\rangle + ||x_0||^2 + 2\rho\langle x_n, u^*\rangle + 2\rho\alpha \\ &= ||x_n||^2 - 2\langle x_n, Jx_0 - \rho u^*\rangle + ||x_0||^2 + 2\rho\alpha \\ &\geq ||x_n||^2 - 2||x_n||||Jx_0 - \rho u^*|| + ||x_0||^2 + 2\rho\alpha \\ &= (||x_n|| - ||Jx_0 - \rho u^*||)^2 + ||x_0||^2 - ||Jx_0 - \rho u^*||^2 + 2\rho\alpha. \end{aligned} \tag{3.2}$$

Since $x_n = \Pi_{C_n}^f x_0$, it follows from (3.2) that

$$G(u, Jx_0) \geq G(x_n, Jx_0) \geq (||x_n|| - ||Jx_0 - \rho u^*||)^2 + ||x_0||^2 - ||Jx_0 - \rho u^*||^2 + 2\rho\alpha$$

$\forall n \geq 0$ and for each $u \in F$. This implies that $\{x_n\}_{n=0}^{\infty}$ is bounded and so is $\{G(x_n, Jx_0)\}_{n=0}^{\infty}$. By the construction of C_n, we have that $C_m \subset C_n$ and $x_n = \Pi_{C_n}^f x_0$ for any positive integer $m \geq n$. Then from Lemma 2.8, we obtain that

$$\phi(x_m, x_n) + G(x_n, Jx_0) \leq G(x_m, Jx_0). \tag{3.3}$$

In particular,

$$\phi(x_{n+1}, x_n) + G(x_n, Jx_0) \leq G(x_{n+1}, Jx_0)$$

In view of (1.2), we have

$$G(x_{n+1}, Jx_0) - G(x_n, Jx_0) \geq \phi(x_{n+1}, x_n) \geq (||x_{n+1}|| - ||x_n||)^2 \geq 0,$$

and so $\{G(x_n, Jx_0)\}_{n=0}^{\infty}$ is nondecreasing. It follows that the limit $\lim_{n\to\infty} G(x_n, Jx_0)$ exists.

We now show that $F \subset C_n, \forall n \geq 0$. Since $\{x_n\}$ is bounded, so is $\{G(u, Jx_n)\}$ for any fixed $u \in C$. Let $B_r \subset \mathbb{R}$ be some ball of positive radius r satisfying $\{G(u, Jx_n)\} \subset B_r$

for every $u \in F$. Also let B be a set defined by $B := \{t \in B_r : t > 0\}$ and define a map $\zeta : \bar{B} \to \mathbb{R}^+$ by $\zeta(t) = \underset{i\geq 1}{sup}\zeta_i(t)$.

For $n = 0$, we clearly have $F \subset C = C_0$. Now let $u \in F$, then we have

$$\begin{aligned}
G(u, Jy_n) &= G(u, \beta_n Jz_n + (1-\beta_n)Jx_n) \\
&= ||u||^2 - 2\beta_n\langle u, Jz_n\rangle - 2(1-\beta_n)\langle u, Jx_n\rangle \\
&+ ||\beta_n Jz_n + (1-\beta_n)Jx_n||^2 + 2\rho f(u) \\
&\leq ||u||^2 - 2\beta_n\langle u, Jz_n\rangle - 2(1-\beta_n)\langle u, Jx_n\rangle \\
&+ \beta_n||Jz_n||^2 + (1-\beta_n)||Jx_n||^2 + 2\rho f(u) \\
&= \beta_n G(u, Jz_n) + (1-\beta_n)G(u, Jx_n) \\
&= (1-\beta_n)G(u, Jx_n) + \beta_n G(u, \alpha_{n0}Jx_n + \sum_{i=1}^{\infty}\alpha_{ni}Jw_n^{(i)})
\end{aligned}$$

$$\begin{aligned}
&= (1-\beta_n)G(u, Jx_n) + \beta_n[||u||^2 - 2\langle u, \alpha_{n0}Jx_n + \sum_{i=1}^{\infty}\alpha_{ni}Jw_n^{(i)}\rangle \\
&+ ||\alpha_{n0}Jx_n + \sum_{i=1}^{\infty}\alpha_{ni}Jw_n^{(i)}||^2 + 2\rho f(u)] \\
&\leq (1-\beta_n)G(u, Jx_n) + \beta_n\alpha_{n0}G(u, Jx_n) + \beta_n\sum_{i=1}^{\infty}G(u, Jw_n^{(i)}) \\
&- \beta_n\alpha_{n0}\alpha_{ni}g(||Jx_n - Jw_n^{(i)}||) \\
&\leq (1-\beta_n)G(u, Jx_n) + \beta_n\alpha_{n0}G(u, Jx_n) \\
&+ \beta_n\sum_{i=1}^{\infty}\alpha_{ni}[G(u, Jx_n) + v_{ni}\zeta(G(u, Jx_n)) + u_{ni}] \\
&- \beta_n\alpha_{n0}\alpha_{ni}g(||Jx_n - Jw_n^{(i)}||) \\
&\leq G(u, Jx_n) + \beta_n\sum_{i=1}^{\infty}\alpha_{ni}[v_{ni}\sup_{u\in F}\zeta(G(u, Jx_n)) + u_{ni}] \\
&- \beta_n\alpha_{n0}\alpha_{ni}g(||Jx_n - Jw_n^{(i)}||) \\
&= G(u, Jx_n) + \sigma_n - \beta_n\alpha_{n0}\alpha_{ni}g(||Jx_n - Jw_n^{(i)}||) \qquad (3.4)
\end{aligned}$$

Therefore

$$G(u, Jy_n) \leq G(u, Jx_n) + \sigma_n, \ \forall\ u \in F. \qquad (3.5)$$

i.e., $u \in C_{n+1}$ and so $F \subset C_{n+1}$ for all $n \geq 0$. Since C_{n+1} is closed and convex and $F \subset C_n, \forall n \geq 0$, it follows that $\Pi^f_{C_{n+1}}x_0$ is well defined for all $n \geq 0$.
By the assumptions on $\{u_{ni}\}$ and $\{v_{ni}\}$ for each $i = 1, 2, ...$ we have

$$\sigma_n = \sum_{i=1}^{\infty}\alpha_{ni}\Big(v_{ni}\sup_{p\in F}\zeta(G(p, Jx_n)) + u_{ni}\Big) \to 0\ (n \to \infty).$$

Now,(3.3) implies that

$$\phi(x_m, x_n) \leq G(x_m, Jx_0) - G(x_n, Jx_0), \tag{3.6}$$

taking the limit as $m, n \to \infty$, we obtain

$$\lim_{n\to\infty} \phi(x_m, x_n) = 0.$$

It then follows from Lemma 2.9 that $||x_m - x_n|| \to 0$ as $m, n \to \infty$. Hence $\{x_n\}_{n=1}^{\infty}$ is Cauchy. As E is a Banach space and C is closed, then there exists $p \in C$ such that $x_n \to p$ as $n \to \infty$.

Now since $\phi(x_m, x_n) \to 0$ as $m, n \to \infty$ we have in particular that

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0$$

and this further implies that

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \tag{3.7}$$

By the fact that $x_{n+1} \in C_{n+1} \subset C_n$, it follows that

$$\phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, x_n) + \sigma_n,$$

hence

$$\lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0.$$

By Lemma 2.9 we obtain

$$\lim_{n\to\infty} ||x_{n+1} - y_n|| = 0.$$

So

$$||x_n - y_n|| \leq ||x_{n+1} - x_n|| + ||x_{n+1} - y_n||,$$

implies

$$\lim_{n\to\infty} ||x_n - y_n|| = 0. \tag{3.8}$$

Since J is uniformly norm-to-norm continuous on bounded sets we also have

$$\lim_{n\to\infty} ||Jx_n - Jy_n|| = 0. \tag{3.9}$$

As $0 < \beta \leq \beta_n < 1$, then

$$||Jz_n - Jx_n|| = \frac{1}{\beta_n}||Jx_n - Jy_n|| \to 0, \quad \text{as} \quad n \to \infty.$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets we obtain

$$\lim_{n\to\infty} ||z_n - x_n|| = 0,$$

so that $z_n \to p$ as $n \to \infty$. From (3.4), for any $u \in F$ and $w_n^{(i)} \in T_i^n x_n,\ i = 1, 2, ...,$ we have

$$G(u, Jy_n) \leq G(u, Jx_n) + \sigma_n - \beta_n \alpha_{n0} \alpha_{ni} g(||Jx_n - Jw_n^{(i)}||),$$

it follows that,

$$\beta_n \alpha_{n0} \alpha_{ni} g(||Jx_n - Jw_n^{(i)}||) \leq G(u, Jx_n) - G(u, Jy_n) + \sigma_n,$$

but

$$\begin{aligned} G(u,Jx_n)-G(u,Jy_n) &= ||x_n||^2-||y_n||^2-2\langle u, Jx_n-Jy_n\rangle \\ &\leq \Big|||x_n||^2-||y_n||^2\Big|+2\Big|\langle u, Jx_n-Jy_n\rangle\Big| \\ &\leq \Big|||x_n||-||y_n||\Big|(||x_n||+||y_n||)+2||u||||Jx_n-Jy_n|| \\ &\leq ||x_n-y_n||(||x_n||+||y_n||)+2||u||||Jx_n-Jy_n|| \end{aligned}$$

from this, (3.8) and (3.9), we obtain $G(u,Jx_n)-G(u,Jy_n)\to 0$ as $n\to\infty$, using condition (i) and (ii) it follows that

$$\lim_{n\to\infty} g(||Jx_n-Jw_n^{(i)}||)=0.$$

By property of g, we have

$$\lim_{n\to\infty}||Jx_n-Jw_n^{(i)}||=0. \tag{3.10}$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets we obtain

$$\lim_{n\to\infty}||x_n-w_n^{(i)}||=0. \tag{3.11}$$

As $w_n^{(i)}\in T_i^n x_n$ it implies that $T_i w_n^{(i)}\subset T_i^{n+1}x_n$, now let $s_{n+1}^{(i)}\in T_i w_n^{(i)}$ which implies that $s_{n+1}^{(i)}\in T_i^{n+1}x_n$. Then, since T_i is uniformly L_i-Lipschitzian, $i\in\mathbb{N}$, we have

$$\begin{aligned} ||s_{n+1}^{(i)}-w_n^{(i)}|| &\leq ||s_{n+1}^{(i)}-w_{n+1}^{(i)}||+||w_{n+1}^{(i)}-x_{n+1}||+||x_{n+1}-x_n||+||x_n-w_n^{(i)}|| \\ &\leq (L_i+1)||x_{n+1}-x_n||+||w_{n+1}^{(i)}-x_{n+1}||+||x_n-w_n^{(i)}||. \end{aligned}$$

From this together with (3.7) and (3.11), we obtain that $\lim_{n\to\infty}||s_{n+1}^{(i)}-w_n^{(i)}||=0$ and so $s_{n+1}^{(i)}\to p$ as $n\to\infty$ for each $i\in\mathbb{N}$. In view of the closure of T_i, we have that $p\in T_i p$, for each $i\in\mathbb{N}$, therefore $p\in F$.

Finally, we show that $p=\Pi_F^f x_0$. Since $F=\cap_{i=1}^{\infty}F(T_i)$ is closed and convex, by Lemma 2.7 Π_F^f is single valued and so if we denote $v=\Pi_F^f x_0$, as $x_n=\Pi_{C_n}^f x_0$ and $v\in F\in C_n$, we have

$$G(x_n,Jx_0)\leq G(v,Jx_0),\ \forall\ n\geq 0.$$

We know that $G(\xi,J\varphi)$ is convex and lower semi-continuous with respect to ξ when φ is fixed, this implies that

$$G(p,Jx_0)\leq \liminf_{n\to\infty} G(x_n,Jx_0)\leq \limsup_{n\to\infty} G(x_n,Jx_0)\leq G(v,Jx_0).$$

From the definition of $\Pi_F^f x_0$ and the fact that $p\in F$, we see that $p=v$. This completes the proof.

The following important corollaries follow from Theorem 3.2 and Lemma 3.1.

Corollary 3.3. Let E be a real uniformly convex and uniformly smooth Banach space, and C be a nonempty closed and convex subset of E. Let $T_i: C\to N(C)$, $i=1,2,3,...$ be an infinite family of uniformly L_i-Lipschitzian, closed and relatively nonexpansive multi-valued mappings. Let $F=\cap_{i=1}^{\infty}F(T_i)\neq\emptyset$ and $f:E\to\mathbb{R}$ be convex and lower semi continuous mapping with $C \subset int(D(f))$. Suppose $\{x_n\}_{n=0}^{\infty}$ is iteratively generated by $x_0 \in C, C_0 = C$,

$$\begin{cases} z_n = J^{-1}(\alpha_{n0}Jx_n + \sum_{i=1}^{\infty} \alpha_{ni}Jw_n^{(i)}); \\ y_n = J^{-1}(\beta_n Jz_n + (1-\beta_n)Jx_n); \\ C_{n+1} = \{v \in C_n : G(v, Jy_n) \le G(v, Jx_n)\}; \\ x_{n+1} = \Pi_{C_{n+1}}^{f} x_0, \ n \ge 0, \end{cases} \tag{3.12}$$

where $\{\alpha_{ni}\}_{n=1}^{\infty}$ are sequences of positive real numbers, $w_n^{(i)} \in T_i x_n, \ i \in \mathbb{N}$ and

(i) $\sum_{i=0}^{\infty} \alpha_{ni} = 1$; $\liminf_{n\to\infty} \alpha_{n0}\alpha_{ni} > 0 \ \forall i \in \mathbb{N}$;

(ii) $0 < \beta < \beta_n \le 1$ for some $\beta \in (0, 1)$.

Then, $\{x_n\}_{n=1}^{\infty}$ converges strongly to $\Pi_F^f x_0$.

Corollary 3.4. Let E be a real uniformly convex and uniformly smooth Banach space, and C be a nonempty closed and convex subset of E. Let $T_i : C \to N(C)$, $i = 1, 2, 3, ...$ be an infinite family of uniformly L_i-Lipschitzian, closed and weak relatively nonexpansive multi-valued mappings. Let $F = \cap_{i=1}^{\infty} F(T_i) \ne \emptyset$ and $f : E \to \mathbb{R}$ be convex and lower semi continuous mapping with $C \subset int(D(f))$. Suppose $\{x_n\}_{n=0}^{\infty}$ is iteratively generated by $x_0 \in C, C_0 = C$,

$$\begin{cases} z_n = J^{-1}(\alpha_{n0}Jx_n + \sum_{i=1}^{\infty} \alpha_{ni}Jw_n^{(i)}); \\ y_n = J^{-1}(\beta_n Jz_n + (1-\beta_n)Jx_n); \\ C_{n+1} = \{v \in C_n : G(v, Jy_n) \le G(v, Jx_n)\}; \\ x_{n+1} = \Pi_{C_{n+1}}^{f} x_0, \ n \ge 0, \end{cases} \tag{3.13}$$

where $\{\alpha_{ni}\}_{n=1}^{\infty}$ are sequences of positive real numbers, $w_n^{(i)} \in T_i x_n, \ i \in \mathbb{N}$ and

(i) $\sum_{i=0}^{\infty} \alpha_{ni} = 1$; $\liminf_{n\to\infty} \alpha_{n0}\alpha_{ni} > 0 \ \forall i \in \mathbb{N}$;

(ii) $0 < \beta < \beta_n \le 1$ for some $\beta \in (0, 1)$.

Then, $\{x_n\}_{n=1}^{\infty}$ converges strongly to $\Pi_F^f x_0$.

Corollary 3.5. Let E be a real uniformly convex and uniformly smooth Banach space, and C be a nonempty closed and convex subset of E. Let $T_i : C \to N(C)$, $i = 1, 2, 3, ...$ be an infinite family of uniformly L_i-Lipschitzian, closed and quasi-ϕ-asymptotically nonexpansive multi-valued mappings with real sequences $\{k_{ni}\}$ $i = 1, 2, ...$ such that $k_{ni} \to 1, \ as \ n \to \infty$, $\forall i \in \mathbb{N}$. Let $F = \cap_{i=1}^{\infty} F(T_i) \ne \emptyset$, bounded and $f : E \to \mathbb{R}$ be convex and lower semi continuous mapping with $C \subset int(D(f))$. Suppose $\{x_n\}_{n=0}^{\infty}$ is a sequence iteratively generated by $x_0 \in C, C_0 = C$.

$$\begin{cases} z_n = J^{-1}(\alpha_{n0}Jx_n + \sum_{i=1}^{\infty} \alpha_{ni}Jw_n^{(i)}); \\ y_n = J^{-1}(\beta_n Jz_n + (1-\beta_n)Jx_n); \\ C_{n+1} = \{v \in C_n : G(v, Jy_n) \le G(v, Jx_n) + \sigma_n\}; \\ x_{n+1} = \Pi_{C_{n+1}}^{f} x_0, \ n \ge 0, \end{cases} \tag{3.14}$$

where $\{\alpha_{ni}\}_{n=1}^{\infty}$ are sequences of positive real numbers, $w_n^{(i)} \in T_i^n x_n$, $i \in \mathbb{N}$,

$$\sigma_n := \sum_{i=1}^{\infty} \alpha_{ni}\Big((k_{ni} - 1)\sup_{p\in F} G(p, Jx_n)\Big),$$

and

(i) $\sum_{i=0}^{\infty} \alpha_{ni} = 1$; $\liminf_{n\to\infty} \alpha_{n0}\alpha_{ni} > 0 \ \forall i \in \mathbb{N}$;

(ii) $0 < \beta < \beta_n \leq 1$ for some $\beta \in (0, 1)$.

Then, $\{x_n\}_{n=1}^{\infty}$ converges strongly to $\Pi_F^f x_0$.

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