COINCIDENCE POINT THEOREMS FOR MULTI-VALUED MAPPINGS IN FUZZY METRIC SPACES

TATJANA DOŠENOVIĆ* AND IVANA ˇSTAJNER-PAPUGA**

*Faculty of Technology
University of Novi Sad, Serbia
E-mail: tatjanad@tf.uns.ac.rs

**Department of Mathematics and Informatics
University of Novi Sad, Serbia
E-mail: stajner.papuga@dmi.uns.ac.rs

Abstract. In this paper, by applying the countable extensions of a $t$-norm, we have proved a coincidence point theorem for the fuzzy Nadler type of contraction mappings. Also, a coincidence point theorem in a fuzzy metric spaces for an implicit relation is given.

Key Words and Phrases: fuzzy metric space, $t$-norm, Nadler contraction mapping, coincidence point, Cauchy sequence, implicit relation.

2010 Mathematics Subject Classification: 47H10, 54H25; 55M20.

1. Introduction

It is a well known fact that a fuzzy metric space is a generalization of the concept of metric space that is based on the theory of fuzzy sets ([24]). Kramosil and Michalek (see [14]) introduced the notion of a fuzzy metric space by translating the concept of probabilistic metric space to the fuzzy environment. Further on, George and Veeramanani ([2, 3]) modified the previous and obtained a Hausdorff topology for this specific kind of fuzzy metric spaces. Recently, many authors observed that various contraction mappings in metric spaces can be transferred into a fuzzy metric spaces endowed with a special $t$-norm. S.B. Nadler ([19]) has proved a generalization of the well known Banach contraction principle for a multi-valued mappings $f : X \rightarrow CB(X)$, where $(X, d)$ is a classical metric space and $CB(X)$ is the family of all non-empty, closed and bounded subsets of $X$. There are various extensions of the Banach contraction mappings for single-valued and multi-valued mappings done in the fuzzy metric spaces’ background (see [1, 4, 6, 7, 8, 9, 12, 15, 16, 17, 18]). V. Popa ([20]) introduced the idea of implicit function to prove a common fixed point theorem in metric spaces. B. Singh and S. Jain ([23]) extended the result of Popa to the fuzzy metric spaces. Many authors (see [11], [21], [23], [25]) proved a common fixed point theorem for a single-valued mappings in the fuzzy metric spaces using implicit relations. S. Sedghi et al. ([22]) proved a common fixed point theorem for a multi-valued mappings that satisfies an implicit relation on a fuzzy metric spaces for a specific $t$-norm.
The focus of this paper is on a coincidence point theorem for three mappings for the fuzzy Nadler type of contraction mappings. Presented result is a generalization of the Theorem 4.13 from [6]. A probabilistic version of the presented result can be found in [26]. Also, a coincidence point theorem for three mappings in a fuzzy metric space with an implicit relation is given.

A short overview of some well-known definitions and facts that are the core of the presented results follows (see [2, 4, 6, 13]).

1.1. Preliminaries - t-norms.

Definition 1.1 A mapping $T : [0, 1] \times [0, 1] \to [0, 1]$ is called a triangular norm (a t-norm) if the following conditions are satisfied:

- $T(a, 1) = a$ for all $a \in [0, 1]$;
- $T(a, b) = T(b, a)$ for all $a, b \in [0, 1]$;
- $a \geq b, c \geq d \Rightarrow T(a, c) \geq T(b, d)$ $(a, b, c, d \in [0, 1])$;
- $T(a, T(b, c)) = T(T(a, b), c)$ $(a, b, c \in [0, 1])$.

The following are the four basic t-norms ([13]):

$T_M(x, y) = \min(x, y)$, $T_P(x, y) = x \cdot y$, $T_L(x, y) = \max(0, x + y - 1 + xy)$, $T_D(x, y) = \begin{cases} \min(x, y) & \text{if } \max(x, y) = 1, \\ 0 & \text{otherwise.} \end{cases}$

Some important families of t-norms are given in the following example ([13]):

Example 1.2

(i) The Dombi family of t-norms $(T^D_\lambda)_{\lambda \in [0, \infty]}$, which is defined by

$$T^D_\lambda(x, y) = \begin{cases} T_D(x, y), & \lambda = 0 \\ T_M(x, y), & \lambda = \infty \\ \frac{\lambda x}{\lambda x + (1 - \lambda)(1 + \frac{1-x}{y})}, & \lambda \in (0, \infty). \end{cases}$$

(ii) The Aczél-Alsina family of t-norms $(T^{AA}_\lambda)_{\lambda \in [0, \infty]}$, which is defined by

$$T^{AA}_\lambda(x, y) = \begin{cases} T_D(x, y), & \lambda = 0 \\ T_M(x, y), & \lambda = \infty \\ e^{-(\log x)^\lambda + (\log y)^\lambda}^{1/\lambda}, & \lambda \in (0, \infty). \end{cases}$$

(iii) Sugeno-Weber family of t-norms $(T^{SW}_\lambda)_{\lambda \in [-1, \infty]}$, which is defined by

$$T^{SW}_\lambda(x, y) = \begin{cases} T_D(x, y), & \lambda = -1 \\ T_P(x, y), & \lambda = \infty \\ \max(0, \frac{x^\lambda y^{1+\lambda x}}{x+y^\lambda}), & \lambda \in (-1, \infty). \end{cases}$$

The following class of t-norms, that has proved itself as a highly useful tool in the fixed point theory, was introduced in [4].

Definition 1.3 [4] Let $T$ be a t-norm and let $(T_n)_{n \in \mathbb{N}}$ be a sequence of t-norms given by the following:

$$T_1(x) = T(x, x) \quad \text{and} \quad T_{n+1}(x) = T(T_n(x), x).$$
A t-norm $T$ is of the $H$-type if $T$ is continuous and the sequence \{$T_n(x)$\}$_{n \in \mathbb{N}}$ is equicontinuous at $x = 1$.

**Remark 1.4** The family \{$T_n(x)$\}$_{n \in \mathbb{N}}$ of t-norms is equicontinuous at $x = 1$, if for all $\lambda \in (0, 1)$ there exists $\delta(\lambda) \in (0, 1)$ such that the following implication holds:

$$x > 1 - \delta(\lambda) \Rightarrow T_n(x) > 1 - \lambda \quad \text{for all} \quad n \in \mathbb{N}.$$  

(see [4]). A trivial example of a t-norm of $H$-type is $T = T_M$. A nontrivial example can be found in [4].

An arbitrary t-norm $T$ can be extended, due to the associativity, to an $n$-ary operation on $[0, 1]$ (see [13]):

$$T(x_1, x_2, \ldots, x_n) = T_n\left(\prod_{i=1}^{n} x_i\right) \quad \text{and} \quad T_1\sum_{i=1}^{n} x_i = x_1.$$

Now, the four basic t-norms are extended in the following manner

$$T_L(x_1, x_2, \ldots, x_n) = \max\{\sum_{i=1}^{n} x_i - (n - 1), 0\},$$

$$T_M(x_1, x_2, \ldots, x_n) = \min\{x_1, x_2, \ldots, x_n\},$$

$$T_P(x_1, x_2, \ldots, x_n) = x_1 \cdot x_2 \cdots x_n$$

and

$$T_D(x_1, x_2, \ldots, x_n) = \left\{ \begin{array}{ll} x_i & \text{if } x_j = 1 \text{ for all } j \neq i, \\
0 & \text{otherwise.} \end{array} \right.$$  

Also, a t-norm $T$ can be extend to a countable case as follows:

$$T^\infty_{\prod_{i=1}^{n} x_i} = \lim_{n \to \infty} T^\infty_{\prod_{i=1}^{n} x_i},$$

where $(x_n)_{n \in \mathbb{N}}$ is an arbitrary sequence from $[0, 1]$. The limit on the right-hand side exists since the sequence $(T^\infty_{\prod_{i=1}^{n} x_i})_{n \in \mathbb{N}}$ is nonincreasing and bounded from below.

The following equivalences and proposition that will be used further on can be found in [6]:

- If $T = T_L$ or $T = T_P$, then

$$\lim_{n \to \infty} T^\infty_{\prod_{i=1}^{n} x_i} = 1 \iff \sum_{i=1}^{\infty} (1 - x_i) < \infty.$$  

- If $(T^\lambda_{\prod_{i=1}^{n} x_i})_{\lambda \in (0, \infty)}$ is the Domfani family of t-norms or the Aczél-Alsina family of t-norms and if $(x_n)_{n \in \mathbb{N}}$ is a sequence of elements from $(0, 1]$ such that $\lim_{n \to \infty} x_n = 1$, then

$$\lim_{n \to \infty} (T^\lambda_{\prod_{i=1}^{n} x_i}) = 1 \iff \sum_{i=1}^{\infty} (1 - x_i)^\lambda < \infty. \quad (1.1)$$

- If $(T^\lambda_{\prod_{i=1}^{n} x_i})_{\lambda \in (-1, \infty]}$ is the Sugeno-Weber family of t-norms and $(x_n)_{n \in \mathbb{N}}$ is a sequence of elements from $(0, 1]$ such that $\lim_{n \to \infty} x_n = 1$, then

$$\lim_{n \to \infty} (T^\lambda_{\prod_{i=1}^{n} x_i}) = 1 \iff \sum_{i=1}^{\infty} (1 - x_i) < \infty. \quad (1.2)$$
Proposition 1.5 [6] Let \((x_n)_{n \in \mathbb{N}}\) be a sequence of numbers from \([0, 1]\) such that \(\lim_{n \to \infty} x_n = 1\) and t-norm \(T\) is of the \(H\)-type. Then,
\[
\lim_{n \to \infty} T_i x_i = \lim_{n \to \infty} T_{i+n} x_{i+n} = 1.
\]

1.2. Preliminaries - fuzzy metric spaces.

Definition 1.6 [2] The 3-tuple \((X, M, T)\) is a fuzzy metric space if \(X\) is an arbitrary set, \(T\) is a continuous t-norm and \(M\) is a fuzzy set on \(X^2 \times (0, \infty)\) satisfying the following conditions:

- \(M(x, y, t) > 0\), for all \(x, y \in X\), \(t > 0\),
- \(M(x, y, t) = 1\) for all \(t > 0 \Leftrightarrow x = y\),
- \(M(x, y, t) = M(y, x, t)\), for all \(x, y \in X\), \(t > 0\),
- \(T(M(x, y, t), M(y, z, s)) \leq M(x, z, t+s)\), for all \(x, y, z \in X\), \(t, s > 0\),
- \(M(x, y, \cdot) : (0, \infty) \to [0, 1]\) is continuous for all \(x, y \in X\).

If \((X, M, T)\) is a fuzzy metric space, then \(M\) generates the Hausdorff topology on \(X\) with base of open sets \(\{U(x, r, t) : x \in X, r \in (0, 1), t > 0\}\), where
\[
U(x, r, t) = \{y : y \in X, M(x, y, t) > 1 - r\}
\]
(see [2]).

Since the focus is on the complete fuzzy metric spaces, the following definition is needed.

Definition 1.7 [2]

(a) A sequence \(\{x_n\}_{n \in \mathbb{N}}\) in a fuzzy metric space \((X, M, T)\) is a Cauchy sequence if for all \(\varepsilon \in (0, 1), t > 0\) there exists \(n_0 \in \mathbb{N}\) such that \(M(x_n, x_m, t) > 1 - \varepsilon\) for all \(n, m \geq n_0\).

(b) A sequence \(\{x_n\}_{n \in \mathbb{N}}\) in a fuzzy metric space \((X, M, T)\) converges to \(x\) if for all \(\varepsilon \in (0, 1), t > 0\) there exists \(n_0 \in \mathbb{N}\) such that \(M(x_n, x, t) > 1 - \varepsilon\) for all \(n \geq n_0\).

(c) A fuzzy metric space in which every Cauchy sequence is convergent is complete.

Additionally, through this paper let us assume that \(\lim_{t \to \infty} M(x, y, t) = 1\).

2. MAIN RESULTS

Let \(X\) be an arbitrary nonempty set, \(CB(X)\) the family of all non-empty, closed and bounded subsets of \(X\), and \(C(X)\) the family of all nonempty and closed subsets of \(X\).

Let \((X, M, T)\) be a fuzzy metric space.

2.1. Fuzzy Nadler \(q\)-contraction. S.B. Nadler ([19]) proved on a metric space \((X, d)\) generalization of the Banach contraction principle for multi-valued mappings \(f : X \to CB(X)\) of the form
\[
D(fx, fy) \leq qd(x, y),
\]
where \(D\) is the Hausdorff metric and \(q \in (0, 1)\).
A generalization of the Nadler contraction principle in a fuzzy metric space \((X, M, T)\) follows.

**Definition 2.1** [6] Let \((X, M, T)\) be a fuzzy metric space, \(A\) nonempty subset of \(X\) and \(F : A \to C(A)\). Mapping \(F\) is the fuzzy Nadler \(q\)-contraction, where \(q \in (0, 1)\), if the following condition is satisfied:

for all \(u, v \in A\), all \(x \in Fu\) and all \(\delta > 0\) there exists \(y \in Fu\) such that for all \(t > \delta\)

\[
M(x, y, t) \geq M(u, v, \frac{t - \delta}{q}).
\]

**Remark 2.2**

(i) Since the function \(M(x, y, \cdot)\) is continuous, for \(f\) being a single valued mapping the fuzzy Nadler \(q\)-contraction coincides with the notion of fuzzy Banach \(q\)-contraction.

(ii) Connection of the probabilistic Nadler \(q\)-contraction with random operators is illustrated by Example 4.10 from [6]. Connection with the fuzzy metric case is analogous.

**Definition 2.3** [10] Let \((X, M, T)\) be a fuzzy metric space, \(\emptyset \neq A \subset X\), \(f : A \to A\) and \(F : A \to C(A)\). Mapping \(F\) is a \(f\)-strongly demicompact if for every sequence \((x_n)_{n \in \mathbb{N}}\) from \(A\), and every \(\varepsilon > 0\) such that

\[
\lim_{n \to \infty} M(fx_n, y_n, \varepsilon) = 1\text{ for some sequence } (y_n)_{n \in \mathbb{N}}, y_n \in Fx_n, n \in \mathbb{N},
\]

there exists convergent subsequences \((fx_{n_k})_{k \in \mathbb{N}}\).

**Example 2.4** Let \(X = D = \mathbb{R}\), \(M(x, y, t) = M_d(x, y, t) = \frac{t}{\pi + d(x, y)}\) where \(d\) is the usual metric on \(\mathbb{R}\) and \(T = T_{\mathbb{R}}\). Then \((X, M, T)\) is a fuzzy metric space. Let \(f(x) = \arctan x\) and \(F(x) = [\arctan x - 1, \arctan x + 1]\). Then for every sequence \((x_n)_{n \in \mathbb{N}}\), we can find a sequence \((y_n)_{n \in \mathbb{N}}\) such that \(y_n \in F(x_n)\), namely \(y_n = \arctan x_n + \frac{1}{n}\), and \(M_d(f(x_n), y_n, t) = \frac{t}{\pi + d(f(x_n), y_n)} \to 1, n \to \infty\). Now, since the sequence \((\arctan x_n)_{n \in \mathbb{N}}\) is bounded and it has a monotone subsequence, there exists a convergent subsequence, e.g., for \(x_n = n(-1)^n\), the sequence \(f(x_n) = \arctan(n(-1)^n)\) has a convergent subsequence obtained by selecting even members.

**Definition 2.5** [5] A mapping \(F : X \to C(X)\) is weakly commuting with \(f : X \to X\) if for all \(x \in X\) it holds

\[f(Fx) \subseteq F(fx).\]

**Theorem 2.6** Let \((X, M, T)\) be a complete fuzzy metric space, and \(A\) nonempty and closed subset of \(X\). Let \(f : A \to A\) be a continuous mapping and \(F, G : A \to C(A)\) such that for \(q \in (0, 1)\) the following holds:

For every \(u, v \in A\), \(x \in Fu\) and \(\delta > 0\), there exist \(y \in Gu\) such that for all \(\varepsilon > \delta\)

\[
M(x, y, \varepsilon) \geq M(fu, fv, \frac{\varepsilon - \delta}{q}). \tag{2.1}
\]

If \(F\) and \(G\) are weakly commuting with \(f\) and the following is satisfied

(i) \(F\) or \(G\) are \(f\)-strongly demicompact

or
(ii) there exists \( x_0, x_1 \in A, f x_1 \in F x_0 \) and \( \mu \in (q, 1) \) such that for a \( t \)-norm \( T \) holds
\[
\lim_{n \to \infty} T_{i=n}^{\infty} M(f x_0, f x_1, \frac{1}{\mu^i}) = 1.
\]

Then there exist \( x \in A \) such that \( f x \in F x \cap G x \).

Proof. Let \( x_0, x_1 \in A \) be such that \( f x_1 \in F x_0 \). From (2.1) for \( x = f x_1, u = x_0, v = x_1 \), \( \delta = q \) it follows that there exist \( x_2 \in A \) such that \( f x_2 \in G x_1 \) and
\[
M(f x_1, f x_2, \varepsilon) \geq M(f x_0, f x_1, \frac{\varepsilon - q}{q}).
\]

Continuing in this way we can construct a sequence \( (x_n)_{n \in \mathbb{N}} \) from \( A \) such the following conditions are satisfied
(a) \( f x_{2n+1} \in F x_{2n} \) and \( f x_{2n+2} \in G x_{2n+1} \)
(b) \( M(f x_n, f x_{n+1}, \varepsilon) \geq M(f x_{n-1}, f x_n, \frac{\varepsilon - q^n}{q^n}) \).

From (b) it follows
\[
M(f x_n, f x_{n+1}, \varepsilon) \geq M(f x_1, f x_0, \frac{\varepsilon - nq^n}{q^n}).
\]

Since \( \varepsilon > 0, \lim_{n \to \infty} M(f x_1, f x_0, \frac{\varepsilon - nq^n}{q^n}) = 1 \), \( \lim_{n \to \infty} (\frac{\varepsilon - nq^n}{q^n}) = \infty \), it follows
\[
\lim_{n \to \infty} M(f x_n, f x_{n+1}, \varepsilon) = 1. \tag{2.2}
\]

If we supposed that \( F \) is \( f \)-strongly demicompact, i.e., condition (i) is fulfilled, using
\[
\lim_{n \to \infty} M(f x_{2n}, f x_{2n+1}, \varepsilon) = 1 \quad \text{and} \quad f x_{2n+1} \in F x_{2n}
\]
we conclude that there exist a convergent subsequence \( (f x_{2n_k})_{k \in \mathbb{N}} \) of a sequence \( (f x_n)_{n \in \mathbb{N}} \).

It remains to be proved that the sequence \( (f x_n)_{n \in \mathbb{N}} \) is convergent if \( T \) satisfies condition (ii).

Let \( \sigma = \frac{2}{\mu} \). Since \( \sigma \in (0, 1), \sum_{i=1}^{\infty} \sigma^i \) is convergent, there exist \( m_0 \in \mathbb{N} \) such that \( \sum_{i=m_0}^{\infty} \sigma^i < 1 \). Therefore, for all \( m > m_0 \) and \( s \in \mathbb{N} \) it holds
\[
\varepsilon > \varepsilon \sum_{i=m_0}^{\infty} \sigma^i > \varepsilon \sum_{i=m}^{m+s} \sigma^i.
\]
Then,

\[
M(fx_{m+s+1}, fx_m, \varepsilon) \geq M(fx_{m+s+1}, fx_m, \varepsilon \sum_{i=m}^{m+s} \sigma^i)
\]

\[
\geq T(T(\ldots T(M(fx_{m+s+1}, fx_{m+s}, \varepsilon \sigma^m)) \ldots, M(fx_{m+1}, fx_m, \varepsilon \sigma^m)))
\]

\[
\geq T(T(\ldots T(M(fx_1, fx_0, \varepsilon \sigma^m - (m + s)q^m + \varepsilon / q^m)))
\]

\[
\geq \ldots M(fx_1, fx_0, \varepsilon \sigma^m - m q^m)
\]

\[
= T^{m+s} M(fx_1, fx_0, \varepsilon / \mu^i - i).
\]

Since \( \mu \in (q, 1) \), there exist \( m_1(\varepsilon) > m_0 \) such that \( \varepsilon / \mu^m - m > \varepsilon / 2 \mu^m \), for every \( m > m_1(\varepsilon) \). Now, for all \( s \in \mathbb{N} \) we have

\[
M(fx_{m+s+1}, fx_m, \varepsilon) \geq T_{i=m}^{m+s} M(fx_1, fx_0, \varepsilon / 2 \mu^i)
\]

\[
\geq T_{i=m}^\infty M(fx_1, fx_0, \varepsilon / 2 \mu^i)
\]

Since \( \lim_{m \to \infty} T_{i=m}^\infty M(fx_1, fx_0, \varepsilon / 2 \mu^i) = 1 \), it gives us \( \lim_{m \to \infty} T_{i=m}^\infty M(fx_1, fx_0, \varepsilon / 2 \mu^i) = 1 \), for all \( \varepsilon > 0 \). The previous implies that for all \( \varepsilon > 0, \lambda \in (0, 1) \), there exist \( m_2(\varepsilon, \lambda) > m_1(\varepsilon) \) such that \( M(fx_{m+s+1}, fx_m, \varepsilon) > 1 - \lambda \), for all \( m > m_2(\varepsilon, \lambda) \) and all \( s \in \mathbb{N} \).

The obtained sequence \( (fx_n)_{n \in \mathbb{N}} \) is a Cauchy sequence and, since \( X \) is complete, the limit \( \lim_{n \to \infty} fx_n \) exists.

Therefore, in both cases (i) and (ii) there exists a subsequence \( (fx_{n_k})_{k \in \mathbb{N}} \) such that

\[
x = \lim_{k \to \infty} fx_{2n_k} \in A.
\]

Also, from (2.2) it follows that \( x = \lim_{k \to \infty} fx_{2n_k+1} \).

Now, let us show that \( fx \in Fx \cap Gx \). Since \( Fx \) and \( Gx \) are closed, it should be prove that \( fx \in Fx \cap Gx \), i.e., that for every \( \varepsilon > 0 \) and \( \lambda \in (0, 1) \) there exists
$r_1(\varepsilon, \lambda) \in Fx$ and $r_2(\varepsilon, \lambda) \in Gx$ such that

$$r_1(\varepsilon, \lambda) \in U(fx, \varepsilon, \lambda) \quad \text{and} \quad r_2(\varepsilon, \lambda) \in U(fx, \varepsilon, \lambda).$$

Since the t-norm $T$ is continuous, we have $\sup x < 1 T(x, x) = 1$ and there exists $\delta(\lambda) \in (0, 1)$ such that

$$T(1 - \delta(\lambda), T(1 - \delta(\lambda), 1 - \delta(\lambda))) > 1 - \lambda.$$  

From the continuity of the mapping $f$ and from $x = \lim_{k \to \infty} fx_{2n_k}$ follows that there exist $k_1 \in \mathbb{N}$ such that

$$M(fx, fx_{2n_k}, \frac{\varepsilon}{3}) > 1 - \delta(\lambda), \quad \text{for every} \quad k \geq k_1.$$ 

Further on, condition (2.2) insures existence of $k_2 \in \mathbb{N}$ such that

$$M(fx_{2n_k}, fx_{2n_k+1}, \frac{\varepsilon}{3}) > 1 - \delta(\lambda), \quad \text{for every} \quad k \geq k_2.$$ 

Let $\delta = \frac{\varepsilon}{12}$. Since $F$ is weakly commuting with $f$ then

$$ffx_{2n_k+1} \in f(Fx_{2n_k}) \subseteq F(fx_{2n_k}),$$

and, due to (2.1), there exist $r_2(\varepsilon, \lambda) \in Gx$ such that

$$M(fx_{2n_k+1}, r_2(\varepsilon, \lambda), \frac{\varepsilon}{3}) \geq M(fx_{2n_k}, fx, \frac{\varepsilon}{4q})$$

$$> 1 - \delta(\lambda).$$

Then

$$M(fx, r_2(\varepsilon, \lambda), \varepsilon) \geq T(M(fx, fx_{2n_k}, \frac{\varepsilon}{3}), T(M(fx_{2n_k}, fx_{2n_k+1}, \frac{\varepsilon}{3})),$$

$$M(fx_{2n_k+1}, r_2(\varepsilon, \lambda), \frac{\varepsilon}{3})) > 1 - \lambda.$$ 

Now we have $r_2(\varepsilon, \lambda) \in U(fx, \varepsilon, \lambda)$.

Similarly we can prove that $r_1(\varepsilon, \lambda) \in Fx \cap U(fx, \varepsilon, \lambda)$.

**Example 2.7** Let $(X, M, T)$ be a fuzzy metric space as in Example 2.4. Let $A = [\frac{1}{2}, 2]$ and for all $x \in A$.

(a) Let $f(x) = 2$, $F(x) = \{1, 2\}$ and $G(x) = \{\frac{1}{2}, 1, 2\}$. Since

$$f(Fx) = \{2\} \subseteq F(fx) = \{1, 2\} \quad \text{and} \quad f(Gx) = \{2\} \subseteq G(fx) = \{\frac{1}{2}, 1, 2\}$$

mappings $F$ and $G$ weakly commuting with $f$. Also, mappings $F$ and $G$ are $f$-strongly demicompact, for every $u, v \in A$ condition (2.1) is satisfied and the claim from the previous theorem holds.

(b) Let $f(x) = x$, $F(x) = \{1\}$ and $G(x) = \{1, x\}$. Then $F$ and $G$ are weakly commuting with $f$, $F$ is $f$-strongly demicompact, for every $u, v \in A$ condition (2.1) is satisfied and the claim from the previous theorem holds.

**Remark 2.8** From Proposition 1.5 follows that the Theorem 2.6 holds if condition (ii) is substituted with the assumption “t-norm $T$ is of $H$-type”.

**Corollary 2.9** Let $(X, M, T^*)$ be a complete fuzzy metric space and $A$ nonempty and closed subset of $X$. Let $f : A \to A$ be a continuous mapping and $F, G : A \to C(A)$ such that for $q \in (0, 1)$ the following holds:
For every \( u, v \in A, x \in Fu \) and \( \delta > 0 \), there exist \( y \in Gv \) such that for all \( \varepsilon > \delta \)
\[
M(x, y, \varepsilon) \geq M(fu, fv, \frac{\varepsilon - \delta}{q}).
\]

If \( F \) and \( G \) are weakly commuting with \( f \) and the following is satisfied
(i) \( F \) or \( G \) are \( f \)-strongly demicompact
or
(ii-a) for \( T^* = T^D_\lambda \) or \( T^* = T^A_\lambda \), \( \lambda > 0 \), there exists \( x_0, x_1 \in A, fx_1 \in Fx_0 \) and \( \mu \in (q, 1) \) such that
\[
\sum_{i=1}^{\infty} (1 - M(fx_0, fx_1, \frac{1}{\mu^i}))^\lambda < \infty;
\]
(iii-b) for \( T^* = T^SW_\lambda \), \( \lambda > -1 \), there exists \( x_0, x_1 \in A, fx_1 \in Fx_0 \) and \( \mu \in (q, 1) \) such that
\[
\sum_{i=1}^{\infty} (1 - M(fx_0, fx_1, \frac{1}{\mu^i})) < \infty.
\]

Then, there exist \( x \in A \) such that \( fx \in Fx \cap Gx \).

**Proof.** Follows directly from equivalences (1.1) and (1.2).

### 2.2. Implicit relation

Let \( \Phi \) be the set of all continuous functions \( \phi : [0,1]^6 \to \mathbb{R} \) such that
1. \( \phi(t_1, t_2, t_3, t_4, t_5, t_6) \) is non-increasing in \( t_6 \).
2. \( \phi(u, v, v, u, 1, T(u, v)) \geq 0 \) imply \( u \geq v \).
3. \( \phi(u, 1, 1, u, 1, u) \geq 0 \) imply \( u \geq 1 \).

**Example 2.10** Let \( T = T_p \) and \( \phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 t_2 t_4 - t_3 t_5 t_6 \).

**Theorem 2.11** Let \( (X, M, T) \) be a complete fuzzy metric space and \( A \) nonempty and closed subset of \( X \). Let \( f : A \to A \) a continuous function and let \( F, G : X \to C(A) \) be a multi-valued functions such that for some \( \phi \in \Phi \) there exists a constant \( k \in (0,1) \) such that for all \( u, v \in A \) and every \( t > 0 \), \( x \in Fu \) there exists \( y \in Gv \) such that
\[
\phi\left(M(x, y, kt), M(fu, fv, t), M(x, fu, t)\right) \geq 0.
\]

If \( F \) and \( G \) are weakly commuting with \( f \) and the following is satisfied
(i) \( F \) or \( G \) are \( f \)-strongly demicompact
or
(ii) there exists \( x_0, x_1 \in A \) such that for \( fx_1 \in Fx_0 \) and \( \mu \in (k, 1) \) the following holds:
\[
\lim_{n \to \infty} T^\infty_n M(fx_0, fx_1, \frac{1}{\mu^n}) = 1.
\]

Then there exists \( x \in A \) such that \( fx \in Fx \cap Gx \).

**Proof.** Let \( x_0 \) and \( x_1 \in A \) such that \( fx_1 \in Fx_0 \). From (2.3) for \( x = fx_1, u = x_0, v = x_1 \) it follows that there exist \( x_2 \in A \) such that \( fx_2 \in Gx_1 \) and
\[
\phi\left(M(fx_1, fx_2, kt), M(fx_0, fx_1, t), M(fx_1, fx_0, t), M(fx_2, fx_1, kt), M(fx_1, fx_1, t), M(fx_2, fx_0, (k + 1)t)\right) \geq 0.
\]
Since $M(f_{x_2}, f_{x_0}, (k+1)t) \geq T(M(f_{x_2}, f_{x_1}, k), M(f_{x_1}, f_{x_0}, t))$, by applying $(\phi_1)$ and $(\phi_2)$, it is obtained

$$
\phi(M(f_{x_2}, f_{x_1}, k), M(f_{x_0}, f_{x_1}, t), M(f_{x_1}, f_{x_0}, t),
M(f_{x_2}, f_{x_1}, k), M(f_{x_1}, f_{x_0}, t), T(M(f_{x_2}, f_{x_1}, k), M(f_{x_1}, f_{x_0}, t))) \geq 0
$$

and $M(f_{x_1}, f_{x_2}, k) \geq M(f_{x_0}, f_{x_1}, t)$, i.e., $M(f_{x_1}, f_{x_2}, t) \geq M(f_{x_0}, f_{x_1}, \frac{t}{k})$. Continuing with the previous procedure, we can obtain a sequence $\{x_n\}_{n \in \mathbb{N}}$ from $A$ such that

(a) $f_{x_{2n+1}} \in F_{x_{2n}}$ and $f_{x_{2n+2}} \in G_{x_{2n+1}}$

(b) $M(f_{x_n}, f_{x_{n+1}}, t) \geq M(f_{x_{n-1}}, f_{x_n}, \frac{t}{k}) \geq \cdots \geq M(f_{x_1}, f_{x_0}, \frac{t}{k^n})$; i.e.,

$$
\lim_{n \to \infty} M(f_{x_n}, f_{x_{n+1}}, t) = 1.
$$

If $F$ is $f$-strongly demicompact, from $\lim_{n \to \infty} M(f_{x_{2n}}, f_{x_{2n+1}}, t) = 1$ and $f_{x_{2n+1}} \in F_{x_{2n}}$ follows existence of a convergent subsequence $(f_{x_{2n_p}})_{p \in \mathbb{N}}$ of a sequence $(f_{x_{2n}})_{n \in \mathbb{N}}$.

It remains to be proved that the sequence $\{f_{x_n}\}$ is a convergent if the $t$-norm $T$ satisfies condition (ii). Let $\sigma = \frac{k}{\mu}$. Since $0 < \sigma < 1$ the series $\sum_{i=1}^{\infty} \sigma^i$ is convergent and there exists $n_0 \in \mathbb{N}$ such that $\sum_{i=n_0}^{\infty} \sigma^i < 1$. Hence, for $n > n_0$ and all $m \in \mathbb{N}$, it holds

$$
t > t \sum_{i=n_0}^{\infty} \sigma^i > t \sum_{i=n}^{n+m-1} \sigma^i.
$$

Then we have,

$$
M(f_{x_{n+m}}, f_{x_n}, t) \geq M(f_{x_{n+m}}, f_{x_n}, t \sum_{i=n}^{n+m-1} \sigma^i) \geq
\begin{array}{c}
T(T(\ldots T(M(f_{x_{n+m}}, f_{x_{n+m-1}}, t\sigma^{n+m-1}), \ldots, M(f_{x_{n+1}}, f_{x_{n}}, t\sigma^n)))
\end{array}
$$

$$
\geq T(T(\ldots T(M(f_{x_0}, f_{x_1}, \frac{t\sigma^{n+m-1}}{k^{n+m-1}}), \ldots, M(f_{x_0}, f_{x_1}, \frac{t\sigma^n}{k^n}))
\begin{array}{c}
= T(T(\ldots T(M(f_{x_0}, f_{x_1}, \frac{t}{\mu^{n+m-1}}), \ldots, M(f_{x_0}, f_{x_1}, \frac{t}{\mu^n}))
\end{array}
$$

$$
= T^{n+m-1}_{t=1} M(f_{x_0}, f_{x_1}, \frac{1}{\mu^t}) \geq T^{\infty}_{i=1} M(f_{x_0}, f_{x_1}, \frac{1}{\mu^t}).
$$

The limit $\lim_{n \to \infty} T^{\infty}_{i=1} M(f_{x_0}, f_{x_1}, \frac{1}{\mu^t}) = 1$ implies $\lim_{n \to \infty} T^{\infty}_{i=1} M(f_{x_0}, f_{x_1}, \frac{1}{\mu^t}) = 1$ for all $t > 0$. Now, for every $t > 0$ and every $\lambda \in (0, 1)$ there exist $n_1(t, \lambda)$ such that $M(f_{x_{n+m}}, f_{x_n}, t) > 1 - \lambda$ for every $n \geq n_1(t, \lambda)$ and every $m \in \mathbb{N}$. Therefore, the sequence $\{f_{x_n}\}$ is a Cauchy sequence and the completeness of $X$ insures that there exists $x \in A$ such that $\lim_{n \to \infty} f_{x_n} = x$. Now, for both cases there exists a subsequence $(f_{x_{n_p}})_{p \in \mathbb{N}}$ such that $x = \lim_{p \to \infty} f_{x_{2n_p}} \in A$. 

76 TATJANA DOŠEVIĆ AND IVANA ŠTAJNER-PAPUGA
The remaining step is to prove that \( fx \in Fx \cap Gx \). Since \( Fx \) and \( Gx \) are closed, it should be proved that \( fx \in Fx \cap Gx \), i.e., that for all \( \varepsilon > 0 \) and \( \lambda \in (0, 1) \) there exist \( r_1(\varepsilon, \lambda) \in Fx \) and \( r_2(\varepsilon, \lambda) \in Gx \) such that

\[
\frac{r_1(\varepsilon, \lambda)}{r_2(\varepsilon, \lambda)} \in U(fx, \varepsilon, \lambda) \quad \text{and} \quad r_2(\varepsilon, \lambda) \in U(fx, \varepsilon, \lambda).
\]

From the continuity of the mapping \( F \)

\[
F \quad \text{by applying (2.3) on} \quad x
\]

\[
\phi
\]

\[
\text{Letting} \quad p, \quad \phi
\]

\[
i.e., \quad \phi
\]

\[
\text{Example 2.12} \quad \text{can prove that}
\]

\[
T \quad \text{is of}
\]

\[
\text{Again the second condition can be substituted with assumption “t-norm}
\]

\[
f \text{fied and the claim holds.}
\]

\[
\text{Provincial Secretariat for Science and Technological Development of Vojvodina.}
\]

\[
\text{The authors are supported by MNTRRS-174009 and by the}
\]

\[
\text{Acknowledgment.} \quad \text{The authors are supported by MNTRRS-174009 and by the}
\]

\[
\text{the Province Secretariat for Science and Technological Development of Vojvodina.}
\]

\[
\text{References}
\]


78 TATJANA DOŠENOVIĆ AND IVANA ŠTAJNER-PAPUGA


Received: May 2, 2012; Accepted: January 24, 2013.