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FIXED POINTS FOR MAPPINGS OF CYCLICAL TYPE IN ORDERED METRIC SPACES

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Abstract. The purpose of this paper is to present some fixed point results for mappings of cyclical type in the context of ordered metric spaces.
Our results extend the ones appearing in [A. Amini-Harandi, H. Emami, *A fixed point theorem for contraction type maps in partially ordered metric spaces and application to ordinary differential equations*, Nonlinear Anal. **72** (2010), 2238-2242].
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1. Introduction

In [2], Geraghty introduces the following class of functions $\Im$ given by:

$$\Im = \{\beta : [0, \infty) \to [0, 1) : \ \beta(t_n) \to 1 \Rightarrow t_n \to 0\},$$

and he proves the following fixed point theorem.

Theorem 1.1. *(Theorem 1.3 of* [2]*). Let (X, d) be a complete metric space and let $T : X \to X$ be an operator satisfying for every $x, y \in X$,*

$$d(Tx, Ty) \le \beta(d(x, y)).d(x, y),$$

where $\beta \in \Im$. Then T has a unique fixed point.

In [3], Kirk. et al. extend Theorem 1.1 for mappings of cyclical type.
Before to present this result, we need the following definition.

Definition 1.2. *(see [4,5]) Let X be a nonempty set, m a positive integer and $T : X \to X$ a mapping. $X = \bigcup_{i=1}^{m} A_i$ is said to be a cyclic representation of X with respect to T if*

(i) A_i, $i = 1, 2, \dots, m$ *are nonempty sets;*
(ii) $T(A_1) \subset A_2, \dots, T(A_{m-1}) \subset A_m, T(A_m) \subset A_1$.

Now, we present the following result which appears in [3].

Theorem 1.3. *(Theorem 2.3 of* [3]*). Let* (X, d) *be a complete metric space,* m *is a positive integer,* $A_1, A_2, \dots, A_m$ *nonempty closed subsets of* X, $X = \bigcup_{i=1}^{m} A_i$ *and* $T : X \to X$ *an operator such that*

(i) $X = \bigcup_{i=1}^{m} A_i$ *is a cyclic representation of* X *with respect to* T,
(ii) *for any* $x \in A_i$ *and* $y \in A_{i+1}$ $(i = 1, 2, \dots, m)$, *where* $A_{m+1} = A_1$

$$d(Tx, Ty) \le \alpha(d(x, y))d(x, y),$$

where $\alpha \in \Im$.

Then T *has a unique fixed point.*

Recently, in [1] the authors prove a version of Theorem 1.3 in the context of ordered metric spaces. The main result in [1] is the following.

Theorem 1.4. *Let* $(X, \le)$ *be a partially ordered set and suppose that there exists a metric* d *in* X *such that* (X, d) *is a complete metric space. Let* $T : X \to X$ *be a increasing operator such that there exists* $x_0 \in X$ *with* $x_0 \le Tx_0$. *Suppose that there exists* $\alpha \in \Im$ *satisfying*

$$d(Tx, Ty) \le \alpha(d(x, y))d(x, y) \quad \text{for } x, y \in X \text{ with } x \ge y.$$

Assume that either T *is continuous or* X *is such that if* $\{x_n\}$ *is an increasing sequence such that* $x_n \to x$ *then* $x_n \le x$ *for all* n.

Besides, if for each $x, y \in X$ *there exists* $z \in X$ *which is comparable to* x *and* y, *then* T *has a unique fixed point.*

The purpose of this paper is to present a version of Theorem 1.4 for mappings of cyclical type.

The existence of fixed point in ordered metric spaces has its starting point in the paper [6] where the authors apply their results to the theory of existence of solutions of matrix equations.

Recently, a lot of papers have appeared in order to study this question (see [1, 6-11], for example).

2. Fixed point theorems: increasing case

We begin this section with the following definition.

Definition 2.1. *Let* $(X, \le)$ *be a partially ordered set and* $T : X \to X$ *an operator. We say that* T *is increasing if for* $x, y \in X$

$$x \le y \Rightarrow Tx \le Ty.$$

One of the main results of this section is the following.

Theorem 2.2. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space.*
Suppose that m is a positive integer, $A_1, A_2, \ldots, A_m$ nonempty closed subsets of X, $X = \bigcup_{i=1}^{m} A_i$, $T : X \to X$ a continuous and increasing mapping satisfying:

(i) *$X = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of X with respect to T,*
(ii) *for any $x \in A_i$ and $y \in A_{i+1}$ $(i = 1, 2, \ldots, m)$, where $A_{m+1} = A_1$ with $x \leq y$*

$$d(Tx, Ty) \leq \alpha(d(x, y))d(x, y),$$

where $\alpha \in \Im$,
(iii) *there exists $x_0 \in X$ with $x_0 \leq Tx_0$.*

Then T has at least a fixed point.

Proof. If $Tx_0 = x_0$ then the proof is finished.
Suppose that $x_0 < Tx_0$. Since T is increasing, we have

$$x_0 < Tx_0 \leq T^2x_0 \leq T^3x_0 \leq \cdots \leq T^nx_0 \leq T^{n+1}x_0 \leq \ldots.$$

Put $x_{n+1} = Tx_n$, for $n = 0, 1, 2, \ldots$.
If $x_n = x_{n+1}$ for some $n \in \mathbb{N}^*$ then $x_{n+1} = Tx_n = x_n$. Thus, x_n is a fixed point of T and the proof is finished.
Suppose that $x_n \neq x_{n+1}$ for any $n \in \mathbb{N}$. By (i), for any $n \geq 1$ there exists $i_n \in \{1, 2, \ldots, m\}$ such that $x_{n-1} \in A_{i_n}$ and $x_n \in A_{i_n+1}$.
Since $x_{n-1} < x_n$ and $\alpha \in \Im$, using (ii) we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq \alpha(d(x_{n-1}, x_n))d(x_{n-1}, x_n) \\ &< d(x_{n-1}, x_n). \end{aligned} \tag{2.1}$$

Consequently, $\{d(x_n, x_{n+1})\}$ is a nonincreasing sequence of nonnegative real numbers, so $\lim_{n\to\infty} d(x_n, x_{n+1}) = r \geq 0$ for certain $r \in [0, \infty)$. suppose $r > 0$.
From (2.1) we obtain

$$\frac{d(x_n, x_{n+1})}{d(x_{n-1}, x_n)} \leq \alpha(d(x_{n-1}, x_n)) \quad \text{for any } n \in \mathbb{N}^*,$$

letting $n \to \infty$ in the last inequality we see that

$$\lim_{n\to\infty} \alpha(d(x_{n-1}, x_n)) = 1.$$

Since $\alpha \in \Im$, we obtain $r = \lim_{n\to\infty} d(x_{n-1}, x_n) = 0$ which is a contradiction. Therefore

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.2}$$

Now, we will prove that $\{x_n\}$ is a Cauchy sequence.
Firstly, we show the following claim.
Claim: For every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that if $p, q \geq n_0$ with $p - q \equiv 1(m)$ then $d(x_p, x_q) < \epsilon$.

In fact, in contrary case, we can find $\epsilon > 0$ such that for any $n \in \mathbb{N}$ there exist $p_n > q_n \geq n$ with $p_n - q_n \equiv 1(m)$ and $d(x_{q_n}, x_{p_n}) \geq \epsilon$.
using the triangle inequality, we have

$$\begin{aligned} \epsilon &\leq d(x_{q_n}, x_{p_n}) \\ &\leq d(x_{q_n}, x_{q_n+1}) + d(x_{q_n+1}, x_{p_n+1}) + d(x_{p_n+1}, x_{p_n}) \\ &= d(x_{q_n}, x_{q_n+1}) + d(Tx_{q_n}, Tx_{p_n}) + d(x_{p_n+1}, x_{p_n}). \end{aligned} \tag{2.3}$$

Since $p_n - q_n \equiv 1(m)$, x_{q_n} and x_{p_n} lie in different adjacently labeled sets A_i and A_{i+1} for certain $1 \leq i \leq m$ and using (ii) in (2.3) we get

$$\begin{aligned} \epsilon &\leq d(x_{q_n}, x_{p_n}) \\ &\leq d(x_{q_n}, x_{q_n+1}) + \alpha(d(x_{q_n}, x_{p_n}))d(x_{q_n}, x_{p_n}) + d(x_{p_n+1}, x_{p_n}). \end{aligned}$$

From the last inequality

$$\begin{aligned} (1 - \alpha(d(x_{q_n}, x_{p_n})))\epsilon &\leq (1 - \alpha(d(x_{q_n}, x_{p_n})))d(x_{q_n}, x_{p_n}) \\ &\leq d(x_{q_n}, x_{q_n+1}) + d(x_{p_n+1}, x_{p_n}). \end{aligned}$$

Letting $n, m \to \infty$ with $p_n - q_n \equiv 1(m)$ and taking into account (2.2) we have

$$\lim_{n\to\infty} 1 - \alpha(d(x_{q_n}, x_{p_n})) = 0.$$

Since $\alpha \in \Im$, this implies $\lim_{n\to\infty} d(x_{q_n}, x_{p_n}) = 0$. Which contradicts the fact that $d(x_{q_n}, x_{p_n}) \geq \epsilon$ for any $n \in \mathbb{N}$.
Therefore, the claim is proved.
In what follows, we prove that $\{x_n\}$ is a Cauchy sequence.
Fix $\epsilon > 0$. Using the claim we can find $n_0 \in \mathbb{N}$ such that if $p, q \geq n_0$ with $p-q \equiv 1(m)$, $d(x_p, x_q) \leq \frac{\epsilon}{2}$.
On the other hand, by (2.2), we find $n_1 \in \mathbb{N}$ such that

$$d(x_n, x_{n+1}) \leq \frac{\epsilon}{2m} \quad \text{for any } n \geq n_1.$$

Now, let $r, s \geq \max(n_0, n_1)$ with $s > r$. Then, there exists $k \in \{1, 2, \dots, m\}$ such that $s - r \equiv k(m)$.
Thus, $s - r + j \equiv 1(m)$ for $j = m - k + 1$, and, so, we have

$$\begin{aligned} d(x_r, x_s) &\leq d(x_r, x_{s+j}) + d(x_{s+j}, x_{s+j-1}) + \cdots + d(x_{s+1}, x_s) \\ &\leq \frac{\epsilon}{2} + j\frac{\epsilon}{2m} \\ &\leq \frac{\epsilon}{2} + m\frac{\epsilon}{2m} = \epsilon. \end{aligned}$$

This proves that $\{x_n\}$ is a Cauchy sequence. Since X is a complete metric space, $\lim_{n\to\infty} x_n = x$ for certain $x \in X$.
Finally, the continuity of T gives that

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = T(\lim_{n\to\infty} x_n) = Tx.$$

This finishes the proof. □

Remark 2.3. *Notice that, as* $X = \bigcup_{i=1}^{m} A_i$ *is a cyclic representation of* X *with respect to* T*, the fixed point* x *appearing in Theorem 2.2 satisfies* $x \in \bigcap_{i=1}^{m} A_i$*.*

Now, we will prove that the conclusion of Theorem 2.2 is true for T not necessarily continuous. The condition of continuity of T is replaced by:

If $\{x_n\}$ is an increasing sequence in X such that $x_n \to x$ then

$$x_n \leq x \text{ for all } n \in \mathbb{N}. \tag{2.4}$$

Theorem 2.4. *If in Theorem 2.2 we replace the continuity of* T *by condition (2.4) we obtain the same conclusion.*

Proof. Following the lines of the proof of Theorem 2.2 we only have to check that x is a fixed point.

In fact, since $\lim_{n\to\infty} x_n = x$ and $X = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of X with respect to T, the sequence $\{x_n\}$ has an infinite number of terms in each A_i ($i = 1, 2, \ldots, m$). We take $\{x_{n_k}\}$ subsequence of $\{x_n\}$ with $x_{n_k} \in A_i$ for $i \in \{1, 2, \ldots, m\}$ fixed.

Since $x \in \bigcap_{i=1}^{m} A_i$ (see Remark 2.3), Using the triangle inequality and (ii) of Theorem 2.2, we get

$$\begin{aligned} d(x, Tx) &\leq d(x, x_{n_k+1}) + d(x_{n_k+1}, Tx) \\ &= d(x, x_{n_k+1}) + d(Tx_{n_k}, Tx) \\ &\leq d(x, x_{n_k+1}) + \alpha(d(x_{n_k}, x))d(x_{n_k}, x) \\ &< d(x, x_{n_k+1}) + d(x_{n_k}, x). \end{aligned}$$

letting $k \to \infty$ in the last inequality and since $\lim_{n\to\infty} x_n = x$, we have

$$d(x, Tx) \leq 0,$$

or, equivalently, $x = Tx$. This finished the proof. □

In the sequel, we present an example where it can be appreciate that assumptions in Theorem 2.2 and 2.4 do not guarantee the uniqueness of the fixed point.

Example 2.5. *Consider* $(\mathbb{R}^2, d_2)$*, where* d_2 *denotes the euclidean distance and the order given by* $R = \{(x, x) : x \in \mathbb{R}^2\}$*. Let* $A_i (i = 1, 2)$ *be the closed subsets of* $\mathbb{R}^2$ *given by*

$$A_1 = \{(x, y) : y \geq 0\} \text{ and } A_2 = \{(x, y) : y \leq 0\}.$$

Obviously, $\mathbb{R}^2 = A_1 \cup A_2$*.*

Now, we consider the operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ *given by* $T(x, y) = (x, -y)$*. It is easily seen that* $\mathbb{R}^2 = A_1 \cup A_2$ *is a cyclic representation of* $\mathbb{R}^2$ *with respect to* T*. Obviously,* T *is continuous and* $(0, 0) \leq T(0, 0) = (0, 0)$*. Since elements in* X *are only comparable to themselves,* T *is increasing and satisfies condition (ii) of Theorem 2.2, for any* $\alpha \in \Im$*.*

Therefore, this example satisfies assumptions of Theorem 2.2 and the operator T has as fixed points the set $\{(x, 0) : x \in \mathbb{R}\}$.

In the sequel, we will prove that the following assumption:

$$\text{For } x, y \in \bigcap_{i=1}^{m} A_i \text{ there exists } z \in X \text{ which is comparable to } x \text{ and } y, \tag{2.5}$$

is a sufficient condition for uniqueness of the fixed point in Theorem 2.2 and 2.4.

Theorem 2.6. *Adding condition (2.5) to the assumptions of Theorems 2.2 and 2.4 we obtain the uniqueness of the fixed point.*

Proof. Suppose that $x, y \in X$ are fixed points of T with $x \neq y$. By Remark 2.3, $x, y \in \bigcap_{i=1}^{m} A_i$. We consider two cases.

Case 1. x and y are comparable.
In this case, Using (ii) of Theorem 2.2 and taking into account that $\alpha \in \Im$, we have

$$d(x, y) = d(Tx, Ty) \leq \alpha(d(x, y))d(x, y) < d(x, y),$$

which is a contradiction. Therefore, $x = y$.

Case 2. x and y are not comparable.
Since $x, y \in \bigcap_{i=1}^{m} A_i$, by condition (2.5) we find $z \in X$ with z is comparable to x and y. Since T is increasing, $x = T^n x$ and $T^n z$ are comparable for $n = 0, 1, 2, \ldots$, and, taking into account that $x \in \bigcap_{i=1}^{m} A_i$ by (ii) of Theorem 2.2, we can obtain

$$\begin{aligned} d(x, T^n z) &= d(T^n x, T^n z) \\ &\leq \alpha(d(T^{n-1}x, T^{n-1}z))d(T^{n-1}x, T^{n-1}z) \\ &< d(T^{n-1}x, T^{n-1}z) = d(x, T^{n-1}z). \end{aligned}$$

This proves that $\{d(x, T^n z)\}$ is a decreasing sequence of nonnegative real numbers and, so, $\lim_{n\to\infty} d(x, T^n z) = r \geq 0$, for certain $r \in \mathbb{R}^+$. Using a similar argument that in Theorem 2.2 it can be proved that $r = 0$. Therefore, $\lim_{n\to\infty} T^n z = x$.
Similarly, we can prove that $\lim_{n\to\infty} T^n z = y$ and the uniqueness of the limit gives us $x = y$.

This finishes the proof. □

In what follows, we present an example which says us the condition (2.5) is not a necessary for the uniqueness of the fixed point.

Example 2.7. *Consider the same metric space that in Example 2.5 and with the same order. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ the operator given by $T(x, y) = (-x, -y)$. It is easily checked that $\mathbb{R}^2 = A_1 \cup A_2$ is a cyclic representation of $\mathbb{R}^2$ with respect to T, where the sets A_i are defined as in Example 2.5. It is easily checked that this example satisfies assumptions of Theorem 2.2. In this case, the operator T has a unique fixed point which is $(0, 0)$.*
On the other hand, since $A_1 \cap A_2 = \{(x, 0) : x \in \mathbb{R}\}$ and the order is given by $R = \{(x, x) : x \in \mathbb{R}^2\}$ we see that condition (2.5) is not satisfied.

3. Fixed point theorems: nonincreasing case

The starting point of this section is the following definition.

Definition 3.1. *Let $(X, \leq)$ be a partially ordered set and $T : X \to X$ an operator. We say that T is nonincreasing if for $x, y \in X$*

$$x \leq y \Rightarrow Tx \geq Ty.$$

The main result of this section is the following.

Theorem 3.2. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space.*
Suppose that m is a positive integer, $A_1, A_2, \ldots, A_m$ nonempty subsets of X,

$$X = \bigcup_{i=1}^{m} A_i,$$

$T : X \to X$ a nonincreasing mapping such that:

(i) *$X = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of X with respect to T,*
(ii) *for any $x \in A_i$ and $y \in A_{i+1}$ with x and y comparable $(i = 1, 2, \ldots, m)$,*

$$d(Tx, Ty) \leq \alpha(d(x, y))d(x, y),$$

where $A_{m+1} = A_1$ and $\alpha \in \Im$,
(iii) *there exists $x_0 \in X$ with x_0 and Tx_0 comparable.*

Then

(a) $\inf\{d(x, Tx) : x \in X\} = 0$.
(b) *If, in addition, X is compact and T is continuous, then T has at least a fixed point. Moreover, if $(X, \leq)$ satisfies condition (2.5) then the fixed point is unique.*

Proof. (a) If $Tx_0 = x_0$ then it is obvious that $\inf\{d(x, Tx) : x \in X\} = 0$. Suppose that $x_0 < Tx_0$ (the same argument works for $x_0 > Tx_0$). Since T is nonincreasing the consecutive terms of the sequence $\{T^n x_0\}$ are comparable and by (ii) we have

$$\begin{aligned} d(T^{n+1}x_0, T^n x_0) &\leq \alpha(d(T^n x_0, T^{n-1}x_0))d(T^n x_0, T^{n-1}x_0) \\ &< d(T^n x_0, T^{n-1}x_0). \end{aligned}$$

This proves that $\{d(T^{n+1}x_0, T^n x_0)\}$ is decreasing and nonnegative, so

$$\lim_{n\to\infty} d(T^{n+1}x_0, T^n x_0) = r$$

for certain $r \in [0, \infty)$.

Using a similar argument that in Theorem 2.2, we prove that

$$\lim_{n\to\infty} d(T^{n+1}x_0, T^n x_0) = 0.$$

This implies that $\inf\{d(x, Tx) : x \in X\} = 0$.

This finishes the proof of (a).

(b) Since X is compact and T is continuous, the mapping $\varphi : X \to \mathbb{R}^+$ given by $\varphi(x) = d(x, Tx)$ is continuous, and since X is compact it attains its minimum. Therefore, we find $z \in X$ such that

$$d(z, Tz) = \inf\{d(x, Tx) : x \in X\}.$$

Now, the part (a) says us that $d(z, Tz) = 0$.
Consequently, z is a fixed point of T.

The uniqueness of the fixed point is proved as in Theorem 2.6. □

Remark 3.3. *Notice that in Theorem 3.2, we do not impose that the subsets A_i $(i = 1, \dots, m)$ are closed.*

4. Examples and Remarks

In what follows we present some examples which prove that if at least one of the assumptions of Theorem 2.2 is not satisfied then the conclusion is false.

Example 4.1. *We consider $\mathbb{N}^*$ (the natural numbers without zero) with the usual distance given by $d(x, y) = |x - y|$ and with the usual order. Obviously, $(\mathbb{N}^*, d)$ is a complete metric space since $\mathbb{N}^*$ is closed subsets of $(\mathbb{R}, d)$.*
Let A_i $(i = 1, 2)$ be the closed subsets of $\mathbb{N}^*$ given by

$$A_1 = \{n \in \mathbb{N}^* : n \text{ even}\},$$

$$A_2 = \{n \in \mathbb{N}^* : n \text{ odd}\}.$$

Consider the operator $T : \mathbb{N}^* \to \mathbb{N}^*$ defined as $T(n) = n + 1$.
Obviously, T is continuous and increasing.
It is easily seen that $\mathbb{N}^* = A_1 \cup A_2$ is a cyclic representation of $\mathbb{N}^*$ with respect to T.
Moreover, for every $n \in \mathbb{N}^*$, $n \le T(n) = n + 1$.
On the other hand, for $p \in A_1$ and $q \in A_2$ with $p < q$ we have for any $\alpha \in \Im$,

$$d(T(p), T(q)) = |p - q|,$$

$$\alpha(d(p, q))d(p, q) = \alpha(d(p, q))|p - q| < |p - q|$$

Consequently, condition (ii) of Theorem 2.2 is not satisfied.
Finally, it is easily seen that T has not fixed point.

This proves that if assumption (ii) of Theorem 2.2 is not satisfied the conclusion of this theorem can be false.

Example 4.2. *Consider the same set* $\mathbb{N}^*$ *that in Example 2.5 with the same distance, and with the order given by*

$$R = \{(n, n) : n \in \mathbb{N}^*\}.$$

We consider the same sets A_i $(i = 1, 2)$ *that in Example 2.5 and the same operator* T.
In this case, assumption (ii) of Theorem 2.2 is satisfied since that elements in $\mathbb{N}^*$ *are only comparable to themselves.*
On the other hand, in this case assumption (iii) of Theorem 2.2 is not satisfied and the operator T *has not fixed point.*

Now, we present an example which can be studied by Theorem 2.2 and it cannot be treated by Theorem 1.4.

Example 4.3. *Consider* $X = \{(0, 2), (0, 0), (1, 0), (1, 1)\} \subset \mathbb{R}^2$ *with the Euclidean distance* d_2 *and the order given by*

$$R = \{(x, x) : x \in X\} \cup \{((1, 0), (1, 1)), ((0, 0), (0, 2))\},$$

obviously, (X, d_2) *is a complete metric space. Let* A_i $(i = 1, 2)$ *be the closed subsets of* X *given by*

$$A_1 = \{(1, 0), (1, 1), (0, 0)\},$$

$$A_2 = \{(0, 0), (0, 2)\}$$

and $T : X \to X$ *the operator defined by*

$$T(1, 0) = (0, 0); T(1, 1) = (0, 2); T(0, 2) = (0, 0) = (0, 0).$$

Obviously, T *is an increasing and continuous and* $X = A_1 \cup A_2$ *is a cyclic representation of* X *with respect to* T. *Moreover, in this case the elements* $x, y \in X$ *satisfying* $x \in A_i$ *and* $y \in A_{i+1}$ *and* x *and* y *are comparable are* $x = y = (0, 0)$ *and, therefore, assumption (ii) of Theorem 2.2 is satisfied.*
Moreover, since $(0, 0) \leq T(0, 0) = (0, 0)$, *Theorem 2.2 says us that the operator* T *has a fixed point (which is* $(0, 0)$*).*
On the other hand, since $(1, 0) \leq (1, 1)$ *and*

$$d(T(1, 0), T(1, 1)) = d((0, 0), (0, 2)) = 2,$$

and

$$\alpha(d((1, 0), (1, 1)))d((1, 0), (1, 1) = \alpha(d((1, 0), (1, 1)) < 1,$$

for any $\alpha \in \Im$.

We see that the contractive condition of Theorem 1.4 is not satisfied and, consequently, this example cannot be treated by Theorem 1.4.

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