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A MODIFIED HOMOTOPY METHOD FOR SOLVING NONCONVEX FIXED POINTS PROBLEMS

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Abstract. In the paper, to compute the fixed point of a self-mapping in a class of nonconvex unbounded sets with both inequality and equality constraints, a modified homotopy is constructed and the existence and convergence of the smooth homotopy pathways is proved under mild conditions. Compared with the previous results, the modified combined homotopy method needs only weaker conditions. Some numerical examples are given to show the feasibility and effectiveness of the proposed method.

Key Words and Phrases: Combined homotopy method, nonconvex sets, fixed point, pseudo cone condition.
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1. Introduction

In 1976, the constructive proof of homotopy for computing the Brouwer fixed points of a continuous differentiable self-mapping in a bounded closed convex set was given by Kellogg, Li and Yorke [1]. In 1978, a more convenient homotopy for computing Brouwer fixed points in a bounded convex set was constructed by Chow, Mallet-Paret and Yorke [2] as follows:

$$(1-t)(x-F(x))+t(x-x_0)=0. \tag{1.1}$$

In 1996, to relax the convex condition, a combined homotopy for computing Brouwer fixed points in nonconvex bounded sets $\Omega=\{x : g_i(x)\le 0, i=1,2,\dots,m\}$ was constructed by Yu and Lin [3] as follows:

$$H(w,t)=\left(\begin{array}{c}(1-t)(x-F(x)+\sum\limits_{i=1}^{m}\nabla g_i(x)y_i)+t(x-x^{(0)})\\ Yg(x)-tY^{(0)}g(x^{(0)})\end{array}\right) \tag{1.2}$$

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where $(x^{(0)}, y^{(0)}) \in \Omega^0 \times R^m_{++}$, $y_i \geq 0$, $t \in [0,1]$, Y and Y^0 denote the diagonal matrices whose ith diagonal element are y_i and y_i^0 respectively, and the strict feasible set $\Omega^0 = \{x : g_i(x) < 0, i = 1,2,\ldots,m\}$. Existence and convergence of a smooth homotopy pathway were proven under the nonemptiness and boundedness of Ω^0, full column rank of the matrix $\{\nabla g_i(x), i \in I(x)\}$ for any $x \in \partial\Omega$, where $I(x) = \{i \in \{1,2,\ldots,m\} : g_i(x) = 0\}$, and the so called normal cone condition (NCC): for any $x \in \partial\Omega = \Omega \setminus \Omega^0$, any normal line of Ω at any $x \in \partial\Omega$ only meets Ω at x, i.e.,

$$\{x + \sum_{i \in I(x)} y_i \nabla g_i(x) : y_i \geq 0, i \in I(x)\} \cap \Omega = \{x\}.$$

In 2003, a modified combined homotopy for computing Brouwer fixed points in nonconvex bounded sets $\Omega = \{x : g_i(x) \leq 0, i = 1,2,\ldots,m\}$ was constructed by Lin, Yu and Zhu [4] as follows:

$$H(w,t) = \begin{pmatrix} (1-t)(x - F(x) + \sum\limits_{i=1}^{m} \xi_i(x) y_i) + t(x - x^{(0)}) \\ Yg(x) - tY^{(0)} g(x^{(0)}) \end{pmatrix} \tag{1.3}$$

where $\xi_i(x) \in R^n, i = 1,2,\ldots,m$ is a system of C^2 mappings. Existence and convergence of a smooth homotopy pathway were proven under the nonemptiness and boundedness of Ω^0, positive linear independence of $\xi_i(x)$, $i = 1,2,\ldots,m$, and the so called quasi normal cone condition (QNCC) which is weaker than NCC: $\forall x \in \partial\Omega$,

$$\{x + \sum_{i \in I(x)} y_i \xi_i(x) : y_i \geq 0,\ i \in I(x), \text{and}, \sum_{i \in I(x)} y_i > 0\} \cap \Omega = \{x\}.$$

In 2008, the combined homotopy method for computing Brouwer fixed point in nonconvex set was generalized to the general nonconvex bounded sets

$$\Omega = \{x : g_i(x) \leq 0, i = 1,2,\ldots,m, h_j(x) = 0, j = 1,2,\ldots,m\}$$

by Su and Liu [5] as follows:

$$H(w,t) = \begin{pmatrix} (1-t)(x - F(x) + \sum\limits_{i=1}^{m} \nabla g_i(x) y_i) + \sum\limits_{j=1}^{l} \nabla h_j(x) z_j + t(x - x^{(0)}) \\ h(x) \\ Yg(x) - tY^{(0)} g(x^{(0)}) \end{pmatrix} \tag{1.4}$$

where $(x^{(0)}, y^{(0)}, z^{(0)}) \in \Omega^0 \times R^m_{++}$, $y_i \geq 0$, $z_j \in R$, $t \in [0,1]$, and the strict feasible set $\Omega^0 = \{x : g_i(x) < 0, i = 1,2,\ldots,m, h_j(x) = 0, j = 1,2,\ldots,l\}$. Existence and convergence of a smooth homotopy pathway were proven under the nonemptiness and boundedness of Ω^0, full column rank of the matrix $\{\nabla h(x), \nabla g_i(x) : i \in I(x)\}$, where $I(x) = \{i \in \{1,2,\ldots,m\} : g_i(x) = 0\}$, and the following NCC: $\forall x \in \Omega$,

$$\{x + \sum_{i \in I(x)} y_i \nabla g_i(x) + \sum_{j=1}^{l} \nabla h_j(x) z_j : y_i \geq 0, i \in I(x), z_j \in R\} \cap \Omega = \{x\}.$$

Since the combined homotopy method has globally convergent property and can be used to solve the nonconvex problems, it has been extensively applied to solve nonlinear programming, complementarity problems, variational inequality problems and so on, see, e.g., [6-10]. In 2000, for solving nonconvex nonlinear programming

$\min\limits_{x\in\Omega} F(x)$, where $\Omega = \{x : g_i(x) \le 0, i = 1, 2, \dots, m\}$, a modified homotopy was constructed by Yu, Liu and Feng [11] as follows:

$$H(x,t)=\begin{pmatrix}(1-t)(\nabla F(x)+\sum\limits_{i=1}^{m} y_i\nabla g_i(x))+\sum\limits_{i=1}^{m}\eta_i(x,t(1-t)y_i^2)+t(x-x^{(0)})\\ Yg(x)-tY^{(0)}g(x^{(0)})\end{pmatrix} \quad (1.5)$$

and existence and convergence of a smooth homotopy pathway were proven under mild conditions. In [11], so called pseudo cone condition (PCC), which is much weaker than QNCC, was firstly proposed by Yu, Liu and Feng as follows: for $i = 1, 2, \dots, m$, a mapping $\eta_i(x, y_i) : R^n \times R \to R^n$ is said to be a hair mapping of $\Omega_i = \{x : g_i(x) \le 0\}$ if

i) $\eta_i(x, 0) = 0$ and for any $x \in \partial\Omega_i = \partial\Omega \cap \Omega_i, \eta_i(x, y_i) = 0 \Leftrightarrow y_i = 0$;

ii) For any $x \in \partial\Omega_i$, $\lim\limits_{y_i\to\infty} \|\eta_i(x, y_i)\| = \infty$.

A set of mappings $\eta(x, y) = (\eta_1(x, y_1), \eta_2(x, y_2), \dots, \eta_m(x, y_m))$ are said to be consistent hair mappings of Ω, if, for every $x \in \partial\Omega$,

i) $\eta_i(x, y_i)$ is a hair mapping of Ω_i, $i = 1, 2, \dots, m$.

ii) $\sum\limits_{i\in I(x)} (\hat{y}_i\nabla g_i(x)) + \eta_i(x, y_i)) = 0$, $\hat{y}_i \ge 0$, $y_i \ge 0$ imply $\hat{y}_i = 0, y_i = 0$, $i \in I(x)$, where $I(x) = \{i \in \{1, 2, \dots, m\} : g_i(x) = 0\}$.

iii) $\|(\hat{y}, y)\| \to \infty$ implies $\|\sum\limits_{i\in I(x)} \hat{y}_i\nabla g_i(x) + \eta_i(x, y_i)\| \to \infty$.

If there exists a set of twice continuous differentiable hair mappings

$$\eta(x, y) = (\eta_1(x, y_1), \eta_2(x, y_2), \dots, \eta_m(x, y_m)),$$

which are consistent hair mappings of Ω, such that for any $x \in \partial\Omega$,

$$\{x + \sum_{i\in I(x)} \eta_i(x, y_i) : y_i \ge 0, i \in I(x)\} \cap \Omega = \{x\},$$

then Ω is said to satisfy the pseudo cone condition (PCC).

To better illustrate the relationship among NCC, QNCC and PCC, some sets are given in the following Figure 1.

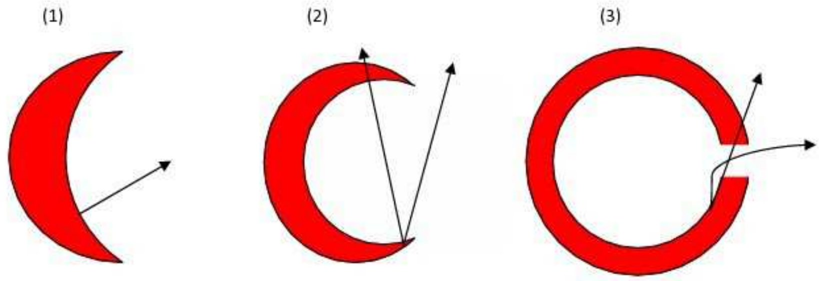

FIGURE 1. the nonconvex sets satisfying NCC, QNCC or PCC

In the Figure 1, the set (1) satisfies NCC, QNCC and PCC; The set (2) satisfies both QNCC and PCC, but it doesn't satisfy NCC; The set (3) satisfies only PCC, and it doesn't satisfy both NCC and QNCC.

Therefore, if a set Ω satisfies NCC $\Rightarrow$ the set Ω satisfies QNCC $\Rightarrow$ the set Ω satisfies PCC. Conversely, the relationship doesn't hold.

Inspired by the above literatures, in this paper, a modified combined homotopy for computing Brouwer fixed points of self-mappings in the general nonconvex unbounded sets with both inequality and equality constraints is constructed, and the existence and global convergence of a smooth homotopy pathway from any given interior initial point to a fixed point of a twice continuous differentiable self-mapping is proven under mild conditions.

In section 2, a modified homotopy for computing Brouwer fixed points of self-mappings is constructed and the existence and global convergence of a smooth homotopy pathway is proved. In section 3, some numerical examples are given to show the feasibility and effectiveness of the method.

2. Main results

In the paper, considering the following general nonconvex set:

$$\Omega = \{x \in R^n : g_i(x) \le 0, i = 1, \dots, m, h_j(x) = 0, j = 1, \dots, l\}, \tag{2.1}$$

where $g_i(x), h_j(x) : R^n \to R$ are all C^2 functions. Let $\Omega^0 = \{x \in R^n : g_i(x) < 0, i = 1, 2, \dots, m, h_j(x) = 0, j = 1, 2, \dots, l\}$ and $\partial\Omega = \Omega \setminus \Omega^0$ be the interior and the boundary of Ω, respectively. Let $\Omega_i = \{x \in R^n : g_i(x) \le 0\}$, $\widehat{\Omega}_j = \{x \in R^n : h_j(x) = 0\}$, $I(x) = \{i \in \{1, 2, \dots, m\} : g_i(x) = 0\}$.

The following assumptions will be used.

Assumption 2.1 Ω^0 is nonempty.

Assumption 2.2 For any $x \in \Omega$, the matrix $\nabla h(x)$ is of full column rank.

Assumption 2.3 (The pseudo cone condition on the set Ω) There exist mappings $\eta_i(x, y_i) : R^n \times R \to R^n$, $i = 1, 2, \dots, m$ and $\zeta_j(x, z_j) : R^n \times R \to R^n$, $j = 1, 2, \dots, l$, called hair mappings (see Definition 1, [11]) of Ω_i and $\widehat{\Omega}_j$ respectively, such that

i) $\eta_i(x, y_i) = 0$ iff $y_i = 0$ $(1 \le i \le m)$, $\zeta_j(x, z_j) = 0$ iff $z_j = 0$ $(1 \le j \le l)$.

ii) $||\eta_i(x, y_i)|| \to \infty$ when $y_i \to +\infty$ $(1 \le i \le m)$, $||\zeta_j(x, z_j)|| \to \infty$ when $|z_j| \to +\infty$ $(1 \le j \le l)$.

iii) (Positive independence) $\forall x \in \partial\Omega$, $\sum\limits_{i \in I(x)} \eta_i(x, y_i) + \sum\limits_{j=1}^{l} \zeta_j(x, z_j) = 0$, $0 \le y_i, z_j \in R$ implies $y_i = 0$, $i \in I(x)$ and $z_j = 0$, $j = 1, 2, \dots, l$.

iv) For any $x \in \Omega$, the matrix $\nabla_z(\sum\limits_{j=1}^{l} \zeta_j(x, z_j))$ is of full row rank.

v) $\forall x \in \Omega$,

$$\{x + \sum_{i \in I(x)} \eta_i(x, y_i) + \sum_{j=1}^{l} \zeta_j(x, z_j) : y_i \ge 0, i \in I(x), z_j \in R\} \cap \Omega = \{x\}.$$

Assumption 2.4 $\forall \{x^k\} \subset \Omega$, $\|x^k\| \to \infty$ (as $k \to \infty$), there exists a $\theta^0 \in \Omega$, such that $\lim\limits_{k\to\infty} (\theta^0 - x^k)^T [\sum\limits_{i=1}^{m} \eta_i(x^k, y_i^k) + \sum\limits_{j=1}^{l} \zeta_j(x^k, z_j^k)] < 0$ and $\lim\limits_{k\to\infty} (\theta^0 - x^k)^T (x^k - F(x^k)) < 0$.

Remark 2.1 *In comparison with the pseudo cone condition given by Yu, Liu and Feng in [11], the requirement that the hair mappings are consistent with $\nabla g(x)$ (and $\nabla h(x)$) is removed in Assumption 2.3.*

The following lemma gives an equivalent condition of the existence of a fixed point.

Lemma 2.1 *Suppose Ω is defined as (2.1), the hair mappings $\eta_i(x, y_i) : R^{n+1} \to R^n, i = 1, 2, \ldots, m$ of Ω_i and $\zeta_j(x, z_j) : R^{n+1} \to R^n, j = 1, 2, \ldots, l$ of $\widehat{\Omega}_j$ are twice continuous differentiable. If Assumptions 2.1-2.4 hold, then for any twice continuous differential mapping $F : \Omega \to \Omega$, $x \in \Omega$ is a fixed point of $F(x)$ in Ω iff there exists a $(y, z) \in R^m_+ \times R^l$, such that (x, y, z) is a solution of*

$$\begin{gathered} x - F(x) + \sum_{i=1}^{m} \eta_i(x, y_i) + \sum_{j=1}^{l} \zeta_j(x, z_j) = 0, \\ h(x) = 0 \\ Y g(x) = 0, \ g(x) \le 0, \ y \ge 0. \end{gathered} \tag{2.2}$$

Proof. If (x^*, y^*, z^*) is a solution of (2.2), then:

(1) when $x^* \in \Omega^0$, we have $g(x^*) < 0$, $h(x^*) = 0$, then we obtain $y^* = 0$ by the third equation of (2.2). From the first equation of (2.2), we have

$$x^* + \sum_{j=1}^{l} \zeta_j(x^*, z_j^*) = F(x^*).$$

By Assumption 2.3 (v) and $F(x^*) \subset \Omega$, we get $\sum\limits_{j=1}^{l} \zeta_j(x^*, z_j^*) = 0$. From Assumption 2.3 (iii), we get $z_j^* = 0$, $j = 1, 2, \ldots, l$, and thus $F(x^*) = x^*$.

(2) when $x^* \in \partial\Omega$, from the third equation of (2.2), we have $g_i(x^*) < 0$ and $y^* = 0$ for $i \notin I(x^*)$, and $g_i(x^*) = 0$ for $i \in I(x^*)$, the first equation of (2.2) becomes

$$x^* + \sum_{i \in I(x^*)} \eta_i(x^*, y_i^*) + \sum_{j=1}^{l} \zeta_j(x^*, z_j^*) = F(x^*).$$

From Assumption 2.3 (v) and $F(x^*) \subset \Omega$, we have

$$\sum_{i \in I(x^*)} \eta_i(x^*, y_i^*) + \sum_{j=1}^{l} \zeta_j(x^*, z_j^*) = 0,$$

and by Assumption 2.3 (iii), we have $z_j^* = 0$, $j = 1, 2, \ldots, l$, and $y_i^* = 0$ for all $i \in I(x^*)$, and thus $x^* = F(x^*)$.

On the other hand, if x^* is a fixed point of $F(x^*)$ in Ω, then for $y^* = 0$ and $z^* = 0$, $(x^*, 0, 0)$ is a solution of (2.2).

The proof is complete. □

To solve the system (2.2), i.e., to compute a fixed point of a twice continuous differentiable self-mapping, the modified homotopy is constructed as follows:

$$H(w,t)=\begin{pmatrix}(1-t)(x-F(x)+\sum\limits_{i=1}^{m}\eta_i(x,y_i))+\sum\limits_{j=1}^{l}\zeta_j(x,z_j)+t(x-x^{(0)})\\ h(x)\\ Yg(x)-tY^{(0)}g(x^{(0)})\end{pmatrix} \tag{2.3}$$

where $w=(x,y,z)$, $w^{(0)}=(x^{(0)},y^{(0)},z^{(0)})$, $y=(y_1,\dots,y_m)^T$, $z=(z_1,\dots,z_l)$, $Y=\mathrm{diag}(y)$, $\eta_i(x,y_i)$ and $\zeta_j(x,z_j)$ are hair mappings.

Remark 2.2 *In comparison with the modified homotopy for solving nonlinear programming in [11], the gradient of inequality constraint $g_i(x), i=1,2,\dots,m$ isn't required in the homotopy (2.3). And the hair mappings $\eta_i(x,y_i)$ and $\zeta_j(x,z_j)$ only need the terms y_i and z_j not y_i^2 or z_j^2 in the modified homotopy (2.3).*

When $t=1$, for any $x^{(0)}\in\Omega^0$, the homotopy equation $H(w,1)=0$ turns into

$$\begin{pmatrix}\sum\limits_{j=1}^{l}\zeta_j(x,z_j)+x-x^{(0)}\\ h(x)\\ Yg(x)-Y^{(0)}g(x^{(0)})\end{pmatrix}=0 \tag{2.4}$$

By the Assumption 2.3 (iii) and (v), it follows from the first equation in (2.4) that $x=x^{(0)}$ and $z_j=0, j=1,2,\dots,l$. Then, from the third equation in (2.4), we have $y=y^{(0)}$.

Therefore, when $t=1$, the homotopy equation $H(w,1)=0$ has only one solution $(x^{(0)},y^{(0)},z^{(0)})=(x^{(0)},y^{(0)},0)$.

When $t=0$, the homotopy equation $H(w,0)=0$ turns into the system (2.2). Thus, we can obtain a fixed point of the twice continuous differentiable self-mapping $F(x)$.

For a given $w^{(0)}\in\Omega^0\times R^m_{++}\times R^l$, the zero-point set of $H(w,t)$ is defined as follows:

$$H^{-1}_{w^{(0)}}=\{(w,t)\in\Omega^0\times R^m_{++}\times R^l\times(0,1]:H(w,t)=0\}.$$

Theorem 2.1 *Suppose Ω is defined as (2.1), the hair mappings $\eta_i(x,y_i):R^{n+1}\to R^n, i=1,2,\dots,m$ of Ω_i and $\zeta_j(x,z_j):R^{n+1}\to R^n, j=1,2,\dots,l$ of $\widehat{\Omega}_j$ are twice continuous differentiable. If Assumptions 2.1-2.4 hold, for any twice continuous differentiable mapping $F:\Omega\to\Omega$, then*

1) $F(x)$ has a fixed point in Ω;

2) For almost all $(x^{(0)},y^{(0)},z^{(0)})\in\Omega^{(0)}\times R^m_{++}\times R^l$, the modified homotopy equation (2.3) determines a smooth curve $\Gamma_{w^{(0)}}\subset\Omega^0\times R^m_{++}\times R^l\times(0,1]$ starting from $(w^{(0)},1)$. When $t\to 0$, the limit set $T\subset\Omega\times R^m_+\times R^l\times 0$ of $\Gamma_{w^{(0)}}$ is nonempty, and the x-component of any point in T is a fixed point of $F(\Omega)$ in Ω.

Proof. Let $\tilde H(w,x^{(0)},y^{(0)},t)$ be the same mapping with $H(w,t)$ but taking $(x^{(0)},y^{(0)})$ as variate. Considering the following submatrix of the Jacobian $D\tilde H(w,x^{(0)},y^{(0)},t)$

of $\tilde{H}(w, x^{(0)}, y^{(0)}, t)$. For any $w^{(0)} \in \Omega^0 \times R^m_{++} \times R^l$ and $t \in (0, 1]$,

$$\frac{\partial \tilde{H}(w, x^{(0)}, y^{(0)}, t)}{\partial(x, x^{(0)}, y^{(0)})} = \begin{pmatrix} * & -tI & 0 \\ \nabla h(x)^T & 0 & 0 \\ Y\nabla g(x)^T & -tY^0 \nabla g(x^{(0)})^T & -tG(x^{(0)}) \end{pmatrix}$$

where I is an identity matrix and $G(x^{(0)}) = \mathrm{diag}(g_1(x^{(0)}), \ldots, g_m(x^{(0)}))$. Since $g_i(x^{(0)}) < 0$ and $\nabla h(x)^T$ is a matrix of full row rank, we have that $\frac{\partial \tilde{H}(w, x^{(0)}, y^{(0)}, t)}{\partial(x, x^{(0)}, y^{(0)})}$ is a matrix of full row rank. Thus, $D\tilde{H}(w, x^{(0)}, y^{(0)}, t)$ is a matrix of full row rank. That is, 0 is a regular value of $\tilde{H}(w, x^{(0)}, y^{(0)}, t)$. By Parameterized Sard's Theorem (Theorem 2.1., [2]), for almost all $w^{(0)} \in \Omega^0 \times R^m_{++} \times R^l$, 0 is a regular value of $H(w, t)$.

For any given $w^{(0)} \in \Omega^0 \times R^m_{++} \times R^l$, by Assumption 2.2 and Assumption 2.3 (iv), the matrix

$$\frac{\partial H(w^{(0)}, 1)}{\partial w} = \begin{pmatrix} * & 0 & \nabla_z(\sum\limits_{j=1}^{l} \zeta_j(x^{(0)}, z_j^{(0)})) \\ \nabla h(x^{(0)})^T & 0 & 0 \\ Y^0 \nabla g(x^{(0)})^T & G(x^{(0)}) & 0 \end{pmatrix}$$

is nonsingular. If 0 is a regular value of $H(w, t)$, from the fact $H(w^{(0)}, 1) = 0$ and the implicit function theorem, $H^{-1}_{w^{(0)}}$ contains a smooth curve $\Gamma_{w^{(0)}}$ which starts from $(x^{(0)}, y^{(0)}, z^{(0)}, 1)$, goes into $\Omega^0 \times R^m_{++} \times R^l \times (0, 1]$ and terminates in or approaches to the boundary of $\Omega \times R^m_{++} \times R^l \times [0, 1]$.

Let (w^*, t_*) be an ending limit point of $\Gamma_{w^{(0)}}$ as $t \to 0$, only the following five cases are possible:

i) $(w^*, t_*) \in \Omega^0 \times R^m_{++} \times R^l \times \{1\}$, $\|(y^*, z^*)\| < \infty$;

ii) $(w^*, t_*) \in \Omega \times R^m_{++} \times R^l \times [0, 1]$, $\|(y^*, z^*)\| = \infty$;

iii) $(w^*, t_*) \in \partial\Omega \times R^m_{++} \times R^l \times (0, 1)$, $\|(y^*, z^*)\| < \infty$;

iv) $(w^*, t_*) \in \Omega \times \partial R^m_{+} \times R^l \times (0, 1)$, $\|(y^*, z^*)\| < \infty$;

v) $(w^*, t_*) \in \Omega \times R^m_{++} \times R^l \times \{0\}$, $\|(y^*, z^*)\| < \infty$.

Since the equation $H(w^{(0)}, 1) = 0$ has only one solution $w^{(0)}$ in $\Omega^0 \times R^m_{++} \times R^l$, case (I) is impossible.

From the continuity of $\Gamma_{w^{(0)}}$ and the third equality of homotopy equation (2.3), cases (III) and (IV) are impossible.

If cases (II) happens, there must exists a sequence of $\{(w^k, t_k)\} \subset \Gamma_{w^{(0)}}$ such that $\|(w^k, t_k)\| \to \infty$. In the following, we will show that the sequence $\{x^k\}$ is bounded. If $\{x^k\}$ is unbounded, i.e., $||x^k|| \to \infty$ as $k \to \infty$, by the Assumption 2.4, there exists a $\theta^0 \in \Omega$ such that $\lim\limits_{k\to\infty} (\theta^0 - x^k)^T (x^k - F(x^k)) < 0$ and $\lim\limits_{k\to\infty} (\theta^0 - x^k)^T [\sum\limits_{i=1}^{m} \eta_i(x^k, y_i^k) + \sum\limits_{j=1}^{l} \zeta_j(x^k, z_j^k)] < 0$. Thus, we have

$$\lim_{k\to\infty} (\theta^0 - x^k)^T [x^k - F(x^k) + \sum_{i=1}^{m} \eta_i(x^k, y_i^k) + \sum_{j=1}^{l} \zeta_j(x^k, z_j^k)] < 0 \tag{2.5}$$

However, since $t_k \in [0,1]$ is bounded, for the above $\theta^0 \in \Omega$, multiplying the two sides of the first equation of (2.3) by $(\theta^0 - x^k)$, we have

$$\begin{aligned} & (1-t_k)(\theta^0 - x^k)^T(x^k - F(x^k) + \sum_{i=1}^m \eta_i(x^k, y_i^k)) \\ + \quad & (\theta^0 - x^k)^T \sum_{j=1}^l \zeta_j(x^k, z_j^k) + t_k(\theta^0 - x^k)^T(x^k - x^{(0)}) = 0. \end{aligned}$$

Thus, we have

$$\begin{aligned} & (\theta^0 - x^k)^T[x^k - F(x^k) + \sum_{i=1}^m \eta_i(x^k, y_i^k) + \tfrac{1}{1-t_k} \sum_{j=1}^l \zeta_j(x^k, z_j^k)] \\ = \quad & \tfrac{t_k}{1-t_k}(x^k - \theta^0)^T(x^k - x^{(0)}) \\ \geq \quad & \tfrac{t_k}{2(1-t_k)}[2(x^k - \theta^0)^T(x^k - x^{(0)}) - \|x^k - x^{(0)}\|^2] \\ = \quad & \tfrac{t_k}{2(1-t_k)}[2(x^k - \theta^0)^T(x^k - x^{(0)}) - (x^k - x^{(0)})^T(x^k - x^{(0)})] \\ = \quad & \tfrac{t_k}{2(1-t_k)}[(x^k - \theta^0)^T(x^k - x^{(0)}) + (x^{(0)} - \theta^0)^T(x^k - x^{(0)})] \\ = \quad & \tfrac{t_k}{2(1-t_k)}[(x^k - \theta^0)^T(x^k - \theta^0 + \theta^0 - x^{(0)}) + (x^{(0)} - \theta^0)^T(x^k - \theta^0 + \theta^0 - x^{(0)})] \\ = \quad & \tfrac{t_k}{2(1-t_k)}(\|x^k - \theta^0\|^2 - \|x^{(0)} - \theta^0\|^2) \end{aligned}$$

i.e.,

$$\begin{aligned} & (\theta^0 - x^k)^T[x^k - F(x^k) + \sum_{i=1}^m \eta_i(x^k, y_i^k) + \tfrac{1}{1-t_k} \sum_{j=1}^l \zeta_j(x^k, z_j^k)] \\ \geq \quad & \tfrac{t_k}{2(1-t_k)}(\|x^k - \theta^0\|^2 - \|x^{(0)} - \theta^0\|^2). \end{aligned} \tag{2.6}$$

We have that the right side of (2.6) is bigger than 0 as $k \to \infty$. Hence, there exists a subsequence of $\{x^k\}$ denoted still by $\{x^k\}$, such that

$$(\theta^0 - x^k)^T[x^k - F(x^k) + \sum_{i=1}^m \eta_i(x^k, y_i^k) + \frac{1}{1-t_k} \sum_{j=1}^l \zeta_j(x^k, z_j^k)]$$

has a limit (finite or infinite), and

$$\begin{aligned} & \lim_{k\to\infty} (\theta^0 - x^k)^T[x^k - F(x^k) + \sum_{i=1}^m \eta_i(x^k, y_i^k) + \tfrac{1}{1-t_k} \sum_{j=1}^l \zeta_j(x^k, z_j^k)] \\ \geq \quad & \lim_{k\to\infty} \tfrac{t_k}{2(1-t_k)}(\|x^k - \theta^0\|^2 - \|x^{(0)} - \theta^0\|^2) \\ \geq \quad & 0, \end{aligned}$$

which contradicts with the inequality (2.5). Therefore, $\{x^k\}$ is bounded.

Then, by the fact that [0,1] is a bounded set, Assumption 2.3, the first and second equalities of (2.3), the component z of $\Gamma_{w^{(0)}}$ is bounded. Therefore, there exists a subsequence of points also denoted by $\{(x^k, y^k, z^k, t_k)\} \subset \Gamma_{w^{(0)}}$ such that $x^k \to x^*$, $z^k \to z^*$, $t_k \to t_*$, $y_i^k \to y_i^*$ for $i \notin I(x^*)$ and $y_i^k \to \infty$ for $i \in I(x^*)$. From the first equation of (2.3), we have

$$(1-t_k)(x^k - F(x^k) + \sum_{i=1}^m \eta_i(x^k, y_i^k)) + \sum_{j=1}^l \zeta_j(x^k, z_j^k) + t_k(x^k - x^{(0)}) = 0. \tag{2.7}$$

And only the following two subcases are possible: (a) $t_* = 1$; (b) $t_* \in [0,1)$.

(a) When $t_* = 1$, rewrite (2.7) as

$$\begin{aligned} & \sum_{i\in I(x^*)}(1-t_k)\eta_i(x^k,y_i^k) + \sum_{j=1}^{l}\zeta_j(x^k,z_j^k) + x^k - x^{(0)} \\ = \; & (1-t_k)[-\sum_{i\notin I(x^*)}\eta_i(x^k,y_i^k) - x^k + F(x^k) + x^k - x^{(0)}]. \end{aligned} \tag{2.8}$$

Since $x^k \in \Omega^0$ and $\{y_i^k\}$, $i \notin I(x^*)$ are bounded, when $k \to \infty$, (2.8) becomes

$$\lim_{k\to\infty}[\sum_{i\in I(x^*)}(1-t_k)\eta_i(x^k,y_i^k) + \sum_{j=1}^{l}\zeta_j(x^k,z_j^k) + x^k - x^{(0)}] = 0. \tag{2.9}$$

From $x^k \to x^*$ and $z^k \to z^*$ as $k \to \infty$, (2.9) becomes

$$x^{(0)} = \sum_{i\in I(x^*)}\lim_{k\to\infty}((1-t_k)\eta_i(x^*,y_i^k)) + \sum_{j=1}^{l}\zeta_j(x^*,z_j^*) + x^*, \tag{2.10}$$

which contradicts with Assumption 2.3 (v).

(b) When $t_* < 1$, rewrite (2.7) as

$$\begin{aligned} & (1-t_k)(x^k - F(x^k) + \sum_{i\notin I(x^*)}\eta_i(x^k,y_i^k)) + \sum_{j=1}^{l}\zeta_j(x^k,z_j^k) \\ + \; & t_k(x^k - x^{(0)}) + (1-t_k)\sum_{i\in I(x^*)}\eta_i(x^k,y_i^k) = 0. \end{aligned} \tag{2.11}$$

As $k \to \infty$, since Ω, $\{z_j^k\}$ and $\{y_i^k\}$ for $i \notin I(x^*)$ are bounded, then the first, second and third parts are bounded. But $y_i^k \to \infty$ for $i \in I(x^*)$ as $k \to \infty$, the fourth part tends to infinity. The equation (2.11) is impossible.

Thus, $\Gamma_{w^{(0)}}$ is a bounded curve in $\Omega^0 \times R^m_{++} \times R^l \times [0,1]$. Therefore, case (II) is impossible.

In conclusion, (V) is the only possible case, hence w^* is a solution of system (2.2). By Lemma 2.1, x^* is a fixed point of $F(x)$ in Ω.

The proof is complete. □

Remark 2.3 *If for any $x \in \Omega$, the matrix $\{\nabla g(x), \nabla h(x)\}$ is full column rank, and Ω satisfies normal cone condition: $\forall x \in \Omega$,*

$$\{x + \sum_{i\in I(x)}\nabla g_i(x)y_i + \sum_{j=1}^{l}\nabla h_j(x)z_j : y_i \ge 0, i \in I(x), z_j \in R\} \cap \Omega = \{x\},$$

we can get Theorem 2.1 of [5] immediately.

Remark 2.4 *If for any $x \in \Omega$, $\alpha_i(x)$ and $\beta_j(x)$ are all C^2 functions and are positive independent with respect to $\nabla g(x)$, i.e.,*

$$\sum_{i\in I(x)}(u_i\nabla g_i(x) + y_i\alpha_i(x)) + \sum_{j=1}^{l}z_j\beta_j(x) = 0, u_i \ge 0, y_i \ge 0, z_j \in R^l$$

implies $u_i = 0$, $y_i = 0$, $\forall i \in I(x)$, *and* $z_j = 0, j = 1, \dots, l$. *And suppose* Ω *satisfies quasi normal cone condition:* $\forall x \in \Omega$,

$$\{x + \sum_{i \in I(x)} \alpha_i(x) y_i + \sum_{j=1}^{l} \beta_j(x) z_j : y_i \geq 0, i \in I(x), z_j \in R\} \cap \Omega = \{x\},$$

we can obtain Lemma 2.3 and Theorem 2.2 of [5] immediately.

3. Numerical test

Theorem 2.1 shows that the zero-point set $H^{-1}_{w^{(0)}}$ determines a smooth curve for any given initial point $w^{(0)} \in \Omega^0 \times R^m_{++} \times R^l$. For numerically tracing the homotopy path $\Gamma_{w^{(0)}}$, we can use standard predictor-corrector procedure, which consists of a succession of three different steps: predictor step, corrector step and adjusting the steplength. The first predictor step is taken by computing the tangent direction, and the midway predictor steps are taken by using secant directions. The corrector steps are taken by Newton iterations for solving an augmented system. If the corrector criteria is satisfied, a successive predictor-corrector step with nondecreasing steplength is performed in the algorithm. If not, the predictor step with decreasing steplength is repeated. At each predictor step and corrector step, we need to check if the computed point is in Ω or not. If not, we take some damping step. More details on the predictor-corrector algorithms see references, e.g., [12,13,14].

Some numerical examples are given to show the effectiveness of the modified homotopy method with termination tolerance $\epsilon = 10^{-6}$. The computations are performed on a computer running the software Matlab R2007b on Microsoft Windows XP Professional with Intel(R) 1.83GHz processor and 1024 megabytes of memory.

In the following tables, CPU denotes the computer time, IT denotes the iteration step which is the summation of the predictor step and the corrector step in the computing process, x^* denotes the fixed point of $F(\Omega) \subseteq \Omega$.

Example 3.1 To find a fixed point of a self-mapping:

$$F(x) = \left(\frac{1}{2}x_1 + \frac{1}{25}x_2, x_1^2 + \frac{1}{4}x_2\right)^T,$$

and the constraint set is

$$\Omega = \{(x_1, x_2) \in R^2 : -x_1 - 5 \leq 0, x_1 - 5 \leq 5, x_2 - 50 \leq 0, -x_1^2 + \frac{1}{2}x_2 = 0\}.$$

In this example, since $h(x) = -x_1^2 + \frac{1}{2}x_2$ is nonlinear, the constraint set Ω is nonconvex. The hair mapping is constructed as follows:

$$\eta_1(x, y_1) = (-1, 0)^T y_1, \ \eta_2(x, y_2) = (1, 0)^T y_2,$$

$$\eta_3(x, y_2) = (0, 1)^T y_2, \ \zeta(x, z) = (-2x_1, \frac{1}{2})^T z.$$

When $t \to 0$, we can get the unique fixed point $x^* = (0, 0)$ of $F(\Omega) \subseteq \Omega$. The numerical results are listed in Table 3.1.

For the four initial points in Table 3.1, the numerically tracing pathways are shown in Figure 2.

Table 3.1: The numerical results of Example 3.1.

$x^{(0)}$	CPU	IT	x^*
(2,8)	0.4601	34	$10^{-10}\times$(-8.4024, -0.3879)
(-2,8)	0.1102	32	$10^{-11}\times$(4.8190, -2.0826)
(-3,18)	0.0701	36	$10^{-11}\times$(-6.2303, -0.2961)
(4,32)	0.0901	37	$10^{-10}\times$(-4.0728, -0.1569)

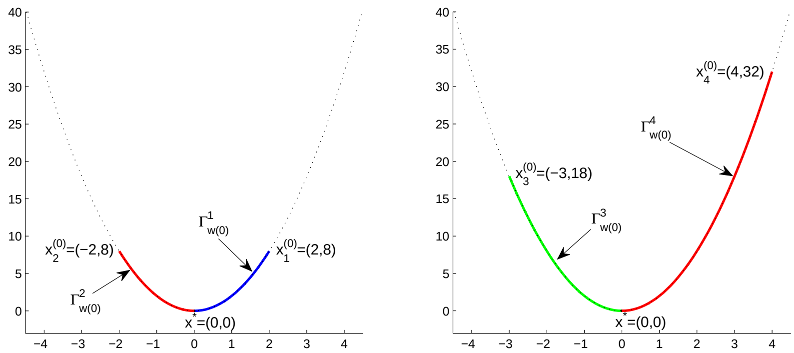

FIGURE 2. The homotopy tracing pathway of Example 3.1

Example 3.2 To find a fixed point of a self-mapping:

$$F(x) = (x_1, -x_2)^T,$$

and the constraint set is

$$\Omega = \{x \in R^2 : -x_1 - 5 \leq 0, x_2 - 10 \leq 0, -x_2 - 10 \leq 0, x_1 - x_2^2 - 5 = 0\}.$$

In this example, the hair mapping is constructed as

$$\eta_1(x, y_1) = (-1, 0)^T y_1,\ \eta_2(x, y_2) = (0, 1)^T y_2,\ \eta_3(x, y_1) = (0, -1)^T y_3,$$
$$\zeta(x, y) = (1, -2x_2)^T z.$$

When $t \to 0$, we can get one fixed point $x^* = (5.0000, 0.0000)$ of $F(\Omega) \subseteq \Omega$. The numerical results are listed in Table 3.2.

Table 3.2: The numerical results of Example 3.2.

$x^{(0)}$	CPU	IT	x^*
(6,1)	0.4907	15	(5.0000, -7.5909$\times 10^{-13}$)
(9,-2)	0.0700	18	(5.0000, 1.1010$\times 10^{-10}$)
(14,-3)	0.1602	28	(5.0000, 3.1286$\times 10^{-10}$)
(9,2)	0.2203	18	(5.0000, -1.1010$\times 10^{-11}$)

For the four initial points in Table 3.2, the numerically tracing pathways are shown in Figure 3.

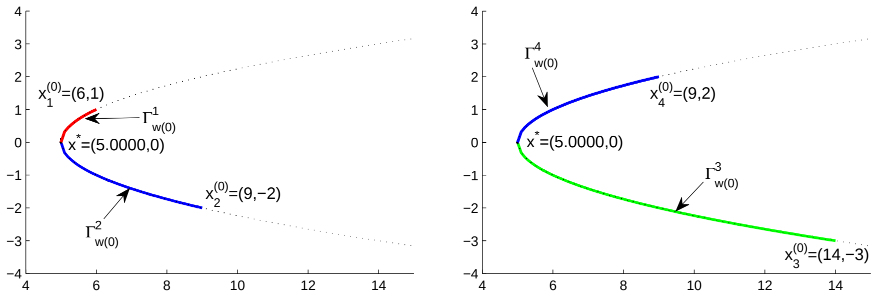

FIGURE 3. The homotopy tracing pathway of Example 3.2

Example 3.3 To find a fixed point of a self-mapping:

$$F(x) = (x_1, -x_2)^T,$$

and the constraint set is

$$\Omega = \{x \in R^2 : x_1^2 + x_2^2 - 4 \leq 0, 2 - x_1^2 - x_2^2 \leq 0, 2x_1 - x_2^2 \leq 0, 1 - (x_1 + 2.7)^2 - x_2^2 \leq 0\}.$$

In this example, the hair mapping $\eta(x, y)$ is constructed as follows:

$$\eta_1(x, y_1) = y_1^2 \nabla g_1(x), \ \eta_2(x, y_2) = (y_2, -\frac{2}{\pi} x_2 \arctan(\frac{y_2}{4}\pi))^T,$$

$$\eta_3(x, y_3) = (y_3, -\frac{2}{\pi} x_2 \arctan(\frac{y_3}{4}\pi))^T, \ \eta_4(x, y_4) = y_4^2 \nabla g_4(x).$$

The numerical results are listed in Table 3.3.

Table 3.3: The numerical results of Example 3.3.

$x^{(0)}$	CPU	IT	x^*
(1,1.5)	0.6214	151	(-1.581013, 0)
(1,-1.5)	0.6267	151	(-1.581013, 0)
(0.3,1.7)	0.4000	64	(-1.663647, 0)
(-0.5,1.5)	0.3594	55	(-1.697863, 0)

For the four initial points in Table 3.3, the numerically tracing pathways are shown in Figure 4.

Example 3.4 To find a fixed point of self-mapping:

$$\begin{aligned} F(x) \quad = \quad & ({x_1}^2 + \cos x_1 \cos x_2 - 1, {x_2}^2 + \cos x_2 \cos x_3 - 1, \ldots, \\ & x_{n-1}^2 + \cos x_{n-1} \cos x_n - 1, x_n^2 + \cos x_n \cos x_1 - 1)^T \end{aligned}$$

and the constraint set is: $\Omega = \{-2n - x_1 \leq 0, -2n - x_2 \leq 0, \ldots, -2n - x_n \leq 0\}$.

In this example, the hair mapping is constructed as follows: $\eta(x, y) = (\nabla g_1(x) y_1, \nabla g_2(x) y_2, \ldots, \nabla g_n(x) y_n)$. The initial point is chosen as $x^{(0)}$=(-2,-2,…,-2), then the numerical results are listed in Table 3.4.

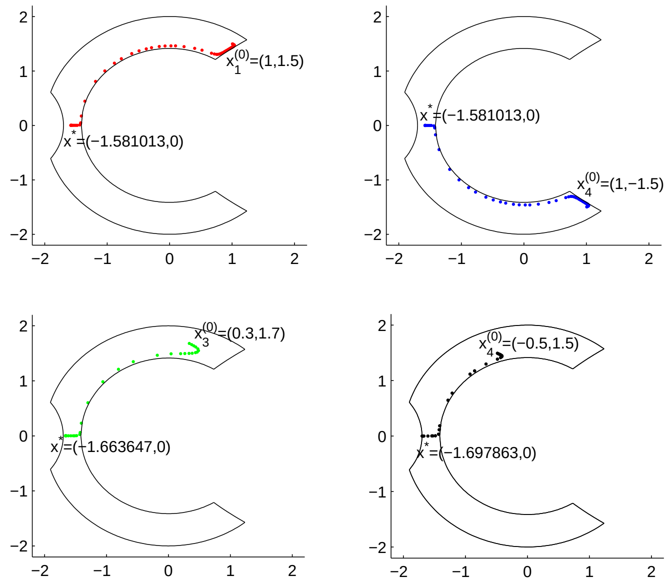

FIGURE 4. The homotopy tracing pathway of Example 3.3

Table 3.4: The numerical results of Example 3.4.

Dimension	CPU	IT	x^*
10	0.2504	9	$10^{-12} \times (1.6867, 1.6867, \ldots, 1.6867)$
30	0.2604	10	$10^{-11} \times (7.7445, 7.7445, \ldots, 7.7445)$
70	0.2604	11	$10^{-15} \times (4.2510, 4.2510, \ldots, 4.2510)$
100	0.2403	11	$10^{-12} \times (4.9919, 4.9919, \ldots, 4.9919)$
200	0.2604	12	$10^{-14} \times (8.6462, 8.6462, \ldots, 8.6462)$
500	0.5207	13	$10^{-16} \times (5.7915, 5.7915, \ldots, 5.7915)$
800	1.0115	13	$10^{-11} \times (-2.8626, -2.8626, \ldots, -2.8626)$
1000	1.4521	13	$10^{-9} \times (-3.2197, -3.2197, \ldots, -3.2197)$
2000	4.8670	14	$10^{-21} \times (-3.3087, -3.3087, \ldots, -3.3087)$

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