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ON THE FIXED POINT THEOREMS OF CARISTI TYPE

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Abstract. In this paper, using the minimal element method, we give some fixed point theorems of Caristi type which extend the previous results due to Amini-Harandi [Nonlinear Anal. 72 (2010) 4661–4665], Khamsi [Nonlinear Anal. 71 (2009) 227–231], Suzuki [J. Math. Anal. Appl. 302 (2005) 502–508] and others. Moreover, variational theorem of Ekeland type is discussed.
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1. Introduction

Throughout the paper, (M, d) is a metric space. Caristi in [1] gave the following fixed point theorem in complete metric spaces (see also [2, 3]).

Theorem 1.1 *Let $T : M \to M$, φ be a lower semi-continuous functional on M and bounded below. If for any $x \in M$, $d(x, Tx) \leq \varphi(x) - \varphi(Tx)$, then T has a fixed point.*

Caristi's fixed point theorem is a generalization of the Banach contraction principle and is equivalent to Ekeland's variational principle [4, 5, 6]. There are many authors who extended Caristi's fixed point theorem in their works to various directions (see, for example, [7]–[18] and the references cited there). In this paper, we are motivated by [8, 13, 17] to give some fixed point theorems of Caristi type, which extend the previous results(see the theorems and the remarks in Section 2), by the minimal element method in [19]. Moreover, variational theorem of Ekeland type is discussed in Section 3.

In order to state and prove our main results, we give some notions and notations.

Let (M, d) be provided with a quasiorder $\preccurlyeq$, i.e. a reflexive and transitive relation. A sequence $\{x_n\} \subset M$ is said to be decreasing with respect to $\preccurlyeq$ iff $x_{n+1} \preccurlyeq x_n (n = 1, 2, \cdots)$. $(M, d, \preccurlyeq)$ is said to be $\preccurlyeq$-complete iff every decreasing Cauchy sequence in M converges to some element of M. Of course, $(M, d, \preccurlyeq)$ is $\preccurlyeq$-complete if it is a complete metric space. A quasiorder $\preccurlyeq$ is called lower closed iff $x \preccurlyeq x_n (n = 1, 2, \cdots)$ for any decreasing sequence $\{x_n\} \subset M$ converging to some $x \in M$. A quasiorder $\preccurlyeq$ on a metric space M is called regular iff every decreasing sequence $\{x_n\} \subset M$ is asymptotic, i.e., $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$. It follows from Proposition 40 in [19] that a regular quasiorder on a metric space is antisymmetric, hence we may assume that the quasiorder on a metric space is partial order without loss of generality if it is regular. The following lemma is the basic minimal element theorem in [19] on metric space.

Lemma 1.1 *Let $\preccurlyeq$ be a quasiorder on (M,d) such that $(M,d,\preccurlyeq)$ is $\preccurlyeq$-complete. If $\preccurlyeq$ is regular and lower closed, then there exists a minimal element $\overline{x}$ in $(M,d,\preccurlyeq)$, that is, if $x \in M$ with $x \preccurlyeq \overline{x}$, then $x = \overline{x}$.*

Let $\gamma : [0,\infty) \to [0,\infty)$ be subadditive, i.e. $\gamma(t+s) \le \gamma(t) + \gamma(s)$ for each $s,t \in [0,\infty)$, a increasing continuous map such that $\gamma^{-1}(\{0\}) = \{0\}$, for example, $\gamma(t) = t^p$ $(0 < p \le 1)$ for $t \in [0,\infty)$. Let Γ consist of all such functions γ.

$\mathcal{A}$ denotes the class of all maps $\eta : [0,\infty) \to [0,\infty)$ for which there exist $\overline{\varepsilon} > 0$ and $\gamma \in \Gamma$ such that if $\eta(t) \le \overline{\varepsilon}$, then $\eta(t) \ge \gamma(t)$.

Let $F : \mathbb{R} \to \mathbb{R}$, $F(0) = 0$, $F^{-1}[0,\infty) \subset [0,\infty)$, and for $t \ge 0$, F is increasing, upper semi-continuous, moreover $F(t) + F(s) \le F(t+s)$ for $t,s \ge 0$. For example,

$$F(t) = \begin{cases} < 0, & t < 0, \\ t^p, & 0 \le t < t_0, \\ t^{p+1}, & t \ge t_0, \end{cases}$$

where $t_0 > 1$ and $p \ge 1$. The class of all these functions F is denoted by $\mathcal{F}$. If $F(t) = t$, $\forall t \in \mathbb{R}$, then trivially $F \in \mathcal{F}$.

2. Fixed point theorems of Caristi type

In the sequel, (M,d) is a complete metric space.

Theorem 2.1 *Let $T : M \to M$, φ be a lower semi-continuous functional on M and bounded below. If there exist $\eta \in \mathcal{A}$ and $F \in \mathcal{F}$ such that for any $x \in M$,*

$$\eta(d(x,Tx)) \le F(\varphi(x) - \varphi(Tx)), \tag{2.1}$$

then T has a fixed point.

Proof. 1. Define in M that for $x,y \in M$,

$$x \le_* y \Leftrightarrow \eta(d(x,y)) \le F(\varphi(y) - \varphi(x)).$$

Notice that $(M, \le_*)$ is not necessarily a partially ordered set. Let $\varphi_0 = \inf_{x \in M} \varphi(x)$. Since F is upper semi-continuous for $t \ge 0$,

$$\limsup_{t \to 0^+} F(t) \le F(0) = 0.$$

Then for $\overline{\varepsilon} > 0$ in the definition of $\mathcal{A}$, there exists $\delta > 0$ such that $F(t) < \overline{\varepsilon}$ for $0 \le t \le \delta$.

Denote $M_\delta = \{x \in M \mid \varphi(x) \le \varphi_0 + \delta\}$. Clearly $M_\delta \ne \emptyset$ and M_δ is a closed set by the lower semi-continuity of φ. Thus (M_δ, d) is a complete metric space.

$\forall x,y \in M_\delta$, we have

$$\varphi_0 \le \varphi(x) \le \varphi_0 + \delta, \quad \varphi_0 \le \varphi(y) \le \varphi_0 + \delta. \tag{2.2}$$

If $x \le_* y$, then $\varphi(y) - \varphi(x) \ge 0$ since $0 \le \eta(d(x,y)) \le F(\varphi(y) - \varphi(x))$ and $F^{-1}[0,\infty) \subset [0,\infty)$. By (2.2) we have $0 \le \varphi(y) - \varphi(x) \le \delta$, and thus $\eta(d(x,y)) \le F(\varphi(y) - \varphi(x)) < \overline{\varepsilon}$. It follows that

$$\gamma(d(x,y)) \le \eta(d(x,y)) \le F(\varphi(y) - \varphi(x)). \tag{2.3}$$

2. Define in M_δ that for $x, y \in M_\delta$,

$$x \le y \Leftrightarrow \gamma(d(x,y)) \le F(\varphi(y) - \varphi(x)).$$

Now we prove that $(M_\delta, \le)$ is a partially ordered set.

(i) For $x \in M_\delta$, it is clear that $x \le x$ from $F(0) = 0$ and $\gamma^{-1}(\{0\}) = \{0\}$.

(ii) Let $x, y \in M_\delta$. If $x \le y,\ y \le x$, then

$$0 \le \gamma(d(x,y)) \le F(\varphi(y) - \varphi(x)), \quad 0 \le \gamma(d(y,x)) \le F(\varphi(x) - \varphi(y)). \tag{2.4}$$

It follows from $F^{-1}[0,\infty) \subset [0,\infty)$ and (2.4) that $\varphi(y) \ge \varphi(x),\ \varphi(x) \ge \varphi(y)$, and hence $\varphi(x) = \varphi(y)$. Therefore, from (2.4), $F(0) = 0$ and $\gamma^{-1}(\{0\}) = \{0\}$, we have $d(x,y) = 0$, i.e. $x = y$.

(iii) Let $x, y, z \in M_\delta$. If $x \le y,\ y \le z$, then

$$0 \le \gamma(d(x,y)) \le F(\varphi(y) - \varphi(x)), \quad 0 \le \gamma(d(y,z)) \le F(\varphi(z) - \varphi(y)). \tag{2.5}$$

It follows from $F^{-1}[0,\infty) \subset [0,\infty)$ and (2.5) that $\varphi(y) - \varphi(x) \ge 0,\ \varphi(z) - \varphi(y) \ge 0$. Since $F(t) + F(s) \le F(t+s)$ for $t, s \ge 0$, we have from the subadditivity of γ and (2.5),

$$\begin{aligned} &\gamma(d(x,z)) \le \gamma(d(x,y)) + \gamma(d(y,z)) \\ \le\ & F(\varphi(y) - \varphi(x)) + F(\varphi(z) - \varphi(y)) \le F(\varphi(z) - \varphi(x)), \end{aligned}$$

and thus $x \le z$.

3. In this step we will prove that $(M_\delta, \le)$ has a minimal element. By Lemma 1.1, we need only to show that $\le$ is regular and lower closed.

(i) Let $\{x_n\}$ be a decreasing sequence in $(M_\delta, \le)$, then

$$0 \le \gamma(d(x_{n+1}, x_n)) \le F(\varphi(x_n) - \varphi(x_{n+1})) \tag{2.6}$$

and $\varphi(x_n) - \varphi(x_{n+1}) \ge 0$ by $F^{-1}[0,\infty) \subset [0,\infty)$. So $\{\varphi(x_n)\}$ is a real number sequence which is decreasing and bounded below, and we may suppose that $\lim_{n\to\infty} \varphi(x_n) = \alpha$. Since F is upper semi-continuous for $t \ge 0$ and $\varphi(x_n) - \varphi(x_{n+1}) \to \alpha - \alpha = 0 (n \to \infty)$, we have from (2.6) that

$$0 \le \limsup_{n\to\infty} \gamma(d(x_{n+1}, x_n)) \le \limsup_{n\to\infty} F(\varphi(x_n) - \varphi(x_{n+1})) \le F(0) = 0.$$

Therefore $\lim_{n\to\infty} \gamma(d(x_{n+1}, x_n)) = 0$ which implies that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0,$$

i.e., $\le$ is regular. In fact, suppose for the contrary that $\lim_{n\to\infty} d(x_{n+1}, x_n) \ne 0$, let $u_n = d(x_{n+1}, x_n)$, then there exist $\varepsilon_0 > 0$ and a subsequence $\{u_{n_i}\}$ such that $u_{n_i} \ge \varepsilon_0$ for any i, and thus $0 < \gamma(\varepsilon_0) \le \gamma(u_{n_i})$ by monotonicity of γ and $\gamma^{-1}(\{0\}) = \{0\}$, $\lim_{i\to\infty} \gamma(u_{n_i}) \ge \gamma(\varepsilon_0)$ which contradicts $\lim_{n\to\infty} \gamma(u_n) = 0$.

(ii) If the decreasing sequence $\{x_n\}$ converges to $x \in M_\delta$, it follows that

$$\varphi(x) \le \liminf_{n\to\infty} \varphi(x_n) = \alpha, \tag{2.7}$$

from the lower semi-continuity of φ. Since F is increasing, upper semi-continuous for $t \ge 0$ and γ is continuous, for $m > n$ we have from $\varphi(x_m) \le \varphi(x_n)$ and (2.7) that

$$\begin{aligned} &\gamma(d(x, x_n)) = \limsup\nolimits_{m\to\infty} \gamma(d(x_m, x_n)) \\ \le\ & \limsup\nolimits_{m\to\infty} F(\varphi(x_n) - \varphi(x_m)) \le F(\varphi(x_n) - \alpha) \le F(\varphi(x_n) - \varphi(x)), \end{aligned}$$

and hence $x \le x_n$ for any n, i.e., $\le$ is lower closed.

4. Suppose that x_* is a minimal element in $(M_\delta, \le)$. In the following, we prove that x_* is also a minimal element in $(M, \le_*)$, i.e. if $x \in M$ with $x \le_* x_*$, then $x = x_*$.

Let $x \in M$ and $x \le_* x_*$. By the definition of $\le_*$,

$$0 \le \eta(d(x, x_*)) \le F(\varphi(x_*) - \varphi(x)). \tag{2.8}$$

It follows from $F^{-1}[0, \infty) \subset [0, \infty)$ and (2.8) that

$$\varphi(x_*) - \varphi(x) \ge 0. \tag{2.9}$$

Since $x_* \in M_\delta$, by the definitions of φ_0 and M_δ we have

$$\varphi_0 \le \varphi(x_*) \le \varphi_0 + \delta. \tag{2.10}$$

(2.9) and (2.10) lead to $\varphi_0 \le \varphi(x) \le \varphi_0 + \delta$ which implies that $x \in M_\delta$ and $0 \le \varphi(x_*) - \varphi(x) \le \delta$. According to the selection of δ, $F(\varphi(x_*) - \varphi(x)) < \bar{\varepsilon}$. It follows from (2.8) that $\eta(d(x, x_*)) < \bar{\varepsilon}$, and hence

$$\gamma(d(x, x_*)) \le \eta(d(x, x_*)) \le F(\varphi(x_*) - \varphi(x)),$$

i.e. $x \le x_*$. Since x_* is a minimal element in $(M_\delta, \le)$, we have that $x = x_*$ and that x_* is also a minimal element in $(M, \le_*)$.

5. Finally, for the minimal element x_* in $(M, \le_*)$, it follows from (2.1) that

$$\eta(d(x_*, Tx_*)) \le F(\varphi(x_*) - \varphi(Tx_*)),$$

then $Tx_* \le_* x_*$, and $Tx_* = x_*$.

Remark 2.1 *By Remark 3 in* [13] *for $\gamma \in \Gamma$, there exist $\bar{\varepsilon}$ and $c > 0$ such that if $\gamma(t) \le \bar{\varepsilon}$, then $\gamma(t) \ge ct$. Therefore, if $F(t) = t$ for $t \in \mathbb{R}$ and $\eta(t) = \gamma(t)$, Theorem 2.1 reduces to Theorem 3 in* [13].

Corollary 2.1 *Let $T : M \to 2^M \backslash \{\emptyset\}$, φ be a lower semi-continuous functional on M and bounded below. If there exist $\eta \in \mathcal{A}$ and $F \in \mathcal{F}$ such that for any $x \in M$, there exists $y \in Tx$ satisfying*

$$\eta(d(x, y)) \le F(\varphi(x) - \varphi(y)), \tag{2.11}$$

then T has a fixed point, i.e. there exists $x_* \in M$ such that $x_* \in Tx_*$.

Proof. From the proof of Theorem 2.1, it follows that $(M, \le_*)$ has a minimal element x_*. (2.11) tells us that there exists $y_* \in Tx_*$ such that $y_* \le_* x_*$, thus $y_* = x_*$, i.e. $x_* \in Tx_*$.

Remark 2.2 *Corollary 2.1 extends Theorem 4 in* [13].

Similar to the proof of Corollary 2.1, it is easy to prove Corollary 2.2 below.

Corollary 2.2 *Let $T : M \to 2^M \backslash \{\emptyset\}$, φ be a lower semi-continuous functional on M and bounded below. If there exist $\eta \in \mathcal{A}$ and $F \in \mathcal{F}$ such that for any $x \in M$ and any $y \in Tx$,*

$$\eta(d(x, y)) \le F(\varphi(x) - \varphi(y)),$$

then there exists $x_* \in M$ such that $Tx_* = \{x_*\}$.

Remark 2.3 *Let Ψ be the class of all the maps $\psi : M \times M \to \mathbb{R}$ satisfying the following conditions: (i) there exists $\hat{x} \in M$ such that $\psi(\hat{x}, \cdot)$ is bounded below and lower semi-continuous, and $\psi(\cdot, y)$ is upper semi-continuous for each $y \in M$; (ii) $\psi(x, x) = 0$ for each $x \in M$; (iii) $\psi(x, y) + \psi(y, z) \le \psi(x, z)$ for each $x, y, z \in M$. Amini-Harandi* *proved in* [8] *that if* $T : M \to M$ *and there exists* $\eta \in \mathcal{A}$ *such that for any* $x \in M$, $\eta(d(x, Tx)) \leq \psi(Tx, x)$, *then* T *has a fixed point. If we define* $\varphi(x) = \psi(\hat{x}, x)$, *then* φ *is a lower semi-continuous functional on* M *and bounded below. Since*

$$\eta(d(x, Tx)) \leq \psi(Tx, x) \leq \psi(\hat{x}, x) - \psi(\hat{x}, Tx) = \varphi(x) - \varphi(Tx),$$

it is easy to see that the main results in [8] *are the spacial cases of Theorem 2.1 and Corollary 2.2 in this paper.*

Theorem 2.2 *Let* $T : M \to M$, φ *be a lower semi-continuous functional on* M *and bounded below. Suppose that* Φ *is a nonnegative functional on* M *and there exists* $\delta > 0$ *such that*

$$\sup\Big\{\Phi(x) \mid x \in M, \varphi(x) \leq \inf_{y \in M} \varphi(y) + \delta\Big\} < +\infty.$$

If there exist $\eta \in \mathcal{A}$ *and* $F \in \mathcal{F}$ *such that for any* $x \in M$,

$$\eta(d(x, Tx)) \leq \Phi(x) F(\varphi(x) - \varphi(Tx)), \tag{2.12}$$

then T *has a fixed point.*

Proof. If $\Phi(x) > 0$, then $F(\varphi(x) - \varphi(Tx)) \geq 0$ and $\varphi(Tx) \leq \varphi(x)$. If $\Phi(x) = 0$, then $\eta(d(x, Tx)) = 0 \leq \bar{\varepsilon}$, and thus $\gamma(d(x, Tx)) \leq \eta(d(x, Tx)) = 0$ which implies that $d(x, Tx) = 0$, i.e. $x = Tx$, and $\varphi(Tx) = \varphi(x)$. Therefore $\varphi(Tx) \leq \varphi(x)$ for all $x \in M$.

Let $\varphi_0 = \inf_{y \in M} \varphi(y)$, $M_\delta = \{x \in M \mid \varphi(x) \leq \varphi_0 + \delta\}$ and $\alpha = \sup_{x \in M_\delta} \Phi(x) < +\infty$. Obviously, $M_\delta \neq \emptyset$ and is closed by the lower semi-continuity of φ, hence (M_δ, d) is a complete metric space.

$\forall x \in M_\delta$, it follows from $\varphi(Tx) \leq \varphi(x) \leq \varphi_0 + \delta$ that $Tx \in M_\delta$, that is, $T : M_\delta \to M_\delta$. By (2.12) we have that $\eta(d(x, Tx)) \leq \alpha F(\varphi(x) - \varphi(Tx))$, $\forall x \in M_\delta$. Clearly, $\alpha F \in \mathcal{F}$. Therefore Theorem 2.1 indicates that there exists $x_0 \in M_\delta \subset M$ such that $Tx_0 = x_0$.

Remark 2.4 *If* $\eta(t) = \gamma(t) = t$ *for* $t \geq 0$ *and* $F(t) = t$ *for* $t \in \mathbb{R}$, *Theorem 2.2 is just Theorem 2 in* [17] *and is also a generalization of the results in* [9, 10].

Corollary 2.3 *Let* $T : M \to 2^M \backslash \{\emptyset\}$, φ *be a lower semi-continuous functional on* M *and bounded below. Suppose that* Φ *is a nonnegative functional on* M *and there exists* $\delta > 0$ *such that*

$$\sup\Big\{\Phi(x) \mid x \in M, \varphi(x) \leq \inf_{y \in M} \varphi(y) + \delta\Big\} < +\infty.$$

If there exist $\eta \in \mathcal{A}$ *and* $F \in \mathcal{F}$ *such that for any* $x \in M$ *and any* $y \in Tx$,

$$\eta(d(x, y)) \leq \Phi(x) F(\varphi(x) - \varphi(y)), \tag{2.13}$$

then there exists $x_* \in M$ *such that* $Tx_* = \{x_*\}$.

Proof. Similar to the proof of Theorem 2.2, we have that $\varphi(y) \leq \varphi(x)$, $\forall x \in M$, $y \in Tx$. Take M_δ and α as the same in the proof of Theorem 2.2. Then $y \in M_\delta$, $\forall x \in M_\delta$, $y \in Tx$, and hence $T : M_\delta \to 2^{M_\delta} \backslash \{\emptyset\}$.

From (2.13) it follows that $\forall x \in M_\delta$, $y \in Tx$,

$$\eta(d(x, y)) \leq \alpha F(\varphi(x) - \varphi(y)). \tag{2.14}$$

Clearly, $\alpha F \in \mathcal{F}$. Define in M_δ that for $x, z \in M_\delta$,

$$x \leq_* z \Leftrightarrow \eta(d(x, z)) \leq \alpha F(\varphi(z) - \varphi(x)).$$

By the analogous proof to Theorem 2.1, $(M_\delta, \leq_*)$ has a minimal element x_*. It follows from (2.14) that $y_* \leq_* x_*,\ \forall y_* \in Tx_*$, and thus $y_* = x_*$, i.e. $Tx_* = \{x_*\}$.

3. Variational theorem of Ekeland type

Theorem 3.1 *Let $\gamma \in \Gamma,\ F \in \mathcal{F}$. If φ is a lower semi-continuous functional on M and is bounded below, then $\forall \varepsilon > 0,\ \tau > 0$ and $x_\varepsilon \in M$ satisfying $\varphi(x_\varepsilon) < \inf_{x\in M} \varphi(x) + \varepsilon$, there exists $y_\varepsilon \in M$ such that*

(i) $\varphi(y_\varepsilon) \leq \varphi(x_\varepsilon)$;

(ii) $\gamma(d(x_\varepsilon, y_\varepsilon)) \leq (\tau/\varepsilon)F(\varepsilon)$;

(iii) $F(\varphi(y_\varepsilon) - \varphi(x)) < (\varepsilon/\tau)\gamma(d(y_\varepsilon, x)),\ \forall x \in M,\ x \neq y_\varepsilon$.

Proof. 1. Assume that there exists $\tau > 0$ such that for any $y \in M$ satisfying

$$F(\varphi(x_\varepsilon) - \varphi(y)) \geq (\varepsilon/\tau)\gamma(d(x_\varepsilon, y)),$$

there exists $x \in M$ with $x \neq y$, and

$$F(\varphi(y) - \varphi(x)) \geq (\varepsilon/\tau)\gamma(d(x, y)).$$

2. Let $X = \{y \in M \mid F(\varphi(x_\varepsilon) - \varphi(y)) \geq (\varepsilon/\tau)\gamma(d(x_\varepsilon, y))\}$. Obviously, $x_\varepsilon \in X$ and $X \neq \emptyset$. Now we show that X is closed.

If $y_n \in X (n = 1, 2, \cdots)$ and $y_n \to y_0$, then

$$0 \leq (\varepsilon/\tau)\gamma(d(x_\varepsilon, y_n)) \leq F(\varphi(x_\varepsilon) - \varphi(y_n)). \tag{3.1}$$

Because φ is lower semi-continuous, $\varphi(y_0) \leq \liminf_{n\to\infty} \varphi(y_n)$. Denote $\alpha = \liminf_{n\to\infty} \varphi(y_n)$, then there is a subsequence $\varphi(y_{n_k}) \to \alpha$. It follows from (3.1) and $F^{-1}[0,\infty) \subset [0,\infty)$ that $\varphi(x_\varepsilon) - \varphi(y_n) \geq 0$. Since γ is continuous and F is increasing, upper semi-continuous for $t \geq 0$, by (3.1) we have

$$\begin{aligned} &(\varepsilon/\tau)\gamma(d(x_\varepsilon, y_0)) = (\varepsilon/\tau)\limsup_{k\to\infty} \gamma(d(x_\varepsilon, y_{n_k})) \\ \leq\ &\limsup_{k\to\infty} F(\varphi(x_\varepsilon) - \varphi(y_{n_k})) \leq F(\varphi(x_\varepsilon) - \alpha) \leq F(\varphi(x_\varepsilon) - \varphi(y_0)), \end{aligned}$$

which implies that $y_0 \in X$ and X is closed. Hence (X, d) is a complete metric space.

3. $\forall y \in X$, let $Ty = \{z \in M \mid z \neq y, F(\varphi(y) - \varphi(z)) \geq (\varepsilon/\tau)\gamma(d(z, y))\}$. $Ty \neq \emptyset$ by step 1. If there exists $z \in Ty$ such that $z \notin X$, then

$$F(\varphi(x_\varepsilon) - \varphi(z)) < (\varepsilon/\tau)\gamma(d(x_\varepsilon, z)). \tag{3.2}$$

It follows from $z \in Ty,\ y \in X$ and $F^{-1}[0,\infty) \subset [0,\infty)$ that $\varphi(y) - \varphi(z) \geq 0$ and $\varphi(x_\varepsilon) - \varphi(y) \geq 0$. Since $F(t) + F(s) \leq F(t + s)$ for $t, s \geq 0$ and γ is subadditive, increasing, we have

$$\begin{aligned} &F(\varphi(x_\varepsilon) - \varphi(z)) \geq F(\varphi(y) - \varphi(z)) + F(\varphi(x_\varepsilon) - \varphi(y)) \\ \geq\ &(\varepsilon/\tau)\gamma(d(z, y)) + (\varepsilon/\tau)\gamma(d(x_\varepsilon, y)) \\ \geq\ &(\varepsilon/\tau)\gamma(d(z, y) + d(x_\varepsilon, y)) \geq (\varepsilon/\tau)\gamma(d(x_\varepsilon, z)), \end{aligned}$$

which contradicts (3.2). Therefore $Ty \subset X,\ \forall y \in X$, i.e. $T : X \to 2^X\backslash\{\emptyset\}$.

4. By the definition of T, $(\varepsilon/\tau)\gamma(d(z, y)) \leq F(\varphi(y) - \varphi(z)),\ \forall y \in X,\ z \in Ty$. Because $(\varepsilon/\tau)\gamma \in \Gamma$, from Corollary 2.2 in which η is replaced with $(\varepsilon/\tau)\gamma$, we have that there exists $x_* \in X$ such that $Tx_* = \{x_*\}$ which is in contradiction with the definition of T.

5. In summary, the assumption in step 1 is false. Then $\forall\tau > 0$, there exists $y_\varepsilon \in M$ satisfying

$$F(\varphi(x_\varepsilon) - \varphi(y_\varepsilon)) \geq (\varepsilon/\tau)\gamma(d(x_\varepsilon, y_\varepsilon)), \tag{3.3}$$

and $F(\varphi(y_\varepsilon) - \varphi(x)) < (\varepsilon/\tau)\gamma(d(x, y_\varepsilon)),\ \forall x \in M,\ x \neq y_\varepsilon$.

From (3.3) and $F^{-1}[0,\infty) \subset [0,\infty)$, it follows that $\varphi(y_\varepsilon) \leq \varphi(x_\varepsilon)$. Finally, by (3.3) and the monotonicity of F for $t \geq 0$, we have

$$(\varepsilon/\tau)\gamma(d(x_\varepsilon, y_\varepsilon)) \leq F(\varphi(x_\varepsilon) - \varphi(y_\varepsilon)) \leq F\Big(\varphi(x_\varepsilon) - \inf_{x\in M} \varphi(x)\Big) \leq F(\varepsilon).$$

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