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SOLVABILITY OF BOUNDARY VALUE PROBLEMS WITH INTEGRAL CONDITIONS FOR FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we study boundary value problems with integral conditions for fractional differential equations of the order $\alpha \in (1, 2)$ in an abstract Banach space. To overcome the difficult from some mixed integral terms in a fractional integral equation, a necessary Gronwall inequality with some mixed integral terms is established to obtain important a priori bounds. Some sufficient conditions for the existence of solutions are presented by means of fractional calculus and fixed point theorems via different conditions and techniques. An example is given to illustrate the results.

Key Words and Phrases: Fractional differential equations, Boundary value problems, Integral conditions, Generalized Gronwall inequality, Fixed point method.

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1. Introduction

In this paper, we extend the earlier work [1] on boundary value problem (BVP for short) with integral conditions for fractional differential equations of the order $\alpha \in (1, 2)$ in R to an abstract Banach space X of the following type

$$\begin{cases} {}^cD^\alpha y(t) = f(t, y(t)),\ \alpha \in (1, 2),\ t \in J = [0, T], \\ y(0) = \int_0^T g(s, y)ds, \\ y(T) = \int_0^T h(s, y)ds, \end{cases} \tag{1.1}$$

where ${}^cD^\alpha$ is the Caputo fractional derivative of order α, $f, g, h : J \times X \to X$ are given functions and satisfy some assumptions that will be specified later. We remark that the BVP model of the order $\alpha \in (1, 2)$ is different from the BVP model of the order $\alpha \in (0, 1)$ since a fractional integral equation which is equivalent to the BVP model of the order $\alpha \in (1, 2)$ is much more complex than the one which is equivalent to the BVP model of the order $\alpha \in (0, 1)$ in some senses.

Fractional differential equations have been proved to be valuable tools in the modelling of many phenomena in various fields of engineering, physics and economics. It draws a great application in nonlinear oscillations of earthquakes, many physical phenomena such as seepage flow in porous media and in fluid dynamic traffic model. Actually, fractional differential equations are considered as an alternative model to integer differential equations. For more details, one can see the monographs of Diethelm [11], Kilbas et al. [14], Miller and Ross [17], Lakshmikantham et al. [18], Podlubny [19], and Tarasov [23]. Particulary, all kinds of fractional differential equations (inclusions) and optimal control problems in Banach spaces are studied by researchers such as Agarwal et al. [1, 2], Ahmad and Nieto [3, 4, 5], Bai [7], Benchohra et al. [8], Chang and Nieto [9], Henderson and Ouahab [13], Wang et al. [26, 27, 28, 29, 30, 31, 32, 33, 34, 35], and Zhou et al. [36, 37].

In the present paper, we show existence and uniqueness results for the fractional BVP (1.1) by virtue of fractional calculus and fixed point method. Compared with the earlier results appeared in [1], there are at least three differences: (i) the work space is not R but the Banach space X; (ii) the assumptions are weakened and easy to check; (iii) a priori bounds is given via sublinear conditions and a new type Gronwall inequality with some mixed integral terms (Lemma 3.2). Compared with the earlier results appeared in [3], different fixed point theorems via the techniques of the generalized Gronwall inequality of mixed integral terms are combined to deal with such problems.

The rest of this paper is organized as follows. In Section 2, we give some notations and recall some concepts and preparation results. In Section 3, we give a generalized Gronwall inequality which can be used to establish the estimate of fixed point set $\{y : y = \lambda F y, \lambda \in [0,1]\}$. In Section 4, we give three main results (Theorems 4.1–4.3), the first result based on Banach contraction principle, the second result based on Schaefer's fixed point theorem, the third result based on nonlinear alternative of Leray-Schauder type. An example is given in Section 5 to demonstrate the application of our main results. These results can be considered as a contribution to this emerging field.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. We denote $C(J,X)$ the Banach space of all continuous functions from J into X with the norm $\|y\|_\infty := \sup\{\|y(t)\| : t \in J\}$. For measurable functions $m : J \to R$, define the norm

$$\|m\|_{L^p(J,R)} := \left(\int_J |m(t)|^p dt\right)^{\frac{1}{p}},\ 1 \le p < \infty.$$

We denote $L^p(J,R)$ the Banach space of all Lebesgue measurable functions m with

$$\|m\|_{L^p(J,R)} < \infty.$$

We need some basic definitions and properties of the fractional calculus theory which are used further in this paper. For more details, see [14].

Definition 2.1. *The fractional order integral of the function $h \in L^1([a,b],R)$ of order $\alpha \in R_+$ is defined by*

$$I_a^\alpha h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds$$

where Γ is the Gamma function.

Definition 2.2. *For a function h given on the interval $[a,b]$, the αth Riemann-Liouville fractional order derivative of h, is defined by*

$$(D_{a+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} h(s)ds,$$

here $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of α.

Definition 2.3. *For a function h given on the interval $[a,b]$, the Caputo fractional order derivative of h, is defined by*

$$({}^cD_{a+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s)ds,$$

where $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of α.

Lemma 2.4. *Let $\alpha \in (n-1,n)$, then the differential equation ${}^cD^\alpha h(t) = 0$ has solutions*

$$h(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

where $c_i \in R$, $i = 0,1,2,\cdots,n-1, n = [\alpha]+1$.

Lemma 2.5. *Let $\alpha \in (n-1,n)$, then*

$$I^\alpha({}^cD^\alpha h)(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

for some $c_i \in R, i = 0,1,2,\cdots,\ n-1,\ n = [\alpha]+1$.

Let us start by defining what we mean by a solution of the fractional BVP (1.1).

Definition 2.6. *(Definition 3.20,* [1]*) A function $y \in C^2(J,X)$ with its α-derivative existing on J is said to be a solution of the fractional BVP (1.1) if y satisfies the equation ${}^cD^\alpha y(t) = f(t,y(t))$ a.e. on J, and the conditions $y(0) = \int_0^T g(s,y(s))ds$ and $y(T) = \int_0^T h(s,y(s))ds$.*

For the existence of solutions for the fractional BVP (1.1), we need the following auxiliary lemma.

Lemma 2.7. *(Lemma 3.21,* [1]*) A function $y \in C(J,X)$ is a solution of the fractional integral equation*

$$\begin{aligned} y(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\sigma(s)ds - \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1}\sigma(s)ds \\ &\quad - \left(\frac{t}{T}-1\right) \int_0^T \rho_1(s)ds + \frac{t}{T}\int_0^T \rho_2(s)ds, \end{aligned}$$

if and only if y is a solution of the following fractional BVP

$$\begin{cases} {}^cD^\alpha y(t) = \sigma(t),\ 1<\alpha<2,\ t\in J, \\ y(0) = \int_0^T \rho_1(s)ds, \\ y(T) = \int_0^T \rho_2(s)ds. \end{cases} \tag{2.1}$$

As a consequence of Lemmas 2.7, we have the following result which is useful in what follows.

Lemma 2.8. *A function $y\in C(J,X)$ is a solution of the fractional integral equation*

$$\begin{aligned} y(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,y(s))ds - \frac{t}{T\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} f(s,y(s))ds \\ &\quad - \left(\frac{t}{T}-1\right)\int_0^T g(s,y(s))ds + \frac{t}{T}\int_0^T h(s,y(s))ds, \end{aligned}$$

if and only if y is a solution of the fractional BVP (1.1).

Lemma 2.9. *(Bochner theorem,* [6]*) A measurable function $f\colon J\to X$ is Bochner integrable if $\|f\|$ is Lebesgue integrable.*

Lemma 2.10. *(Mazur lemma,* [6]*) If $\mathcal{K}$ is a compact subset of X, then its convex closure $\overline{conv}\mathcal{K}$ is compact.*

Lemma 2.11. *(Ascoli-Arzela theorem,* [22]*) Let $\mathcal{S} = \{s(t)\}$ is a function family of continuous mappings $s:[a,b]\to X$. If $\mathcal{S}$ is uniformly bounded and equicontinuous, and for any $t^*\in[a,b]$, the set $\{s(t^*)\}$ is relatively compact, then there exists a uniformly convergent function sequence $\{s_n(t)\}(n=1,2,\cdots,t\in[a,b])$ in $\mathcal{S}$.*

Theorem 2.12. *(Schaefer's fixed point theorem,* [22]*) Let $F:X\to X$ completely continuous operator. If the set*

$$E(F) = \{x\in X : x = \lambda Fx \ for\ some\ \lambda\in[0,1]\}$$

is bounded, then F has fixed points.

Theorem 2.13. *(Nonlinear alternative of Leray-Schauder type,* [12]*) Let $\mathcal{C}$ a nonempty convex subset of X. Let U a nonempty open subset of $\mathcal{C}$ with $0\in U$ and $F:\overline{U}\to\mathcal{C}$ compact and continuous operators. Then either*

(i) F has fixed points.

(ii) There exist $y\in\partial U$ and $\lambda^*\in[0,1]$ with $y=\lambda^*F(y)$.

3. Gronwall inequality with some mixed integral terms for a priori bounds

The method of a priori bounds has been often used together with the coincidence fixed point theorems in order to prove the existence of solutions for some BVP (or IVP) for integer nonlinear differential equations or nonlinear partial differential equations. See, for example, Crăciun and Lungu [10], Lungu [15, 16], Rus [20, 21], Wang et al. [24, 25]. To apply the Schaefer fixed point theorem to prove the existence of solutions of the fractional BVP (1.1), we need a new generalized Gronwall inequality with some mixed integral terms. It will play the key role in the study of BVP for a class of fractional differential equations of the order $\alpha \in (1,2)$ via fixed point methods.

Recall a generalized Gronwall inequality which appeared in our earlier work [24].

Lemma 3.1. *(Lemma 2, [24]) Let $y \in C(J,X)$ satisfy the following inequality:*

$$\|y(t)\| \le a + b\int_0^t \|y(\theta)\|^{\lambda_1} d\theta + c\int_0^T \|y(\theta)\|^{\lambda_2} d\theta,\ t \in J,$$

where $\lambda_1 \in [0,1], \lambda_2 \in [0,1), a,b,c \ge 0$ are constants. Then there exists a constant $M^* > 0$ such that

$$\|y(t)\| \le M^*.$$

Using Lemma 3.1, we can obtain the following generalized Gronwall inequality with some mixed integral terms.

Lemma 3.2. *Let $y \in C(J,X)$ satisfy the following inequality:*

$$\begin{aligned}\|y(t)\| \quad \le \quad & a + b\int_0^t (t-s)^{\alpha-1}\|y(s)\|^{\lambda_1} ds \\ & + c\int_0^T (T-s)^{\alpha-1}\|y(s)\|^{\lambda_2} ds \\ & + d\int_0^T \|y(s)\|^{\lambda_3} ds + e\int_0^T \|y(s)\|^{\lambda_4} ds,\end{aligned} \tag{3.1}$$

where $\alpha \in (1,2)$, $a,b,c,d,e \ge 0$ are constants, $\lambda_1 \in [0, 1-\frac{1}{p}]$, $\lambda_2 \in [0, 1-\frac{1}{p})$, $\lambda_3, \lambda_4 \in [0,1)$, and for some $p > 1, T > 0$ such that $\frac{T^{p(\alpha-1)+1}}{p(\alpha-1)+1} \ge 1$.

Then there exists a constant $M^{**} > 0$ such that

$$\|y(t)\| \le M^{**}.$$

Proof. Let

$$x(t) = \begin{cases} 1, & \|y(t)\| \le 1, \\ y(t), & \|y(t)\| > 1. \end{cases} \tag{3.2}$$

It follows from (3.1), (3.2) and Hölder inequality that

$$\begin{aligned} \|y(t)\| \le \|x(t)\| \le (a+1) + b\int_0^t (t-s)^{\alpha-1}\|x(s)\|^{\lambda_1} ds \\ + c\int_0^T (T-s)^{\alpha-1}\|x(s)\|^{\lambda_2} ds + d\int_0^T \|x(s)\|^{\lambda_3} ds + e\int_0^T \|x(s)\|^{\lambda_4} ds. \\ \le (a+1) + b\left(\int_0^t (t-s)^{p(\alpha-1)} ds\right)^{\frac{1}{p}} \left(\int_0^t \|x(s)\|^{\frac{\lambda_1 p}{p-1}} ds\right)^{\frac{p-1}{p}} \\ + c\left(\int_0^T (T-s)^{p(\alpha-1)} ds\right)^{\frac{1}{p}} \left(\int_0^T \|x(s)\|^{\frac{\lambda_2 p}{p-1}} ds\right)^{\frac{p-1}{p}} \\ + d\int_0^T \|x(s)\|^{\lambda_3} ds + e\int_0^T \|x(s)\|^{\lambda_4} ds \end{aligned}$$

$$\leq (a+1) + b\left[\frac{T^{p(\alpha-1)+1}}{p(\alpha-1)+1}\right]^{\frac{1}{p}} \int_0^t \|x(s)\|^{\frac{\lambda_1 p}{p-1}} ds$$

$$+c\left[\frac{T^{p(\alpha-1)+1}}{p(\alpha-1)+1}\right]^{\frac{1}{p}} \int_0^T \|x(s)\|^{\frac{\lambda_2 p}{p-1}} ds + d\int_0^T \|x(s)\|^{\lambda_3} ds + e\int_0^T \|x(s)\|^{\lambda_4} ds$$

$$\leq (a+1) + b\left[\frac{T^{p(\alpha-1)+1}}{p(\alpha-1)+1}\right]^{\frac{1}{p}} \int_0^t \|x(s)\|^{\frac{\lambda_1 p}{p-1}} ds \ + \overline{c}\int_0^T \|x(s)\|^{\overline{\lambda}} ds$$

where $\overline{c} = \max\left\{c\left[\frac{T^{p(\alpha-1)+1}}{p(\alpha-1)+1}\right]^{\frac{1}{p}}, d, e\right\}$, $\overline{\lambda} = \max\left\{\frac{\lambda_2 p}{p-1}, \lambda_3, \lambda_4\right\}$.

Applying Lemma 3.1, there exists a constant $M^{**} > 0$ such that

$$\|y(t)\| \leq \|x(t)\| \leq M^{**}.$$

The proof is completed. □

Remark 3.3. *For $\alpha \in (0,1)$, we say that the inequality (3.1) is a new type Gronwall inequality with some singular integral terms. In this case, the above priori estimate result is also hold for some $1 < p < \frac{1}{1-\alpha}$ and $T \geq 1$.*

4. Existence results via fixed point method

Before stating and proving the main results, we introduce the following hypotheses.

(H1) For any $u \in X$, $f(t,u)$ is strongly measurable with respect to t on J and for any $t \in J$, $f(t,u)$ is continuous with respect to u on X. $g, h : J \times X \to X$ are continuous functions.

(H2) There exists a constant $\alpha_1 \in (0, \alpha - 1)$ and real valued function $m_1(t) \in L^{\frac{1}{\alpha_1}}(J, R_+)$ such that

$$\|f(t,u_1) - f(t,u_2)\| \leq m_1(t)\|u_1 - u_2\|, \ \textit{for each } t \in J, \textit{ and all } u_1, u_2 \in X.$$

(H3) There exists a constant $\alpha_2 \in (0, \alpha - 1)$ and real valued function $m_2(t) \in L^{\frac{1}{\alpha_2}}(J, R_+)$ such that

$$\|g(t,u_1) - g(t,u_2)\| \leq m_2(t)\|u_1 - u_2\|, \ \textit{for each } t \in J, \textit{ and all } u_1, u_2 \in X.$$

(H4) There exists a constant $\alpha_3 \in (0, \alpha - 1)$ and real valued function $m_3(t) \in L^{\frac{1}{\alpha_3}}(J, R_+)$ such that

$$\|h(t,u_1) - h(t,u_2)\| \leq m_3(t)\|u_1 - u_2\|, \ \textit{for each } t \in J, \textit{ and all } u_1, u_2 \in X.$$

(H5) There exists a constant $\alpha_4 \in (0, \alpha - 1)$ and real valued function $m_4(t) \in L^{\frac{1}{\alpha_4}}(J, R_+)$ such that

$$\|f(t,u)\| \leq m_4(t), \ \textit{for each } t \in J, \textit{ and all } u \in X.$$

For brevity, let

$$M_i = \|m_i\|_{L^{\frac{1}{\alpha_i}}(J,R_+)}, \ i = 1,2,3,4.$$

Our first result is based on Banach contraction principle.

Theorem 4.1. *Assume that (H1)–(H5) hold. Then the system (1.1) has a unique solution on J provided that the function $\Omega : J \to R_+$ such that*

$$\Omega(t) = \frac{2M_1 t^{\alpha-\alpha_1}}{\Gamma(\alpha)(\frac{\alpha-\alpha_1}{1-\alpha_1})^{1-\alpha_1}} + M_2 + M_3 \leq \omega < 1, \ t \in J. \tag{4.1}$$

Proof. Firstly, according to the conditions (H1), it is easy to obtain that $f(t, y(t))$ is a measurable function on J. In light of Hölder inequality and (H5), we obtain that

$$\int_0^t \big\|(t-s)^{\alpha-1} f(s, y(s))\big\| \, ds \quad \leq \quad \frac{T^{\alpha-\alpha_4} M_4}{(\frac{\alpha-\alpha_4}{1-\alpha_4})^{1-\alpha_4}}.$$

Thus, $\|(t-s)^{\alpha-1} f(s, y(s))\|$ is Lebesgue integrable with respect to $s \in [0, t]$ for all $t \in J$ and $y \in C(J, X)$. Then $(t-s)^{\alpha-1} f(s, y(s))$ is Bochner integrable with respect to $s \in [0, t]$ for all $t \in J$ due to Lemma 2.9.

Hence, the fractional BVP (1.1) is equivalent to the following fractional integral equation

$$\begin{aligned} y(t) \quad = \quad & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s)) ds - \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, y(s)) ds \\ & -(\frac{t}{T} - 1) \int_0^T g(s, y(s)) ds + \frac{t}{T} \int_0^T h(s, y(s)) ds. \end{aligned}$$

Since $g, h : J \times X \to X$ are continuous functions, we set

$$G = \int_0^T \|g(s, y(s))\| ds, \ H = \int_0^T \|h(s, y(s))\| ds.$$

Choose

$$r \ \geq \ \frac{2T^{\alpha-\alpha_4} M_4}{\Gamma(\alpha)(\frac{\alpha-\alpha_4}{1-\alpha_4})^{1-\alpha_4}} + G + H,$$

and define $B_r := \{y \in C(J, X) : \|y\| \leq r\}$.

Now we define the operator F on B_r as follows

$$\begin{aligned} (Fy)(t) \quad = \quad & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s)) ds - \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, y(s)) ds \\ & - (\frac{t}{T} - 1) \int_0^T g(s, y(s)) ds + \frac{t}{T} \int_0^T h(s, y(s)) ds. \end{aligned} \tag{4.2}$$

Therefore, the existence of a solution of the fractional BVP (1.1) is equivalent to that the operator F has a fixed point on B_r. We shall use the Banach contraction principle to prove that F has a fixed point. The proof is divided into two steps.

Step 1. $Fy \in C(J, X)$ for every $y \in B_r$.

For every $y \in B_r$ and any $\delta > 0$, by the condition (H5) and Hölder inequality, we get

$$\begin{aligned}
&\|(Fy)(t+\delta) - (Fy)(t)\| \\
\leq\ & \frac{1}{\Gamma(\alpha)} \left(\int_0^t [(t-s)^{\alpha-1} - (t+\delta-s)^{\alpha-1}]^{\frac{1}{1-\alpha_4}} ds \right)^{1-\alpha_4} \left(\int_0^t (m_4(s))^{\frac{1}{\alpha_4}} ds \right)^{\alpha_4} \\
&+ \frac{1}{\Gamma(\alpha)} \left(\int_t^{t+\delta} (t+\delta-s)^{\frac{\alpha-1}{1-\alpha_4}} ds \right)^{1-\alpha_4} \left(\int_t^{t+\delta} (m_4(s))^{\frac{1}{\alpha_4}} ds \right)^{\alpha_4} \\
&+ \frac{\delta}{T\Gamma(\alpha)} \left(\int_0^T (T-s)^{\frac{\alpha-1}{1-\alpha_4}} ds \right)^{1-\alpha_4} \left(\int_0^T (m_4(s))^{\frac{1}{\alpha_4}} ds \right)^{\alpha_4} \\
&+ \frac{\delta}{T} G + \frac{\delta}{T} H \\
\leq\ & \frac{M_4}{\Gamma(\alpha)(\frac{\alpha-\alpha_4}{1-\alpha_4})^{1-\alpha_4}} \left(3\delta^{\alpha-\alpha_4} + T^{\alpha-\alpha_4-1}\delta \right) + \frac{\delta}{T} G + \frac{\delta}{T} H.
\end{aligned}$$

As $\delta \to 0$, the right-hand side of the above inequality tends to zero. Therefore, F is continuous on J, i.e., $Fy \in C(J, X)$.

Moreover, for $y \in B_r$ and all $t \in J$, we get

$$\begin{aligned}
\|(Fy)(t)\| \leq\ & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} m_4(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} m_4(s) ds \\
&+ \int_0^T \|g(s, y(s))\| ds + \int_0^T \|h(s, y(s))\| ds \\
\leq\ & \frac{1}{\Gamma(\alpha)} \left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_4}} ds \right)^{1-\alpha_4} \left(\int_0^t (m_4(s))^{\frac{1}{\alpha_4}} ds \right)^{\alpha_4} \\
&+ \frac{1}{\Gamma(\alpha)} \left(\int_0^T (T-s)^{\frac{\alpha-1}{1-\alpha_4}} ds \right)^{1-\alpha_4} \left(\int_0^T (m_4(s))^{\frac{1}{\alpha_4}} ds \right)^{\alpha_4} \\
&+ \int_0^T \|g(s, y(s))\| ds + \int_0^T \|h(s, y(s))\| ds \\
\leq\ & \frac{2T^{\alpha-\alpha_4} M_4}{\Gamma(\alpha)(\frac{\alpha-\alpha_4}{1-\alpha_4})^{1-\alpha_4}} + G + H \\
\leq\ & r,
\end{aligned}$$

which implies that $\|Fy\|_\infty \leq r$. Thus, we can conclude that for all $y \in B_r$, $Fy \in B_r$. i.e., $F : B_r \to B_r$.

Step 2. F is a contraction mapping on B_r.

For $x, y \in B_r$ and any $t \in J$, using the conditions (H2)–(H4) and Hölder inequality,

$$\|(Fx)(t) - (Fy)(t)\| \leq \frac{\|x-y\|_\infty}{\Gamma(\alpha)} \left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_1}} ds \right)^{1-\alpha_1} \left(\int_0^t (m_1(s))^{\frac{1}{\alpha_1}} ds \right)^{\alpha_1}$$

$$+\frac{\|x-y\|_\infty}{\Gamma(\alpha)}\left(\int_0^T (T-s)^{\frac{\alpha-1}{1-\alpha_1}}ds\right)^{1-\alpha_1}\left(\int_0^T (m_1(s))^{\frac{1}{\alpha_1}}ds\right)^{\alpha_1}$$

$$+\|x-y\|_\infty\int_0^T m_2(s)ds+\|x-y\|_\infty\int_0^T m_3(s)ds$$

$$\leq \frac{2}{\Gamma(\alpha)}\frac{M_1T^{\alpha-\alpha_1}}{(\frac{\alpha-\alpha_1}{1-\alpha_1})^{1-\alpha_1}}\|x-y\|_\infty+M_2\|x-y\|_\infty+M_3\|x-y\|_\infty\leq\omega\|x-y\|_\infty.$$

So we obtain

$$\|Fx-Fy\|_\infty\leq\omega\|x-y\|_\infty.$$

Thus, F is a contraction due to the condition (4.1).

By Banach contraction principle, we can deduce that F has an unique fixed point which is just the unique solution of the fractional BVP (1.1). □

Our second result is based on the well known Schaefer's fixed point theorem via sublinear growth conditions and Lemma 3.2.

We make the following assumptions:

(H6) The functions $f, g, h : J\times X\to X$ are continuous.

(H7) There exist constants $\lambda_1\in[0,1-\frac{1}{p})$ for some p, T satisfy $\frac{T^{p(\alpha-1)+1}}{p(\alpha-1)+1}\geq 1$ and $N_1>0$ such that

$$\|f(t,u)\|\leq N_1(1+\|u\|^{\lambda_1})\ \mathit{for\ each}\ t\in J\ \mathit{and\ all}\ u\in X.$$

(H8) There exist constants $\lambda_2\in[0,1)$ and $N_2>0$ such that

$$\|g(t,u)\|\leq N_2(1+\|u\|^{\lambda_2})\ \mathit{for\ each}\ t\in J\ \mathit{and\ all}\ u\in X.$$

(H9) There exist constants $\lambda_3\in[0,1)$ and $N_3>0$ such that

$$\|h(t,u)\|\leq N_3(1+\|u\|^{\lambda_3})\ \mathit{for\ each}\ t\in J\ \mathit{and\ all}\ u\in X.$$

(H10) For every $t\in J$, the sets $K_f=\{(t-s)^{\alpha-1}f(s,y(s)) : y\in C(J,X), s\in[0,t]\}$, and $K_g=\{g(s,y(s)) : y\in C(J,X), s\in[0,t]\}$, and $K_h=\{h(s,y(s)) : y\in C(J,X), s\in[0,t]\}$ are relatively compact.

Theorem 4.2. *Assume that (H6)–(H10) hold. Then the fractional BVP (1.1) has at least one solution on J.*

Proof. Transform the fractional BVP (1.1) into a fixed point problem. Consider the operator $F : C(J,X)\to C(J,X)$ defined as (4.2). It is obvious that F is well defined due to (H6).

For the sake of convenience, we subdivide the proof into several steps.

Step 1. F is continuous.

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C(J,X)$. Then for each $t \in J$, we have

$$\begin{aligned}
&\|(Fy_n)(t) - (Fy)(t)\| \\
\leq\ & \frac{\|f(\cdot,y_n(\cdot)) - f(\cdot,y(\cdot))\|_\infty}{\Gamma(\alpha)}\left[\int_0^t (t-s)^{\alpha-1}ds + \int_0^T (T-s)^{\alpha-1}ds\right] \\
&+T\|g(\cdot,y_n(\cdot)) - g(\cdot,y(\cdot))\|_\infty + T\|h(\cdot,y_n(\cdot)) - h(\cdot,y(\cdot))\|_\infty \\
\leq\ & \frac{2T^\alpha}{\Gamma(\alpha+1)}\|f(\cdot,y_n(\cdot)) - f(\cdot,y(\cdot))\|_\infty + T\|g(\cdot,y_n(\cdot)) - g(\cdot,y(\cdot))\|_\infty \\
&+T\|h(\cdot,y_n(\cdot)) - h(\cdot,y(\cdot))\|_\infty.
\end{aligned}$$

Since f, g, h are continuous, we have

$$\begin{aligned}
\|Fy_n - Fy\|_\infty \quad &\leq \quad \frac{2T^\alpha}{\Gamma(\alpha+1)}\|f(\cdot,y_n(\cdot)) - f(\cdot,y(\cdot))\|_\infty \\
&\quad\quad + T\|g(\cdot,y_n(\cdot)) - g(\cdot,y(\cdot))\|_\infty + T\|h(\cdot,y_n(\cdot)) - h(\cdot,y(\cdot))\|_\infty \\
&\to \quad 0 \ as\ n \to \infty.
\end{aligned}$$

Step 2. F maps bounded sets into bounded sets in $C(J,X)$.

Indeed, it is enough to show that for any $\eta^* > 0$, there exists a $\ell > 0$ such that for each $y \in B_{\eta^*} = \{y \in C(J,X) : \|y\|_\infty \leq \eta^*\}$, we have $\|Fy\|_\infty \leq \ell$.

For each $t \in J$, using the conditions (H7)–(H9), we get

$$\begin{aligned}
\|(Fy)(t)\| \quad &\leq \quad \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}N_1(1+\|y(s)\|^{\lambda_1})ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}N_1(1+\|y(s)\|^{\lambda_1})ds \\
&\quad + \int_0^T N_2(1+\|y(s)\|^{\lambda_2})ds + \int_0^T N_3(1+\|y(s)\|^{\lambda_3})ds \\
&\leq \quad \frac{2N_1T^\alpha}{\Gamma(\alpha+1)}[1+(\eta^*)^{\lambda_1}] + N_2T[1+(\eta^*)^{\lambda_2}] + N_3T[1+(\eta^*)^{\lambda_3}],
\end{aligned}$$

which implies that

$$\|Fy\|_\infty \leq \frac{2N_1T^\alpha}{\Gamma(\alpha+1)}[1+(\eta^*)^{\lambda_1}] + N_2T[1+(\eta^*)^{\lambda_2}] + N_3T[1+(\eta^*)^{\lambda_3}] := \ell.$$

Step 3. F maps bounded sets into equicontinuous sets of $C(J,X)$.

Let $0 \leq t_1 < t_2 \leq T$, $y \in B_{\eta^*}$, using the condition (H7), we have

$$\begin{aligned}
\|(Fy)(t_2) - (Fy)(t_1)\| &\leq \frac{N_1(1+(\eta^*)^{\lambda_1})}{\Gamma(\alpha)}\int_0^{t_1}[(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}]ds \\
&\quad + \frac{N_1(1+(\eta^*)^{\lambda_1})}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds \\
&\quad + \frac{(t_2-t_1)N_1(1+(\eta^*)^{\lambda_1})}{T\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}ds \\
&\quad + (t_2-t_1)N_2(1+(\eta^*)^{\lambda_2}) + (t_2-t_1)N_3(1+(\eta^*)^{\lambda_3})
\end{aligned}$$

$$\leq \frac{N_1(1+(\eta^*)^{\lambda_1})}{\Gamma(\alpha+1)}(t_2^\alpha - t_1^\alpha + 2(t_2-t_1)^\alpha) + \frac{(t_2-t_1)N_1(1+(\eta^*)^{\lambda_1})T^{\alpha-1}}{\Gamma(\alpha+1)}$$
$$+ (t_2-t_1)N_2(1+(\eta^*)^{\lambda_2}) + (t_2-t_1)N_3(1+(\eta^*)^{\lambda_3}).$$

As $t_2 \to t_1$, the right-hand side of the above inequality tends to zero, therefore F is equicontinuous.

Now, let $\{y_n\}$, $n = 1, 2, \cdots$ be a sequence on B_{η^*}, and

$$(Fy_n)(t) = (F_1y_n)(t) + (F_2y_n)(t) + (F_3y_n)(t),\ t \in J.$$

where

$$\begin{aligned}
(F_1y_n)(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, y_n(s))ds,\ t \in J, \\
(F_2y_n)(t) &= -\frac{t}{T\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} f(s, y_n(s))ds,\ t \in J, \\
(F_3y_n)(t) &= -(\frac{t}{T} - 1)\int_0^T g(s, y_n(s))ds + \frac{t}{T}\int_0^T h(s, y_n(s))ds,\ t \in J.
\end{aligned}$$

In view of the condition (H10) and Lemma 2.10, we know that $\overline{conv}K_f$ is compact. For any $t^* \in J$,

$$\begin{aligned}
(F_1y_n)(t^*) &= \frac{1}{\Gamma(\alpha)}\int_0^{t^*} (t^*-s)^{\alpha-1} f(s, y_n(s))ds \\
&= \frac{1}{\Gamma(\alpha)}\lim_{k\to\infty}\sum_{i=1}^{k}\frac{t^*}{k}(t^* - \frac{it^*}{k})^{\alpha-1} f(\frac{it^*}{k}, y_n(\frac{it^*}{k})) \\
&= \frac{t^*}{\Gamma(\alpha)}\widetilde{\xi_n},
\end{aligned}$$

where

$$\widetilde{\xi_n} = \lim_{k\to\infty}\sum_{i=1}^{k}\frac{1}{k}(t^* - \frac{it^*}{k})^{\alpha-1} f(\frac{it^*}{k}, y_n(\frac{it^*}{k})).$$

Since $\overline{conv}K_f$ is convex and compact, we know that $\widetilde{\xi_n} \in \overline{conv}K_f$. Hence, for any $t^* \in J$, the set $\{(F_1y_n)(t^*)\}$ is relatively compact. From Lemma 2.11, every $\{(F_1y_n)(t)\}$ contains a uniformly convergent subsequence $\{(F_1y_{n_k})(t)\}$, $k = 1, 2, \cdots$ on J. Thus, the set $\{F_1y : y \in B_{\eta^*}\}$ is relatively compact.

Set

$$(\overline{F}_2y_n)(t) = -\frac{t}{T\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, y_n(s))ds,\ t \in J,$$

and for any $t^* \in J$,

$$(\overline{F}_2y_n)(t^*) = -\frac{t^*}{T\Gamma(\alpha)}\int_0^{t^*} (t^*-s)^{\alpha-1} f(s, y_n(s))ds = -\frac{(t^*)^2}{T\Gamma(\alpha)}\widetilde{\xi_n}.$$

Since $\widetilde{\xi_n} \in \overline{conv}K_f$, for any $t^* \in J$, the set $\{(\overline{F}_2y_n)(t^*)\}$ is relatively compact. From Lemma 2.11 again, every $\{(\overline{F}_2y_n)(t)\}$ contains a uniformly convergent subsequence

$\{(\overline{F}_2 y_{n_k})(t)\}$, $k=1,2,\cdots$ on J. Particulary, $\{(F_2 y_n)(t)\}$ contains a uniformly convergent subsequence $\{(F_2 y_{n_k})(t)\}$, $k=1,2,\cdots$ on J. Thus, the set $\{F_2 y : y \in B_{\eta^*}\}$ is relatively compact.

Similarly, one can verifty the set $\{F_3 y : y \in B_{\eta^*}\}$ is relatively compact due to the condition (H10) again. As a result, the set $\{Fy : y \in B_{\eta^*}\}$ is relatively compact.

As a consequence of Step 1–3, we can conclude that F is continuous and completely continuous.

Step 4. A priori bounds.

Now it remains to show that the set

$$E(F) = \{y \in C(J, X) : y = \lambda^* F y, \text{ for some } \lambda^* \in [0,1]\}$$

is bounded.

Let $y \in E(F)$, then $y = \lambda^* F y$ for some $\lambda^* \in [0,1]$. Thus, for each $t \in J$, we have

$$\begin{aligned} y(t) &= \lambda^* \bigg(\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s)) ds \\ &\quad - \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, y(s)) ds \\ &\quad - (\frac{t}{T} - 1) \int_0^T g(s, y(s)) ds + \frac{t}{T} \int_0^T h(s, y(s)) ds \bigg). \end{aligned}$$

For each $t \in J$, we have

$$\begin{aligned} \|y(t)\| &\le \frac{2N_1 T^\alpha}{\Gamma(\alpha+1)} + TN_2 + TN_3 + \frac{N_1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|y(s)\|^{\lambda_1} ds \\ &\quad + \frac{N_1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \|y(s)\|^{\lambda_1} ds \\ &\quad + N_2 \int_0^T \|y(s)\|^{\lambda_2} ds + N_3 \int_0^T \|y(s)\|^{\lambda_3} ds. \end{aligned}$$

Applying the Lemma 3.2, there exists a $M^{**} > 0$ such that

$$\|y(t)\| \le M^{**}, \ t \in J.$$

which implies that

$$\|y\|_\infty \le M^{**}.$$

This shows that the set $E(F)$ is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that F has a fixed point which is a solution of the fractional BVP (1.1). □

In the following theorem we apply the nonlinear alternative of Leray-Schauder type to derive the existence results for the solution of the fractional BVP (1.1).

We need the following conditions.

(H11) There exist a constant $\beta_1 \in (0, \alpha - 1)$, real valued function $\phi_f(t) \in L^{\frac{1}{\beta_1}}(J, R_+)$ and continuous and nondecreasing $\psi : [0, +\infty) \to (0, +\infty)$ such that

$$\|f(t,u)\| \le \phi_f(t) \psi(\|u\|) \ \textit{for each } t \in J \textit{ and all } u \in X.$$

(H12) There exist a constant $\beta_2 \in (0, \alpha - 1)$, real valued function $\phi_g(t) \in L^{\frac{1}{\beta_2}}(J, R_+)$ and continuous and nondecreasing $\psi^* : [0, +\infty) \to (0, +\infty)$ such that

$$\|g(t, u)\| \leq \phi_g(t)\psi^*(\|u\|) \text{ for each } t \in J \text{ and all } u \in X.$$

(H13) There exist a constant $\beta_3 \in (0, \alpha - 1)$, real valued function $\phi_h(t) \in L^{\frac{1}{\beta_3}}(J, R_+)$ and continuous and nondecreasing $\psi^{**} : [0, +\infty) \to (0, +\infty)$ such that

$$\|h(t, u)\| \leq \phi_h(t)\psi^{**}(\|u\|) \text{ for each } t \in J \text{ and all } u \in X.$$

(H14) There exists a constant $N^* > 0$ such that

$$\frac{N^*}{\frac{2\psi(N^*)T^{\alpha-\beta_1}(1-\beta_1)^{1-\beta_1}\vartheta}{\Gamma(\alpha)(\alpha-\beta_1)^{1-\beta_1}} + a\psi^*(N^*) + b\psi^{**}(N^*)} > 1, \tag{4.3}$$

where $\vartheta = \|\phi_f\|_{L^{\frac{1}{\beta_1}}(J,R_+)}$, $a = \int_0^T \phi_g(s)ds$, $b = \int_0^T \phi_h(s)ds$.

Theorem 4.3. *Assume that (H6), (H10)–(H14) hold. Then the fractional BVP (1.1) has at least one solution.*

Proof. Consider the operator F defined in Theorem 4.1. In Theorem 4.2, we have shown that F is continuous and completely continuous. Repeating the same process in Step 4 in Theorem 4.2, using Hölder inequality again, for each $t \in J$, we have

$$\begin{aligned} \|y(t)\| &\leq \frac{\psi(\|y\|_\infty)}{\Gamma(\alpha)} \left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\beta_1}} ds \right)^{1-\beta_1} \left(\int_0^t (\phi_f(s))^{\frac{1}{\beta_1}} ds \right)^{\beta_1} \\ &\quad + \frac{\psi(\|y\|_\infty)}{\Gamma(\alpha)} \left(\int_0^T (T-s)^{\frac{\alpha-1}{1-\beta_1}} ds \right)^{1-\beta_1} \left(\int_0^T (\phi_f(s))^{\frac{1}{\beta_1}} ds \right)^{\beta_1} \\ &\quad + \psi^*(\|y\|_\infty) \int_0^T \phi_g(s)ds + \psi^{**}(\|y\|_\infty) \int_0^T \phi_h(s)ds \\ &\leq \frac{2\psi(\|y\|_\infty)T^{\alpha-\beta_1}(1-\beta_1)^{1-\beta_1}\vartheta}{\Gamma(\alpha)(\alpha-\beta_1)^{1-\beta_1}} + a\psi^*(\|y\|_\infty) + b\psi^{**}(\|y\|_\infty). \end{aligned}$$

Thus we have

$$\frac{\|y\|_\infty}{\frac{2\psi(\|y\|_\infty)T^{\alpha-\beta_1}(1-\beta_1)^{1-\beta_1}\vartheta}{\Gamma(\alpha)(\alpha-\beta_1)^{1-\beta_1}} + a\psi^*(\|y\|_\infty) + b\psi^{**}(\|y\|_\infty)} \leq 1.$$

Because of the condition (H14), there exists a $N^* > 0$ such that $\|y\|_\infty \neq N^*$.

Let $U = \{y \in C(J, X) : \|y\|_\infty < N^*\}$. The operator $F : \overline{U} \to C(J, X)$ is continuous and completely continuous. From the choice of U, there is no $y \in \partial U$ such that $y = \lambda^* F(y)$, $\lambda^* \in [0, 1]$. As a consequence of the nonlinear alternative of Leray-Schauder type, we deduce that F has a fixed point $y \in \overline{U}$, which implies that the fractional BVP (1.1) has at least one solution $y \in C(J, X)$. $\square$

5. Example

We consider the following BVP with integral conditions of fractional differential equation

$$\begin{cases} {}^cD^\alpha y(t) = \frac{1}{L+e^t}\frac{|y(t)|}{1+|y(t)|},\ \alpha\in(1,2),\ t\in J=[0,T],\ L>0, \\ y(0) = \int_0^T \frac{\lambda_1|y(s)|}{1+|y(s)|}ds,\ \lambda_1>0, \\ y(T) = \int_0^T \frac{\lambda_2|y(s)|}{1+|y(s)|}ds,\ \lambda_2>0. \end{cases} \tag{5.1}$$

For all $(t,y)\in J\times X$, set

$$f(t,y) = \frac{1}{L+e^t}\frac{|y|}{1+|y|},\ g(t,y) = \frac{\lambda_1|y|}{1+|y|},\ h(t,y) = \frac{\lambda_2|y|}{1+|y|}.$$

Let $y_1,y_2\in X$ and $t\in J$, we have

$$\begin{aligned} \|f(t,y_1)-f(t,y_2)\| &\le m_1(t)\|y_1-y_2\|,\ m_1(t):=\frac{1}{1+L}, \\ \|g(y_1)-g(y_2)\| &\le m_2(t)\|y_1-y_2\|,\ m_2(t):=\lambda_1, \\ \|h(y_1)-h(y_2)\| &\le m_3(t)\|y_1-y_2\|,\ m_3(t):=\lambda_2, \\ \|f(t,y)\| &\le m_4(t),\ m_4(t):=\frac{1}{1+L},\ \text{for all } y\in X \text{ and each } t\in J. \end{aligned}$$

It is obviously that our assumptions in Theorem 4.1 can be satisfied by choosing a sufficient large L, small enough T, λ_1, λ_2 and some $\alpha_i\in(0,\alpha-1)$, $i=1,2,3$ such that

$$\frac{2\|\frac{1}{1+L}\|_{L^{\frac{1}{\alpha_1}}(J,R_+)}T^{\alpha-\alpha_1}}{\Gamma(\alpha)(\frac{\alpha-\alpha_1}{1-\alpha_1})^{1-\alpha_1}} + \|\lambda_1\|_{L^{\frac{1}{\alpha_2}}(J,R_+)} + \|\lambda_2\|_{L^{\frac{1}{\alpha_3}}(J,R_+)} < 1. \tag{5.2}$$

For example: set

$$T=\sqrt{\pi},\ \alpha=\frac{3}{2},\ \alpha_1=\alpha_2=\alpha_3=\frac{1}{3},\ L=99,\ \lambda_1=\lambda_2=\frac{2}{25}\times\left(\frac{4}{7}\right)^{\frac{2}{3}}\times\pi^{\frac{1}{4}}$$

and note that $\Gamma(\frac{3}{2})=\frac{\sqrt{\pi}}{2}$, then

$$\text{the condition (5.2)} \Longleftrightarrow \frac{5}{25}\times\left(\frac{4}{7}\right)^{\frac{2}{3}}\times\pi^{\frac{1}{4}}<1 \Longleftrightarrow \frac{4}{7}<\left(\frac{5}{\sqrt[4]{\pi}}\right)^{\frac{3}{2}}.$$

It is obvious that $5^4>\pi$ which implies that $\left(\frac{5}{\sqrt[4]{\pi}}\right)^{\frac{3}{2}}>1$, then the condition (5.2) holds. Therefore, the problem (5.1) has an unique solution. □

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