Fixed Point Theory, 14(2013), No. 2, 497-506
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

SUZUKI TYPE COMMON FIXED POINT THEOREMS AND APPLICATIONS

SHYAM LAL SINGH*, RENU CHUGH** AND RAJ KAMAL***

*L. M. S. Govt. Autonomous Postgraduate College
Rishikesh 249201, India
E-mail: vedicmri@gmail.com

,*Department of Mathematics, Maharshi Dayanand University
Rohtak 124001, India
E-mail: **chughrenu@yahoo.com, ***rajkamalpillania@yahoo.com

Abstract. Common fixed point theorems for Suzuki type conditions for a pair of maps on a metric space are obtained. Existence of a common solution for a class of functional equations arising in dynamic programming is also discussed.

Key Words and Phrases: Fixed point; Banach contraction theorem, functional equations, dynamic programming.

2010 Mathematics Subject Classification: 47H10, 54H25.

1. Introduction

Generalizing the classical Banach contraction theorem (Bct) and some other results by Chatterjea [5], Hardy and Rogers [8], Kannan [10], Reich [15], Rus [17] and others, Wong [24] obtained the following common fixed point theorem for a pair of maps on a complete metric space.

Theorem 1.1 *Let S and T be maps from a complete metric space (X, d) to itself. Suppose that their exist nonnegative real numbers a_1, a_2, a_3, a_4, a_5 which satisfy*

(i) $a_1 + a_2 + a_3 + a_4 + a_5 < 1$;
(ii) $a_2 = a_3$ *or* $a_4 = a_5$.

Assume for each $x, y \in X$,

(iii) $d(Sx, Ty) \leq a_1 d(x, y) + a_2 d(x, Sx) + a_3 d(y, Ty) + a_4 d(x, Ty) + a_5 d(y, Sx)$.

Then S and T have a unique common fixed point.

Recently Suzuki [23] obtained a forceful generalization of the Bct. It has several important outcomes and applications (see, for instance, [6, 7, 11, 12, 14, 20, 22]). The following generalization of the Bct is essentially due to Moţ and Petruşel [12].

*Corresponding author's address: 21, Govind Nagar, Rishikesh 249201, India.
Tel.:+91-135-2431624 E-mail: `vedicmri@gmail.com`.

Theorem 1.2 *Let T be a map from a complete metric space X into itself. Suppose that their exist nonnegative real numbers a_1, a_2, a_3 such that $a_1+a_2+a_3 < 1$. Assume, for each $x, y \in X$,*

$$\left(\frac{(1-a_2-a_3)}{(1+a_1)}\right) d(x,Tx) \le d(x,y)$$

implies

$$d(Tx,Ty) \le a_1 d(x,y) + a_2 d(x,Tx) + a_3 d(y,Ty).$$

Then T has a unique fixed point.

The purpose of this paper is to obtain a generalization of Theorems 1.1 and 1.2. Applications regarding the existence of common solutions of certain functional equations are also discussed.

2. Common fixed point theorems

We begin with the following result.

Theorem 2.1 *Let S and T be maps from a complete metric space X to itself. Suppose that there exist nonnegative real numbers a_1, a_2, a_3, a_4, a_5 which satisfy (i) and (ii). Assume for each $x, y \in X$,*

$$\beta \min\{d(x,Sx), d(y,Ty)\} \le d(x,y) \quad \textit{implies (iii)}, \tag{2.1}$$

where $\beta = \left(\dfrac{(1-a_2-a_3-a_4-a_5)}{(1+a_1+a_4+a_5)}\right)$.

Then S and T have a unique common fixed point.

Proof. Pick $x_0 \in X$. Construct a sequence $x_n$in X such that

$$x_{2n+1} = Sx_{2n},\ x_{2n+2} = Tx_{2n+1}, \quad n = 0,1,2,\ldots.$$

If, for any n, $d(x_{2n}, x_{2n+1}) \le d(x_{2n+1}, x_{2n+2})$, then

$$\begin{aligned}\beta \min\{d(x_{2n},Sx_{2n}), d(x_{2n+1},Tx_{2n+1})\} &= \beta \min\{d(x_{2n},x_{2n+1}), d(x_{2n+1},x_{2n+2})\}\\ &= \beta d(x_{2n},x_{2n+1})\\ &\le d(x_{2n},x_{2n+1}).\end{aligned}$$

If, for any n, $d(x_{2n+1}, x_{2n+2}) \le d(x_{2n}, x_{2n+1})$, then

$$\begin{aligned}\beta \min\{d(x_{2n},Sx_{2n}), d(x_{2n+1},Tx_{2n+1})\} &= \beta \min\{d(x_{2n},x_{2n+1}), d(x_{2n+1},x_{2n+2})\}\\ &= \beta d(x_{2n+1},x_{2n+2})\\ &\le d(x_{2n+1},x_{2n+2})\\ &\le d(x_{2n},x_{2n+1}).\end{aligned}$$

Hence, in either case by (2.1),

$$\begin{aligned}d(x_{2n+1},x_{2n+2}) &= d(Sx_{2n},Tx_{2n+1})\\ &\le a_1 d(x_{2n},x_{2n+1}) + a_2 d(x_{2n},Sx_{2n}) + a_3 d(x_{2n+1},Tx_{2n+1})\\ &\quad + a_4 d(x_{2n},Tx_{2n+1}) + a_5 d(x_{2n+1},Sx_{2n})\\ &= a_1 d(x_{2n},x_{2n+1}) + a_2 d(x_{2n},x_{2n+1}) + a_3 d(x_{2n+1},x_{2n+2})\\ &\quad + a_4 d(x_{2n},x_{2n+1}) + a_5 d(x_{2n+1},x_{2n+2}).\end{aligned}$$

This yields

$$d(x_{2n+1}, x_{2n+2}) \le r\ d(x_{2n}, x_{2n+1}), \tag{2.2}$$

where $r = \dfrac{(a_1 + a_2 + a_4)}{(1 - a_3 - a_4)}$.
Analogously

$$d(x_{2n+2}, x_{2n+3}) \le s\ d(x_{2n+1}, x_{2n+2}), \tag{2.3}$$

where $s = \dfrac{(a_1 + a_3 + a_5)}{(1 - a_2 - a_5)}$.
Therefore inductively by (2.2) and (2.3),

$$d(x_{2n+1}, x_{2n+2}) \le r(rs)^n d(x_0, x_1) \quad \text{and} \quad d(x_{2n+2}, x_{2n+3}) \le (rs)^{n+1} d(x_0, x_1).$$

Since $rs < 1$ and

$$\sum_{n=0}^{\infty} d(x_n, x_{n+1}) \le (1 + r) \sum_{n=0}^{\infty} (rs)^n d(x_0, x_1),$$

$\{x_n\}$ is Cauchy sequence. Since the space X is complete, it has a limit in X. Call it z.
Now we show that z is fixed point of S. Let N be the set of natural numbers.
Since $x_n \to z$, there exists $n_0 \in N$ such that

$$d(z, x_n) \le \frac{1}{3} d(z, y) \quad \text{for} \ \ y \ne z \ \ \text{and all} \ \ n \ge n_0.$$

Then as in [23, p. 1862] and [12, p. 3376],

$$\begin{aligned} \beta d(x_{2n}, Sx_{2n}) &\le d(x_{2n}, Sx_{2n}) \\ &= d(x_{2n}, x_{2n+1}) \\ &\le d(x_{2n}, z) + d(z, x_{2n+1}) \\ &\le \frac{2}{3} d(z, y) \\ &= d(z, y) - \frac{1}{3} d(z, y) \\ &\le d(z, y) - d(x_{2n}, z) \\ &\le d(x_{2n}, y). \end{aligned} \tag{2.4}$$

If $d(x_{2n}, Sx_{2n}) \le d(y, Ty)$ then by (2.4),

$$\beta \min\{d(x_{2n}, Sx_{2n}), d(y, Ty)\} \le d(x_{2n}, y).$$

If $d(y, Ty) \le d(x_{2n}, Sx_{2n})$, then again by (2.4),

$$\begin{aligned} \beta \min\{d(x_{2n}, Sx_{2n}), d(y, Ty)\} &= \beta d(y, Ty) \\ &\le \beta d(x_{2n}, Sx_{2n}) \\ &\le d(x_{2n}, y). \end{aligned}$$

So for any $n \in N$,

$$\beta \min\{d(x_{2n}, Sx_{2n}), d(y, Ty)\} \le d(x_{2n}, y).$$

Hence by (2.1),

$$\begin{aligned} d(Sx_{2n}, Ty) &\leq a_1 d(x_{2n}, y) + a_2 d(x_{2n}, Sx_{2n}) + a_3 d(y, Ty) \\ &\quad + a_4 d(x_{2n}, Ty) + a_5 d(y, Sx_{2n}) \\ &= a_1 d(x_{2n}, y) + a_2 d(x_{2n}, x_{2n+1}) + a_3 d(y, Ty) \\ &\quad + a_4 d(x_{2n}, Ty) + a_5 d(y, x_{2n+1}). \end{aligned}$$

Making $n \to \infty$,

$$d(z, Ty) \leq (a_1 + a_5) d(z, y) + a_3 d(y, Ty) + a_4 d(z, Ty).$$

Therefore

$$\begin{aligned} d(y, Ty) &\leq d(z, y) + d(z, Ty) \\ &\leq (1 + a_1 + a_5) d(z, y) + a_3 d(y, Ty) + a_4 d(z, Ty) \\ &\leq (1 + a_1 + a_5) d(z, y) + a_3 d(y, Ty) + a_4 d(z, y) + a_4 d(y, Ty). \end{aligned}$$

This gives

$$\left(\frac{(1 - a_3 - a_4)}{(1 + a_1 + a_4 + a_5)} \right) d(y, Ty) \leq d(z, y).$$

Therefore

$$\beta d(y, Ty) \leq \left(\frac{(1 - a_3 - a_4)}{(1 + a_1 + a_4 + a_5)} \right) d(y, Ty) \leq d(z, y). \tag{2.5}$$

Now we consider two cases.
(I) Suppose $\min\{d(z, Sz), d(y, Ty)\} = d(y, Ty)$.
Then by (2.5),

$$\beta \min\{d(z, Sz), d(y, Ty)\} \leq d(z, y),$$

and by (2.1),

$$d(Sz, Ty) \leq a_1 d(z, y) + a_2 d(z, Sz) + a_3 d(y, Ty) + a_4 d(z, Ty) + a_5 d(y, Sz).$$

Taking $y = x_{2n+1}$, this gives

$$\begin{aligned} d(Sz, Tx_{2n+1}) &\leq a_1 d(z, x_{2n+1}) + a_2 d(z, Sz) + a_3 d(x_{2n+1}, Tx_{2n+1}) \\ &\quad + a_4 d(z, Tx_{2n+1}) + a_5 d(x_{2n+1}, Sz). \end{aligned}$$

Making $n \to \infty$,

$$d(z, Sz) \leq a_2 d(z, Sz) + a_5 d(z, Sz).$$

This proves $Sz = z$.
(II) Suppose $\min\{d(z, Sz), d(y, Ty)\} = d(z, Sz)$.
Then by (2.5),

$$\beta \min\{d(z, Sz), d(y, Ty)\} = \beta d(z, Sz) \leq \beta d(y, Ty) \leq d(z, y).$$

We shall use the obvious fact of this case, viz., $\beta d(z, Sz) \leq d(z, y)$ as well.

Therefore by (2.1),

$$\begin{aligned} d(Sz,Ty) &\le a_1 d(z,y) + a_2 d(z,Sz) + a_3 d(y,Ty) + a_4 d(z,Ty) + a_5 d(y,Sz) \\ &\le a_1 d(z,y) + \frac{a_2}{\beta} d(z,y) + a_3 d(y,Ty) + a_4 d(z,Ty) + a_5 d(y,Sz). \end{aligned}$$

Taking $y = x_{2n+1}$ in this relation and passing to the limit, we obtain $d(Sz,z) \le a_5 d(z,Sz)$, and z is a fixed point of S.

Thus we have shown that z is a fixed point of S in both the cases. Analogously, we can prove that z is a fixed point of T as well.

Assume that y is another common fixed point of S and T. Then as

$$\beta \min\{d(z,Sz), d(y,Ty)\} = 0 \le d(z,y).$$

We have by (2.1),

$$\begin{aligned} d(z,y) = d(Sz,Ty) &\le a_1 d(z,y) + a_2 d(z,Sz) + a_3 d(y,Ty) + a_4 d(z,Ty) + a_5 d(y,Sz) \\ &\le (a_1 + a_4 + a_5) d(z,y), \end{aligned}$$

yielding $z = y$.

This completes the proof.

Corollary 2.1 *Theorem 1.2.*

Proof. It comes from Theorem 2.1 when $S = T$ and $a_4 = a_5 = 0$.

The following result is a generalization of Theorem 1.2 and certain results from [8, 9, 15, 16, 18] and others.

Corollary 2.2 *Let T be a map from a complete metric space X to itself. Suppose that there exist nonnegative real numbers a_1, a_2, a_3, a_4, a_5 which satisfy (i) and (ii). Assume, for each $x, y \in X$,*

$$\beta d(x,Tx) \le d(x,y)$$

implies

$$d(Tx,Ty) \le a_1 d(x,y) + a_2 d(x,Tx) + a_3 d(y,Ty) + a_4 d(x,Ty) + a_5 d(y,Tx),$$

where $\beta = \left(\dfrac{(1 - a_2 - a_3 - a_4 - a_5)}{(1 + a_1 + a_4 + a_5)} \right)$.

Then T has a unique fixed point.

Proof. It comes from Theorem 2.1 when $S = T$.

The following example shows the generality of Theorem 2.1 over Theorem 1.1.

Example 2.1 Let $X = \{(1,1), (1,4), (4,1), (4,5), (5,4)\}$ and d is defined by $d[(x_1,x_2),(y_1,y_2)] = |x_1 - y_1| + |x_2 - y_2|$.

Let S and T be such that

$$S(x_1,x_2) = \begin{cases} (x_1,1) & \text{if } x_1 \le x_2 \\ (1,x_2) & \text{if } x_1 > x_2 \end{cases} \quad \text{and} \quad T(x_1,x_2) = \begin{cases} (1,x_1) & \text{if } x_1 \le x_2 \\ (x_2,1) & \text{if } x_1 > x_2 \end{cases}$$

Then S and T do not satisfy the condition (iii) of Theorem 1.1 at $x = (4,5)$, $y = (4,5)$. However, this is readily verified that all the hypotheses of Theorem 2.1 are satisfied for the maps S and T.

We remark that, by virtue of symmetry in the contractive condition (iii), one may take $a_2 = a_3$ and $a_4 = a_5$ (see, for instance, [8, 16, 17] and [18, p. 98]). So, we have the following result generalizing several common fixed point theorems (see, for instance, Rus [18, Theorem 9.8.1]).

Theorem 2.2 *Let S and T be maps from a complete metric space X to itself. Suppose that there exist nonnegative real numbers a, b, c which satisfy*

(iv) $a + 2b + 2c < 1$.

Assume for each $x, y \in X$,

$$\gamma \min\{d(x, Sx), d(y, Ty)\} \leq d(x, y)$$

implies

(v) $d(Sx, Ty) \leq ad(x, y) + b[d(x, Sx) + d(y, Ty)] + c[d(x, Ty) + d(y, Sx)]$,

where $\gamma = \left(\dfrac{(1 - 2b - 2c)}{(1 + a + 2c)}\right)$.

Then S and T have a unique common fixed point.

3. Applications

In all that follows we assume that Y and Z are Banach spaces, $W \subseteq Y$ and $D \subseteq Z$. Let R denote the field of reals, $g_1, g_2 : W \times D \to R$ and $H_1, H_2 : W \times D \times R \to R$. Viewing W and D as the state and decision spaces respectively, the problem of dynamic programming reduces to the problem of solving functional equations:

$$p_i = \sup_{y \in D}\{g_i(x, y) + H_i(x, y, p_i(x, y))\}, \quad x \in W,\ i = 1, 2. \tag{3.1}$$

In the multistage process, some functional equations arise in a natural way (cf. Bellman [2] and Bellman and Lee [3] ; see also [1, 4, 13, 19]). The intent of this section is to study the existence of the common solution of the functional equations (3.1).

Let $B(W)$ denote the set of all bounded real-valued functions on W. For an arbitrary $h \in B(W)$, define $\|h\| = \sup_{x \in W} |h(x)|$. Then $(B(W), \|\cdot\|)$ is a Banach space. Suppose that the following conditions hold:

(DP-1) H_1, H_2, g_1 and g_2 are bounded.

(DP-2) Let a, b, c and γ be defined as in Theorem 2.2. Assume for every $(x, y) \in W \times D$, $h, k \in B(W)$ and $t \in W$,

$$\gamma \min\{|h(t) - A_1h(t)|, |k(t) - A_2k(t)|\} \leq |h(t) - k(t)|$$

implies

$$\begin{aligned} &|H_1(x, y, h(t)) - H_2(x, y, k(t))| \\ &\quad \leq a|h(t) - k(t)| + b[|h(t) - A_1h(t)| + |k(t) - A_2k(t)|] \\ &\qquad + c[|h(t) - A_2k(t)| + |k(t) - A_1h(t)|] \end{aligned}$$

where A_1, A_2 are defined as follows:

$$A_ih(x) = \sup_{y \in D} H_i(x, y, h(x, y)), \quad x \in W,\ h \in B(W),\ i = 1, 2.$$

Theorem 3.1 *Assume the conditions (DP-1) and (DP-2). Then the functional equations (3.1), $i = 1, 2$, have a unique common solution in $B(W)$.*

Proof. For any $h, k \in B(W)$, let $d(h,k) := \sup\{|h(x) - k(x)| : x \in W\}$. Let λ be any arbitrary positive number and $h_1, h_2 \in B(W)$. Pick $x \in W$ and choose $y_1, y_2 \in D$ such that

$$A_i h_i < H_i(x, y_i, h_i(x_i)) + \lambda, \tag{3.2}$$

where $x_i = (x, y_i)$, $i = 1, 2$.
Further,

$$A_1 h_1 \geq H_1(x, y_2, h_1(x_2)), \tag{3.3}$$

$$A_2 h_2 \geq H_2(x, y_1, h_2(x_1)). \tag{3.4}$$

Therefore, the first inequality in (DP-2) becomes

$$\gamma \min\{|h_1(x) - A_1 h_1(x)|, |h_2(x) - A_2 h_2(x)|\} \leq |h_1(x) - h_2(x)|, \tag{3.5}$$

and this together with (3.2) and (3.4) implies

$$\begin{aligned} A_1 h_1 - A_2 h_2 &< H_1(x, y_1, h_1(x_1)) - H_2(x, y_1, h_2(x_1)) + \lambda \\ &\leq |H_1(x, y_1, h_1(x_1)) - H_2(x, y_1, h_2(x_1))| + \lambda \\ &\leq a|h_1(t) - h_2(t)| + b[|h_1(t) - A_1 h_1(t)| + |h_2(t) - A_2 h_2(t)|] \\ &\quad + c[|h_1(t) - A_2 h_2(t)| + |h_2(t) - A_1 h_1(t)|] + \lambda. \end{aligned} \tag{3.6}$$

Similarly, (3.2), (3.3) and (3.5) imply

$$\begin{aligned} A_2 h_2(x) - A_1 h_1(x) &\leq a|h_1(t) - h_2(t)| + b[|h_1(t) - A_1 h_1(t)| + |h_2(t) - A_2 h_2(t)|] \\ &\quad + c[|h_1(t) - A_2 h_2(t)| + |h_2(t) - A_1 h_1(t)|] + \lambda. \end{aligned} \tag{3.7}$$

So from (3.6) and (3.7), we obtain

$$\begin{aligned} |A_1 h_1(x) - A_2 h_2(x)| &\leq a|h_1(t) - h_2(t)| + b[|h_1(t) - A_1 h_1(t)| + |h_2(t) - A_2 h_2(t)|] \\ &\quad + c[|h_1(t) - A_2 h_2(t)| + |h_2(t) - A_1 h_1(t)|] + \lambda. \end{aligned} \tag{3.8}$$

Since this inequality is true for any $x \in W$, and $\lambda > 0$ is arbitrary, on taking supremum we find from (3.5) and (3.8) that

$$\gamma \min\{d(h_1, A_1 h_1), d(h_2, A_2 h_2)\} \leq d(h_1, h_2)$$

implies

$$\begin{aligned} d(A_1 h_1, A_2 h_2) &\leq a|h_1(t) - h_2(t)| + b[|h_1(t) - A_1 h_1(t)| + |h_2(t) - A_2 h_2(t)|] \\ &\quad + c[|h_1(t) - A_2 h_2(t)| + |h_2(t) - A_1 h_1(t)|], \end{aligned}$$

that is

$$\begin{aligned} d(A_1 h_1, A_2 h_2) &\leq a d(h_1, h_2) + b[d(h_1, A_1 h_1) + d(h_2, A_2 h_2)] \\ &\quad + c[d(h_1, A_2 h_2) + d(h_2, A_1 h_1)]. \end{aligned}$$

Therefore, Theorem 2.1 applies, wherein A_1 and A_2 correspond respectively to the maps S and T. So A_1 and A_2 have a unique common fixed point h^*, that is, $h^*(x)$ is the unique bounded common solution of the functional equations (3.1), $i = 1, 2$.

The following result is a variant of Singh and Mishra [21, Corollary 4.2] which in turn extends certain results from [1, 3, 4].

Corollary 3.1 *Suppose that the following conditions hold.*

(i) G and g are bounded.

(ii) There exists $r \in [0,1)$ such that for every $(x,y) \in W \times D$, $h,k \in B(W)$ and $t \in W$,

$$\gamma \min |h(t) - Kh(t)| \leq |h(t) - k(t)|$$

implies

$$|G(x,y,h(t)) - G(x,y,k(t))| \leq a|h(t) - k(t)| + 2b[|h(t) - Kh(t)| + |k(t) - Kk(t)|] + 2c[|h(t) - Kk(t)| + |k(t) - Kh(t)|].$$

where K is defined as

$$Kh(t) = \sup_{y \in D}\{g(t,y) + G(t,y,h(t,y))\}, \quad t \in W, h \in B(W).$$

Then the functional equation (3.1) with $H_1 = H_2 = G$ and $g_1 = g_2 = g$ possesses a unique bounded solution in W.

Proof. It comes from Theorem 3.1 when $g_1 = g_2 = g$ and $H_1 = H_2 = G$.

Acknowledgement. The authors thank the referee for his appreciation and suggestions to improve upon the original typescript. The first author (SLS) acknowledges the support by the UGC, New Delhi under Emeritus Fellowship.

References

[1] R. Baskaran, P.V. Subrahmanyam, *A note on the solution of a class of functional equations*, Applicable Anal., **22**(1986), no. 3-4, 235-241.

[2] R. Bellman, *Methods of Nonlinear Analysis*, Vol. II, Academic Press, New York, 1973.

[3] R. Bellman, E.S. Lee, *Functional equations in dynamic programming*, Aequations Math., **17**(1978), no. 1, 1-18.

[4] P.C. Bhakta, S. Mitra, *Some existence theorems for functional equations arising in dynamic programming*, J. Math. Anal. Appl., **98**(1984), no. 2, 348-362.

[5] S.K. Chatterjea, *Fixed-point theorems*, C.R. Acad. Bulgare Sci., **25**(1972), 727-730.

[6] S. Dhompongsa, H. Yingtaweesittikul, *Fixed points for multivalued mappings and the metric completeness*, Fixed Point Theory Appl., **2009**(2009), Art. ID 972395, 15 pp.

[7] D. Dorić, R. Lazović, *Some Suzuki-type fixed point theorems for generalized multivalued mappings and applications*, Fixed Point Theory Appl., **2011**(2011), 2011:40, 13 pp.

[8] G.E. Hardy, T.D. Rogers, *A generalization of a fixed point theorem of Reich*, Canad. Math. Bull., **16**(1973), 201-206.

[9] R. Kannan, *Some results on fixed points*, Bull. Calcutta Math. Soc., **60**(1968), 71-76.

[10] R. Kannan, *Some results on fixed points. II*, Amer. Math. Monthly, **76**(1969), 405-408.

[11] M. Kikkawa, T. Suzuki, *Three fixed point theorems for generalized contractions with constants in complete metric spaces*, Nonlinear Anal., **69**(2008), no. 9, 2942-2949.

[12] G. Mot, A. Petruşel, *Fixed point theory for a new type of contractive multi-valued operators*, Nonlinear Anal., **70**(2009), no. 9, 3371-3377.

[13] H.K. Pathak, Y.J. Cho, S.M. Kang, B.S. Lee, *Fixed point theorems for compatible mappings of type (P) and applications to dynamic programming*, Le Mathematiche, **50**(1995), no. 1, 15-33.

[14] O. Popescu, *Two fixed point theorems for generalized contractions with constants in complete metric space*, Central Europ. J. Math., **7**(2009), no. 3, 529-538.

[15] S. Reich, *Remarks on fixed points II*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., **53**(1972), no. 8, 250—254.

[16] B.E. Rhoades, *A comparison of various definitions of contractive mappings*, Trans. Amer. Math. Soc., **226**(1977), 257-290.
[17] I.A. Rus, *On common fixed points*, Studia Univ. Babes-Bolyai Ser. Math.-Mech., **18**(1973), 31-33.
[18] I.A. Rus, *Generalized Contractions and Applications*, Cluj-Napoca, 2001.
[19] S.L. Singh, S.N. Mishra, *On a Ljubomic Ćirić fixed point theorem for nonexpansive type maps with applications*, Indian J. Pure Appl. Math., **33**(2002), no. 4, 531-542.
[20] S.L. Singh, H.K. Pathak, S.N. Mishra, *On a Suzuki type general fixed point theorem with applications*, Fixed Point Theory Appl., **2010**(2010), 15 pp.
[21] S.L. Singh, S.N. Mishra, *Coincidence theorems for certain classes of hybrid contractions*, Fixed Point Theory Appl., **2010**(2010), Art. ID 898109, 14 pp.
[22] S.L. Singh, S.N. Mishra, *Remarks on recent fixed point theorems*, Fixed Point Theory Appl., **2010**(2010), Art. ID 452905, 18 pp.
[23] T. Suzuki, *A generalized Banach contraction principle that characterizes metric completeness*, Proc. Amer. Math. Soc., **136**(2008), no. 5, 1861-1869.
[24] C.S. Wong, *Common fixed points of two mappings*, Pacific J. Math., **48**(1973), 299-312.

Received: October 24, 2011; Accepted: March 9, 2012.