SUZUKI TYPE COMMON FIXED POINT THEOREMS AND APPLICATIONS

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Abstract. Common fixed point theorems for Suzuki type conditions for a pair of maps on a metric space are obtained. Existence of a common solution for a class of functional equations arising in dynamic programming is also discussed.

Key Words and Phrases: Fixed point; Banach contraction theorem, functional equations, dynamic programming.

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1. Introduction

Generalizing the classical Banach contraction theorem (Bct) and some other results by Chatterjea [5], Hardy and Rogers [8], Kannan [10], Reich [15], Rus [17] and others, Wong [24] obtained the following common fixed point theorem for a pair of maps on a complete metric space.

**Theorem 1.1** Let $S$ and $T$ be maps from a complete metric space $(X,d)$ to itself. Suppose that their exist nonnegative real numbers $a_1, a_2, a_3, a_4, a_5$ which satisfy

(i) $a_1 + a_2 + a_3 + a_4 + a_5 < 1$;
(ii) $a_2 = a_3$ or $a_4 = a_5$.

Assume for each $x, y \in X$,

(iii) $d(Sx, Ty) \leq a_1d(x, y) + a_2d(x, Sx) + a_3d(y, Ty) + a_4d(x, Ty) + a_5d(y, Sx)$.

Then $S$ and $T$ have a unique common fixed point.

Recently Suzuki [23] obtained a forceful generalization of the Bct. It has several important outcomes and applications (see, for instance, [6, 7, 11, 12, 14, 20, 22]). The following generalization of the Bct is essentially due to Moţ and Petruşel [12].

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Theorem 1.2 Let $T$ be a map from a complete metric space $X$ into itself. Suppose that there exist nonnegative real numbers $a_1, a_2, a_3$ such that $a_1 + a_2 + a_3 < 1$. Assume, for each $x, y \in X$,

$$\left( \frac{1 - a_2 - a_3}{1 + a_1} \right) d(x, Tx) \leq d(x, y)$$

implies

$$d(Tx, Ty) \leq a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty).$$

Then $T$ has a unique fixed point.

The purpose of this paper is to obtain a generalization of Theorems 1.1 and 1.2. Applications regarding the existence of common solutions of certain functional equations are also discussed.

2. Common fixed point theorems

We begin with the following result.

Theorem 2.1 Let $S$ and $T$ be maps from a complete metric space $X$ to itself. Suppose that there exist nonnegative real numbers $a_1, a_2, a_3, a_4, a_5$ which satisfy (i) and (ii). Assume for each $x, y \in X$,

$$\beta \min\{d(x, Sx), d(y, Ty)\} \leq d(x, y) \quad \text{implies (iii)}, \quad (2.1)$$

where $\beta = \left( \frac{1 - a_2 - a_3 - a_4 - a_5}{1 + a_1 + a_4 + a_5} \right)$. Then $S$ and $T$ have a unique common fixed point.

Proof. Pick $x_0 \in X$. Construct a sequence $x_n$ in $X$ such that

$$x_{n+1} = Sx_n, \quad x_{n+2} = Tx_{n+1}, \quad n = 0, 1, 2, \ldots$$

If, for any $n$, $d(x_{2n}, x_{2n+1}) \leq d(x_{2n+1}, x_{2n+2})$, then

$$\beta \min\{d(x_{2n}, Sx_{2n}), d(x_{2n+1}, Tx_{2n+1})\} = \beta \min\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\} = \beta d(x_{2n}, x_{2n+1}) \leq d(x_{2n}, x_{2n+1}).$$

If, for any $n$, $d(x_{2n+1}, x_{2n+2}) \leq d(x_{2n}, x_{2n+1})$, then

$$\beta \min\{d(x_{2n}, Sx_{2n}), d(x_{2n+1}, Tx_{2n+1})\} = \beta \min\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\} = \beta d(x_{2n+1}, x_{2n+2}) \leq d(x_{2n+1}, x_{2n+2}) \leq d(x_{2n}, x_{2n+1}).$$

Hence, in either case by (2.1),

$$d(x_{2n+1}, x_{2n+2}) = d(Sx_{2n}, Tx_{2n+1}) \leq a_1 d(x_{2n}, x_{2n+1}) + a_2 d(x_{2n}, Sx_{2n}) + a_3 d(x_{2n+1}, Tx_{2n+1}) + a_4 d(x_{2n}, Tx_{2n+1}) + a_5 d(x_{2n+1}, Sx_{2n}) = a_1 d(x_{2n}, x_{2n+1}) + a_2 d(x_{2n}, x_{2n+1}) + a_3 d(x_{2n+1}, x_{2n+2}) + a_4 d(x_{2n}, x_{2n+1}) + a_5 d(x_{2n+1}, x_{2n+2}).$$
This yields
\[ d(x_{2n+1}, x_{2n+2}) \leq r \cdot d(x_{2n}, x_{2n+1}), \]
where \( r = \frac{a_1 + a_2 + a_4}{1 - a_3 - a_4}. \)

Analogously
\[ d(x_{2n+2}, x_{2n+3}) \leq s \cdot d(x_{2n+1}, x_{2n+2}), \]
where \( s = \frac{a_1 + a_3 + a_5}{1 - a_2 - a_5}. \)

Therefore inductively by (2.2) and (2.3),
\[ d(x_{2n+1}, x_{2n+2}) \leq r^n \cdot d(x_0, x_1) \quad \text{and} \quad d(x_{2n+2}, x_{2n+3}) \leq (rs)^n \cdot d(x_0, x_1). \]

Since \( rs < 1 \) and
\[ \sum_{n=0}^{\infty} d(x_n, x_{n+1}) \leq (1 + r) \sum_{n=0}^{\infty} (rs)^n d(x_0, x_1), \]
\( \{x_n\} \) is Cauchy sequence. Since the space \( X \) is complete, it has a limit in \( X \). Call it \( z \).

Now we show that \( z \) is fixed point of \( S \). Let \( N \) be the set of natural numbers.
Since \( x_n \to z \), there exists \( n_0 \in N \) such that
\[ d(z, x_n) \leq \frac{1}{3} d(z, y) \quad \text{for} \quad y \neq z \quad \text{and all} \quad n \geq n_0. \]

Then as in [23, p. 1862] and [12, p. 3376],
\[ \beta d(x_{2n}, Sx_{2n}) \leq d(x_{2n}, Sx_{2n}) \]
\[ = d(x_{2n}, x_{2n+1}) \]
\[ \leq d(x_{2n}, z) + d(z, x_{2n+1}) \]
\[ \leq \frac{2}{3} d(z, y) \]
\[ = d(z, y) - \frac{1}{3} d(z, y) \]
\[ \leq d(z, y) - d(x_{2n}, z) \]
\[ \leq d(x_{2n}, y). \]

If \( d(x_{2n}, Sx_{2n}) \leq d(y, Ty) \) then by (2.4),
\[ \beta \min\{d(x_{2n}, Sx_{2n}), d(y, Ty)\} \leq d(x_{2n}, y). \]

If \( d(y, Ty) \leq d(x_{2n}, Sx_{2n}) \), then again by (2.4),
\[ \beta \min\{d(x_{2n}, Sx_{2n}), d(y, Ty)\} = \beta d(y, Ty) \]
\[ \leq \beta d(x_{2n}, Sx_{2n}) \]
\[ \leq d(x_{2n}, y). \]

So for any \( n \in N \),
\[ \beta \min\{d(x_{2n}, Sx_{2n}), d(y, Ty)\} \leq d(x_{2n}, y). \]
Hence by (2.1),
\[
d(Sx_{2n}, Ty) \leq a_1d(x_{2n}, y) + a_2d(x_{2n}, Sx_{2n}) + a_3d(y, Ty) + a_4d(x_{2n}, Ty) + a_5d(y, Sx_{2n})
\]
\[
= a_1d(x_{2n}, y) + a_2d(x_{2n}, x_{2n+1}) + a_3d(y, Ty) + a_4d(x_{2n}, Ty) + a_5d(y, x_{2n+1}).
\]
Making \(n \to \infty\),
\[
d(z, Ty) \leq (a_1 + a_5)d(z, y) + a_3d(y, Ty) + a_4d(z, Ty).
\]
Therefore
\[
d(y, Ty) \leq d(z, y) + d(z, Ty)
\]
\[
\leq (1 + a_1 + a_5)d(z, y) + a_3d(y, Ty) + a_4d(z, Ty)
\]
\[
\leq (1 + a_1 + a_5)d(z, y) + a_3d(y, Ty) + a_4d(z, y) + a_4d(y, Ty).
\]
This gives
\[
\left(\frac{1 - a_3 - a_4}{1 + a_1 + a_4 + a_5}\right)d(y, Ty) \leq d(z, y).
\]
Therefore
\[
\beta d(y, Ty) \leq \left(\frac{1 - a_3 - a_4}{1 + a_1 + a_4 + a_5}\right)d(y, Ty) \leq d(z, y). \tag{2.5}
\]
Now we consider two cases.
(I) Suppose \(\min\{d(z, Sz), d(y, Ty)\} = d(y, Ty)\).
Then by (2.5),
\[
\beta \min\{d(z, Sz), d(y, Ty)\} \leq d(z, y),
\]
and by (2.1),
\[
d(Sz, Ty) \leq a_1d(z, y) + a_2d(z, Sz) + a_3d(y, Ty) + a_4d(z, Ty) + a_5d(y, Sz).
\]
Taking \(y = x_{2n+1}\), this gives
\[
d(Sz, Tx_{2n+1}) \leq a_1d(z, x_{2n+1}) + a_2d(z, Sz) + a_3d(x_{2n+1}, Ty) + a_4d(z, Ty) + a_5d(x_{2n+1}, Sz).
\]
Making \(n \to \infty\),
\[
d(z, Sz) \leq a_2d(z, Sz) + a_5d(z, Sz).
\]
This proves \(Sz = z\).
(II) Suppose \(\min\{d(z, Sz), d(y, Ty)\} = d(z, Sz)\).
Then by (2.5),
\[
\beta \min\{d(z, Sz), d(y, Ty)\} = \beta d(z, Sz) \leq \beta d(y, Ty) \leq d(z, y).
\]
We shall use the obvious fact of this case, viz., \(\beta d(z, Sz) \leq d(z, y)\) as well.
Therefore by (2.1),
\[ d(Sz, Ty) \leq a_1 d(z, y) + a_2 d(z, Sz) + a_3 d(y, Ty) + a_4 d(z, Ty) + a_5 d(y, Sz) \]
\[ \leq a_1 d(z, y) + \frac{a_2}{\beta} d(z, y) + a_3 d(y, Ty) + a_4 d(z, Ty) + a_5 d(y, Sz). \]

Taking \( y = x_{2n+1} \) in this relation and passing to the limit, we obtain \( d(Sz, z) \leq a_5 d(z, Sz) \), and \( z \) is a fixed point of \( S \).

Thus we have shown that \( z \) is a fixed point of \( S \) in both the cases. Analogously, we can prove that \( z \) is a fixed point of \( T \) as well.

Assume that \( y \) is another common fixed point of \( S \) and \( T \). Then as
\[ \beta \min\{d(z, Sz), d(y, Ty)\} = 0 \leq d(z, y). \]

We have by (2.1),
\[ d(z, y) = d(Sz, Ty) \leq a_1 d(z, y) + a_2 d(z, Sz) + a_3 d(y, Ty) + a_4 d(z, Ty) + a_5 d(y, Sz) \]
\[ \leq (a_1 + a_4 + a_5) d(z, y), \]
yielding \( z = y \).

This completes the proof.

**Corollary 2.1 Theorem 1.2.**

*Proof.* It comes from Theorem 2.1 when \( S = T \) and \( a_4 = a_5 = 0 \).

The following result is a generalization of Theorem 1.2 and certain results from [8, 9, 15, 16, 18] and others.

**Corollary 2.2** Let \( T \) be a map from a complete metric space \( X \) to itself. Suppose that there exist nonnegative real numbers \( a_1, a_2, a_3, a_4, a_5 \) which satisfy (i) and (ii).

Assume, for each \( x, y \in X \),
\[ \beta d(x, Tx) \leq d(x, y) \]
implies
\[ d(Tx, Ty) \leq a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + a_5 d(y, Tx), \]
where
\[ \beta = \left( \frac{1 - a_2 - a_3 - a_4 - a_5}{1 + a_1 + a_4 + a_5} \right). \]

Then \( T \) has a unique fixed point.

*Proof.* It comes from Theorem 2.1 when \( S = T \).

The following example shows the generality of Theorem 2.1 over Theorem 1.1.

**Example 2.1** Let \( X = \{(1, 1), (1, 4), (4, 1), (4, 5), (5, 4)\} \) and \( d \) is defined by \( d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2| \).

Let \( S \) and \( T \) be such that
\[ S(x_1, x_2) = \begin{cases} (x_1, 1) & \text{if } x_1 \leq x_2 \\ (1, x_2) & \text{if } x_1 > x_2 \end{cases} \]
and
\[ T(x_1, x_2) = \begin{cases} (1, x_1) & \text{if } x_1 \leq x_2 \\ (x_2, 1) & \text{if } x_1 > x_2 \end{cases} \]

Then \( S \) and \( T \) do not satisfy the condition (iii) of Theorem 1.1 at \( x = (4, 5) \), \( y = (4, 5) \). However, this is readily verified that all the hypotheses of Theorem 2.1 are satisfied for the maps \( S \) and \( T \).

We remark that, by virtue of symmetry in the contractive condition (iii), one may take \( a_2 = a_3 \) and \( a_4 = a_5 \) (see, for instance, [8, 16, 17] and [18, p. 98]). So, we
have the following result generalizing several common fixed point theorems (see, for instance, Rus [18, Theorem 9.8.1]).

**Theorem 2.2** Let $S$ and $T$ be maps from a complete metric space $X$ to itself. Suppose that there exist nonnegative real numbers $a$, $b$, $c$ which satisfy

(iv) $a + 2b + 2c < 1$.

Assume for each $x, y \in X$,

$$
\gamma \min\{d(x, Sx), d(y, Ty)\} \leq d(x, y)
$$

implies

(v) $d(Sx, Ty) \leq ad(x, y) + b[d(x, Sx) + d(y, Ty)] + c[d(x, Ty) + d(y, Sx)],$

where $\gamma = \left( \frac{1 - 2b - 2c}{1 + a + 2c} \right)$.

Then $S$ and $T$ have a unique common fixed point.

3. Applications

In all that follows we assume that $Y$ and $Z$ are Banach spaces, $W \subseteq Y$ and $D \subseteq Z$. Let $R$ denote the field of reals, $g_1, g_2 : W \times D \rightarrow R$ and $H_1, H_2 : W \times D \times R \rightarrow R$. Viewing $W$ and $D$ as the state and decision spaces respectively, the problem of dynamic programming reduces to the problem of solving functional equations:

$$
p_i = \sup_{y \in D} \{g_i(x, y) + H_i(x, y, p_i(x, y))\}, \quad x \in W, \ i = 1, 2.
$$

(3.1)

In the multistage process, some functional equations arise in a natural way (cf. Bellman [2] and Bellman and Lee [3]; see also [1, 4, 13, 19]). The intent of this section is to study the existence of the common solution of the functional equations (3.1).

Let $B(W)$ denote the set of all bounded real-valued functions on $W$. For an arbitrary $h \in B(W)$, define $\|h\| = \sup_{x \in W} |h(x)|$. Then $(B(W), \| \cdot \|)$ is a Banach space.

Suppose that the following conditions hold:

(DP-1) $H_1, H_2, g_1$ and $g_2$ are bounded.

(DP-2) Let $a, b, c$ and $\gamma$ be defined as in Theorem 2.2. Assume for every $(x, y) \in W \times D$, $h, k \in B(W)$ and $t \in W$,

$$
\gamma \min\{|h(t) - A_1h(t)|, |k(t) - A_2k(t)|\} \leq |h(t) - k(t)|
$$

implies

$$
|H_1(x, y, h(t)) - H_2(x, y, k(t))|
\leq a|h(t) - k(t)| + b[H_1(x, y, h(t)) - A_1h(t)| + |k(t) - A_2k(t)|
+ c[|h(t) - A_2k(t)| + |k(t) - A_1h(t)|]
$$

where $A_1, A_2$ are defined as follows:

$$
A_i h(x) = \sup_{y \in D} H_i(x, y, h(x, y)), \quad x \in W, \ h \in B(W), \ i = 1, 2.
$$
Theorem 3.1 Assume the conditions (DP-1) and (DP-2). Then the functional equations (3.1), \(i = 1, 2\), have a unique common solution in \(B(W)\).

Proof. For any \(h, k \in B(W)\), let \(d(h, k) := \sup \{|h(x) - k(x)| : x \in W\}\). Let \(\lambda\) be any arbitrary positive number and \(h_1, h_2 \in B(W)\). Pick \(x \in W\) and choose \(y_1, y_2 \in D\) such that

\[ A_i h_i < H_i(x, y_i, h_i(x_i)) + \lambda, \quad (3.2) \]

where \(x_i = (x, y_i), i = 1, 2\).

Further,

\[ A_1 h_1 \geq H_1(x, y_2, h_1(x_2)), \quad (3.3) \]

\[ A_2 h_2 \geq H_2(x, y_1, h_2(x_1)). \quad (3.4) \]

Therefore, the first inequality in (DP-2) becomes

\[ \gamma \min \{|h_1(x) - A_1 h_1(x)|, |h_2(x) - A_2 h_2(x)|\} \leq |h_1(x) - h_2(x)|, \quad (3.5) \]

and this together with (3.2) and (3.4) implies

\[ A_1 h_1 - A_2 h_2 < H_1(x, y_1, h_1(x_1)) - H_2(x, y_1, h_2(x_1)) + \lambda \]

\[ \leq |H_1(x, y_1, h_1(x_1)) - H_2(x, y_1, h_2(x_1))| + \lambda \]

\[ \leq a|h_1(t) - h_2(t)| + b|h_1(t) - A_1 h_1(t)| + |h_2(t) - A_2 h_2(t)| \]

\[ + c|[h_1(t) - A_2 h_2(t)] + |h_2(t) - A_1 h_1(t)| + \lambda. \quad (3.6) \]

Similarly, (3.2), (3.3) and (3.5) imply

\[ A_2 h_2(x) - A_1 h_1(x) \leq a|h_1(t) - h_2(t)| + b|[h_1(t) - A_1 h_1(t)] + |h_2(t) - A_2 h_2(t)| \]

\[ + c|[h_1(t) - A_2 h_2(t)] + |h_2(t) - A_1 h_1(t)| + \lambda. \quad (3.7) \]

So from (3.6) and (3.7), we obtain

\[ |A_1 h_1(x) - A_2 h_2(x)| \leq a|h_1(t) - h_2(t)| + b|[h_1(t) - A_1 h_1(t)] + |h_2(t) - A_2 h_2(t)| \]

\[ + c|[h_1(t) - A_2 h_2(t)] + |h_2(t) - A_1 h_1(t)| + \lambda. \quad (3.8) \]

Since this inequality is true for any \(x \in W\), and \(\lambda > 0\) is arbitrary, on taking supremum we find from (3.5) and (3.8) that

\[ \gamma \min \{d(h_1, A_1 h_1), d(h_2, A_2 h_2)\} \leq d(h_1, h_2) \]

implies

\[ d(A_1 h_1, A_2 h_2) \leq a|h_1(t) - h_2(t)| + b|[h_1(t) - A_1 h_1(t)] + |h_2(t) - A_2 h_2(t)| \]

\[ + c|[h_1(t) - A_2 h_2(t)] + |h_2(t) - A_1 h_1(t)|, \]

that is

\[ d(A_1 h_1, A_2 h_2) \leq ad(h_1, h_2) + b[d(h_1, A_1 h_1) + d(h_2, A_2 h_2)] \]

\[ + c[d(h_1, A_2 h_2) + d(h_2, A_1 h_1)]. \]

Therefore, Theorem 2.1 applies, wherein \(A_1\) and \(A_2\) correspond respectively to the maps \(S\) and \(T\). So \(A_1\) and \(A_2\) have a unique common fixed point \(h^*\), that is, \(h^*(x)\) is the unique bounded common solution of the functional equations (3.1), \(i = 1, 2\).
The following result is a variant of Singh and Mishra [21, Corollary 4.2] which in turn extends certain results from [1, 3, 4].

**Corollary 3.1** Suppose that the following conditions hold.

(i) $G$ and $g$ are bounded.

(ii) There exists $r \in [0, 1)$ such that for every $(x, y) \in W \times D$, $h, k \in B(W)$ and $t \in W$,

\[
\gamma \min |h(t) - Kh(t)| \leq |h(t) - k(t)|
\]

implies

\[
|G(x, y, h(t)) - G(x, y, k(t))| \leq a|h(t) - k(t)| + 2b|h(t) - Kh(t)| + |k(t) - Kk(t)| + 2c|h(t) - Kk(t)| + |k(t) - Kh(t)|.
\]

where $K$ is defined as

\[
Kh(t) = \sup_{y \in D} \{g(t, y) + G(t, y, h(t))\}, \quad t \in W, h \in B(W).
\]

Then the functional equation (3.1) with $H_1 = H_2 = G$ and $g_1 = g_2 = g$ possesses a unique bounded solution in $W$.

**Proof.** It comes from Theorem 3.1 when $g_1 = g_2 = g$ and $H_1 = H_2 = G$.

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**References**


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