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FIRST ORDER MULTIVALUED PROBLEMS ON TIME SCALES

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Abstract. We present an existence result for first order dynamic inclusions on time scales with a periodic boundary value condition, assuming the existence of a solution tube for the considered inclusion and the Henstock integrability of the multifunction on the right-hand side.
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1. Introduction

The time scales theory, introduced in 1988 in the PhD Thesis of S. Hilger (see [24]), allows a unified treatment of continuous and discrete problems, therefore it has recently received a lot of attention (for a survey, we refer to [4], [5] and the references therein).

At the same time, the Henstock integral (see [22], [9], [31] for functions defined on a real interval or [12] for functions on time scales) enlarges the spectrum of considered problems, since the case of very oscillating functions can be covered by this theory of integration (but not by classical theories).

In the present paper, we obtain in the Henstock integrability setting an existence result for periodic dynamic inclusions of first order

$$\begin{array}{ll} x^{\Delta}(t) \in F(t,x(t)), & \Delta - a.e.\ t \in [0,1]_{\mathbb{T}}, \\ x(0) = x(1). & \end{array} \tag{1.1}$$

where $F : [0,1]_{\mathbb{T}} \times \mathbb{R}^n \to \mathcal{P}_{kc}(\mathbb{R}^n)$ and the symbol $\mathcal{P}_{kc}(\mathbb{R}^n)$ stands for the family of nonempty, compact convex subsets of $\mathbb{R}^n$.

Inclusions of first order on time scales have been studied in the particular case $n = 1$ in [11] (via Leray-Schauder nonlinear alternative for multi-valued maps) or in [2] (using the method of upper and lower solutions), while the multi-dimensional case has been treated in [20]. In all these references, Bochner integrability conditions have been imposed.

After introducing the notations and giving some basic properties of (single- or set-valued) functions on time scales as well as some results concerning the Henstock integral in this setting, we will give the main result by assuming the integrability in Henstock sense for the right-hand side and the existence of a solution tube. This notion, that generalizes to the multi-dimensional case the notions of upper and lower solutions on the real line, has been introduced in the study of first order differential inclusions in [26], see also [17], [18].

On time scales, for existence results obtained by the solution tube method we refer to [21] (in the particular single-valued case) or to the more recent papers [19], [20]. To the best of our knowledge, our result is the only one concerning first order differential inclusions on time scales using the Henstock integral.

2. Preliminaries

We start with some basic elements of time scale theory; for a survey on this subject, we refer the reader to [4] or [5] and to references therein.

A time scale $\mathbb{T}$ is a nonempty closed set of real numbers $\mathbb{R}$, with the subspace topology inherited from the standard topology of $\mathbb{R}$ (for example $\mathbb{T} = \mathbb{R}, \mathbb{T} = \mathbb{N}$ or $\mathbb{T} = q^{\mathbb{Z}} = \{q^t : t \in \mathbb{Z}\}$, where $q > 1$). For two points a, b in $\mathbb{T}$ we denote by $[a,b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}$, respectively $[a,b)_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t < b\}$ the time scale intervals. The key elements in developing the time scales theory are presented in the sequel.

Definition 1.1 The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ are defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$, respectively $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$. Also, $\inf \emptyset = \sup \mathbb{T}$ (i.e. $\sigma(M) = M$ if $\mathbb{T}$ has a maximum M) and $\sup \emptyset = \inf \mathbb{T}$ (i.e. $\rho(m) = m$ if $\mathbb{T}$ has a minimum m).

A point $t \in \mathbb{T}$ is called right dense, right scattered, left dense, left scattered, dense, respectively isolated if $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$, $\rho(t) < t$, $\rho(t) = t = \sigma(t)$ and $\rho(t) < t < \sigma(t)$, respectively. Also, we will use the function $\mu(t) = \sigma(t) - t$ that is called the graininess function.

When considering σ one obtains the Δ part of the theory, while ρ is used for the ∇ part. We will be concerned only with the Δ-theory.
We give an auxiliary result.

Lemma 1.2 The function $g : [a,b]_{\mathbb{T}} \to \mathbb{R}_+$, $g(t) = \dfrac{1}{1+\mu(t)}$ is of bounded variation.

Proof. For any partition $a = t_0 < ... < t_n = b$ of $[a,b]_{\mathbb{T}}$,

$$\begin{aligned}\sum_{i=1}^{n} |g(t_i) - g(t_{i-1})| &= \sum_{i=1}^{n} \left| \frac{1}{1+\mu(t_i)} - \frac{1}{1+\mu(t_{i-1})} \right| \\ &\leq 2 \sum_{i=1}^{n} \mu(t_i).\end{aligned}$$

As $(t_i)_i \subset [a,b]$ it follows that the series $\sum_{i=1}^{\infty} \mu(t_i)$ is convergent and so, the function g is indeed of bounded variation.

Definition 1.3 Let $f : \mathbb{T} \to \mathbb{R}^n$ and $t \in \mathbb{T}$. The Δ-derivative $f^{\Delta}(t)$ is the element of $\mathbb{R}^n$ (if it exists) with the property that for any $\varepsilon > 0$ there exists a neighborhood of t on which

$$\left\| f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s] \right\| \leq \varepsilon |\sigma(t) - s|.$$

In this situation, f is called Δ-differentiable at the point t.

Several simple properties of Δ-derivatives were proved in [5] (Theorem 1.3):

i) f is continuous at the points where it is Δ-differentiable;

ii) if f is continuous at the right-scattered point t, then f is Δ-differentiable at t and

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)};$$

iii) if t is right-dense, then f is Δ-differentiable at t if and only if the limit

$$\lim_{s \to t, s > t} \frac{f(s) - f(t)}{s - t}$$

exists and is finite. In this case, its value equals to $f^{\Delta}(t)$.

The following chain rule is available on time scales:

Lemma 1.4 (Theorem 1.87 in [4]) Assume $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, $g : \mathbb{T} \to \mathbb{R}$ is continuous and Δ-differentiable. Then there exists $c \in [t, \sigma(t)]$ such that

$$(f \circ g)^{\Delta}(t) = f'(g(c)) g^{\Delta}(t).$$

Notice that the time scale calculus allows the unification (and a generalization) of treatment of differential and difference equations since, in particular,

(i) $f^{\Delta} = f'$ is the usual derivative if $\mathbb{T} = \mathbb{R}$,

(ii) $f^{\Delta} = \Delta f$ is the usual forward difference operator if $\mathbb{T} = \mathbb{Z}$.

The space $C([a,b]_{\mathbb{T}}, \mathbb{R}^n)$ of continuous functions is endowed with the usual (Banach space) norm $\|f\|_C = \sup\limits_{t \in [a,b]_{\mathbb{T}}} \|f(t)\|$.

Recall the following notion:

Definition 1.5 A function $f : \mathbb{T} \to \mathbb{R}^n$ is said to be rd-continuous provided it is continuous at right-dense points and its left-sided limits exist (and are finite) at all left-dense points. The set of such functions is denoted by $C_{rd}(\mathbb{T}, \mathbb{R}^n)$ and the set of functions that are Δ-differentiable with rd-continuous Δ-derivative by $C^1_{rd}(\mathbb{T}, \mathbb{R}^n)$.

The symbol μ_{Δ} stands for the Lebesgue measure on $\mathbb{T}$ (for its definition and properties we refer the reader to [8]). For properties of Riemann delta-integral we refer to [23] and for Lebesgue integral on time scales to [3], [4], [5] or [23].

Concerning the Henstock-type integrals, as in the case where $\mathbb{T} = \mathbb{R}$ (see [9]), in general Banach spaces two different vector-valued integrals of Henstock-type where introduced in literature. However, in our (finite-dimensional) case, these two notions are equivalent.

In order to recall them, let $\delta = (\delta_L, \delta_R)$ be a Δ-gauge, that is a pair of positive functions such that $\delta_L(t) > 0$ on $(a,b]$, $\delta_R(t) > 0$ and $\delta_R(t) \geq \sigma(t) - t$ on $[a,b)$.

A division $\mathcal{D} = \{[x_{i-1}, x_i]_{\mathbb{T}}; \xi_i \ , i = 1, 2, \dots n\}$ of $[a,b]_{\mathbb{T}}$ (that is, a partition together with an associate system of intermediary points) is δ-fine whenever:

$$\xi_i \in [x_{i-1}, x_i] \subset [\xi_i - \delta_L(\xi_i), \xi_i + \delta_R(\xi_i)], \forall 1 \leq i \leq n.$$

The Cousin's Lemma for time scale domains (Lemma 1.9 in [27]) yields that such a division exists for arbitrary positive pair of functions.

Definition 1.6([12], see also [9], [31], [6], [13] for the particular case $\mathbb{T} = \mathbb{R}$)
i) A function $f \ : \ [a,b]_{\mathbb{T}} \to \mathbb{R}^n$ is Henstock-Δ-integrable on $[a,b]_{\mathbb{T}}$ if there exists an element $(H) \int_{[a,b]_{\mathbb{T}}} f(s)\Delta s \in \mathbb{R}^n$ satisfying the following property: given $\varepsilon > 0$, there exists a Δ-gauge δ_ε on $[a,b]_{\mathbb{T}}$ such that

$$\left\| \sum_{i=1}^{n} f(\xi_i)\mu_\Delta([x_{i-1}, x_i]_{\mathbb{T}}) - (H) \int_{[a,b]_{\mathbb{T}}} f(s)\Delta s \right\| < \varepsilon$$

for every δ_ε-fine division $\mathcal{D} = \{[x_{i-1}, x_i]_{\mathbb{T}}, \xi_i\}$ of $[a,b]_{\mathbb{T}}$. We call it the Henstock-Δ-integral of f on $[a,b]_{\mathbb{T}}$.
ii) A function $f \ : \ [a,b)_{\mathbb{T}} \to \mathbb{R}^n$ is Henstock-Δ-integrable on $[a,b)_{\mathbb{T}}$ if there exists $(H) \int_{[a,b)_{\mathbb{T}}} f(s)\Delta s \in \mathbb{R}^n$ such that for every $\varepsilon > 0$, there exists a Δ-gauge δ_ε with the property that

$$\left\| \sum_{i=1}^{n} f(\xi_i)\mu_\Delta([x_{i-1}, x_i)_{\mathbb{T}}) - (H) \int_{[a,b)_{\mathbb{T}}} f(s)\Delta s \right\| < \varepsilon$$

for every δ_ε-fine division $\mathcal{D} = \{[x_{i-1}, x_i)_{\mathbb{T}}, \xi_i\}$ of $[a,b)_{\mathbb{T}}$. We call it the Henstock-Δ-integral of f on $[a,b)_{\mathbb{T}}$.

If the space $\mathbb{R}^n$ is replaced by $\mathbb{R}$ in the preceding definition, we obtain the Henstock-Kurzweil (shortly, HK) Δ-integral.

On the other hand, a family of Henstock Δ-integrable functions is said to be uniformly Henstock Δ-integrable if the Δ-gauge δ_ε can be chosen to be the same for all elements of the family.

Remark 1.7 It was proved (in [9], see also [31]) that in the particular case where $\mathbb{T} = \mathbb{R}$ if f is Henstock-integrable, then it is measurable and its primitive (H) $\int_0^{\cdot} f(s)ds$ is continuous and a.e. differentiable.

In order to present a similar result on time scales, we need to refer to [8]. As said there, the integrability of a function on time scales is equivalent to the integrability of its extension (defined below) to a real interval. More precisely, if the time scale $\mathbb{T}$ is contained in a real interval $[a,b]$, then a function $f : \mathbb{T} \to \mathbb{R}^n$ is integrable if and only if the function $\hat{f} : [a,b] \to \mathbb{R}^n$ given by

$$\hat{f}(t) = \begin{cases} f(t), \text{ if } t \in \mathbb{T}; \\ f(t_i), \text{ if } t \in (t_i, \sigma(t_i)) \text{ for } i \in R_{\mathbb{T}}. \end{cases}$$

is integrable (here the set $R_{\mathbb{T}}$ is the set of all right-scattered points that is, by Lemma 3.1 in [8], at most countable) and in this case

$$\int_{\mathbb{T}} f(s)\Delta s = \int_{[a,b]} \hat{f}(s)ds.$$

Notice that in [8] the Lebesgue integral is considered, but the discussion is identical for any other integral. Using this property and Remark 1.7 one can prove, following the same method as in Proposition 2.19 in [21], the announced result.

Proposition 1.8 Let $g : [a, b]_{\mathbb{T}} \to \mathbb{R}^n$ be Henstock-Δ-integrable. Then its primitive

$$G(t) = (\mathrm{H}) \int_{[a,t)_{\mathbb{T}}} g(s)\Delta s$$

is Δ-a.e. differentiable and $G^{\Delta} = g$, Δ-a.e.

We denote the space of Henstock-Δ-integrable $\mathbb{R}^n$-valued functions by $\mathcal{H}([a, b]_{\mathbb{T}}, \mathbb{R}^n)$ and we provide it with the Alexiewicz norm:

$$\|f\|_A = \sup_{t\in[a,b]_{\mathbb{T}}} \left\| (\mathrm{H}) \int_{[a,t]_{\mathbb{T}}} f(s)\Delta s \right\|.$$

In [28] it was proved that T is a linear continuous functional on the space of real HK-integrable functions if and only if there exists a real function g of bounded variation such that, for every HK-integrable function f, $T(f) = (\mathrm{HK}) \int_a^b f(s)g(s)ds$. Similarly, it can be shown that

Proposition 1.9 If $f : [a, b)_{\mathbb{T}} \to \mathbb{R}^n$ is Henstock-Δ-integrable and $g : [a, b)_{\mathbb{T}} \to \mathbb{R}$ is of bounded variation, then fg is Henstock-Δ-integrable.

Before passing to the set-valued case, let us give a convergence result on time scales:

Theorem 1.10 Let $(g_n)_{n\in\mathbb{N}} \subset \mathcal{H}([a, b]_{\mathbb{T}}, \mathbb{R}^n)$ be a pointwisely bounded sequence such that:

i) $g_n(t) \to g(t)$ for $t \in [a, b]_{\mathbb{T}} \setminus E$, where $E \subset [a, b]_{\mathbb{T}}$ a Δ-null measure set;

ii) $(g_n)_n$ is uniformly Henstock-Δ-integrable.

Then $g \in \mathcal{H}([a, b]_{\mathbb{T}}, \mathbb{R}^n)$ and $\|g_n - g\|_A \to 0$.

Proof. Let $R_{\mathbb{T}}$ be the set of right-scattered points of $[a, b]_{\mathbb{T}}$. For each $n \in \mathbb{N}$, extend the functions g_n and g on the real interval $[a, b]$ as follows:

$$\hat{g}_n(t) = \begin{cases} g_n(t), \text{ if } t \in \mathbb{T}; \\ g_n(t_i), \text{ if } t \in (t_i, \sigma(t_i)) \text{ for } i \in R_{\mathbb{T}}, \end{cases}$$

respectively

$$\hat{g}(t) = \begin{cases} g(t), \text{ if } t \in \mathbb{T}; \\ g(t_i), \text{ if } t \in (t_i, \sigma(t_i)) \text{ for } i \in R_{\mathbb{T}}. \end{cases}$$

We emphasize that any element of E cannot be a right-scattered point, since right-scattered points are of Δ measure $\sigma(t) - t > 0$.

Then $(\hat{g}_n)_{n\in\mathbb{N}}$ is a sequence of Henstock-integrable functions on the interval $[a, b] \subset \mathbb{R}$ satisfying the following conditions:

1) it is pointwisely bounded;

2) $\hat{g}_n(t) \to \hat{g}(t)$ a.e.;

3) $(\hat{g}_n)_n$ is uniformly Henstock-integrable.

Applying Theorem 4 in [15] we obtain that $\hat{g} \in \mathcal{H}([a, b], \mathbb{R}^n)$ and that $\|\hat{g}_n - \hat{g}\|_A \to 0$. Otherwise said, $g \in \mathcal{H}([a, b]_{\mathbb{T}}, \mathbb{R}^n)$ and $\|g_n - g\|_A \to 0$.

In the set-valued setting, for all concepts of measurability, we refer the reader to [10]. Following [16] (on real intervals), we introduce the Henstock-Δ-integrability.

Definition 1.11 A $\mathcal{P}_{kc}(\mathbb{R}^n)$-valued function Γ is Henstock-Δ-integrable if there exists (H) $\int_{[a,b]_{\mathbb{T}}} \Gamma(t)\Delta t \in \mathcal{P}_{kc}(\mathbb{R}^n)$ satisfying that, for every $\varepsilon > 0$, there is a Δ-gauge δ_ε such that for any δ_ε-fine division of $[a,b]_{\mathbb{T}}$,

$$D\left((\mathrm{H})\int_{[a,b]_{\mathbb{T}}} \Gamma(t)\Delta t,\ \sum_{i=1}^{n} \Gamma(\xi_i)\mu_\Delta([x_{i-1},x_i]_{\mathbb{T}})\right) < \varepsilon.$$

Here D denotes the Pompeiu-Hausdorff distance.

Let us remind the following:

Theorem 1.12 Let $\Gamma : [a,b] \to \mathcal{P}_{kc}(\mathbb{R}^n)$ be Henstock-integrable. Then the following conditions hold:
i) each measurable selection of Γ is Henstock-integrable (Theorem 1 in [16]);
ii) the collection $\{\sigma(x^*, \Gamma(\cdot)); x^* \in B^*\}$ is uniformly HK-integrable (Proposition 1 in [16]).

A classical result will be very useful (see e.g. [29]):

Lemma 1.13 For any sequence $(y_n)_n$ of measurable selections of a $\mathcal{P}_{kc}(\mathbb{R}^n)$-valued measurable multifunction, there exists a sequence $z_n \in conv\,\{y_m, m\geq n\}$ a.e. convergent to some measurable selection.

We recall that a set-valued function $F : [a,b]_{\mathbb{T}} \times \mathbb{R}^n \to \mathcal{P}_{kc}(\mathbb{R}^n)$ is called Carathéodory if it is measurable with respect to the first variable and upper semi-continuous with respect to the second one Δ-a.e.

3. Main results

We first state a new result for the single-valued case, that generalizes Proposition 2.29 in [21] (where the right hand side is Bochner-Δ-integrable).

We will make use of two first order Sobolev-type spaces.

Definition 2.1 A function $u : \mathbb{T} \to \mathbb{R}$ belongs to the space $W^{1,1}(\mathbb{T},\mathbb{R})$ if and only if $u \in L^1(\mathbb{T},\mathbb{R})$ and there exists a function $g \in L^1(\mathbb{T},\mathbb{R})$ such that

$$\int_{\mathbb{T}} u(s)\phi^{\Delta}(s)\Delta s = -\int_{\mathbb{T}} g(s)\phi(\sigma(s))\Delta s$$

for every

$$\phi \in C^1_{0,rd}(\mathbb{T}) = \left\{f : \mathbb{T} \to \mathbb{R}, f \in C^1_{rd}(\mathbb{T}), f(\min\mathbb{T}) = f(\max\mathbb{T}) = 0\right\}.$$

As it will be seen below, this space is closely related to the notion of absolute continuity.

Definition 2.2 A function $f : \mathbb{T} \to \mathbb{R}$ is said to be absolutely continuous on $\mathbb{T}$ if for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that

$$\sum_{i=1}^{n}(t_i - t_{i-1}) < \delta_\varepsilon \implies \sum_{i=1}^{n}|f(t_i) - f(t_{i-1})| < \varepsilon$$

for any finite pairwise disjoint family $([t_{i-1}, t_i))_{i=1}^n$ of subintervals of $\mathbb{T}$.

It was proved (in [1]) that if u and g are like in the preceding definition, then there exists a unique absolutely continuous function $x : \mathbb{T} \to \mathbb{R}$ such that Δ-a.e. $x = u$ and $x^\Delta = g$ (recall that, by Theorem 4.1 in [7], any absolutely continuous function on time scales is Δ-a.e. differentiable and has L^1-derivative).

The other Sobolev-type space involved is

$$W^{1,H}(\mathbb{T},\mathbb{R}^n)=\{v:\mathbb{T}\to\mathbb{R}^n; v \text{ continuous, } \Delta\text{−a.e. differentiable and } v^{\Delta}\in\mathcal{H}(\mathbb{T},\mathbb{R}^n)\}$$

endowed with the supremum norm, that is a Banach space.

Theorem 2.3 Let $\alpha\in\mathcal{H}([0,1]_{\mathbb{T}},\mathbb{R}^n)$. Then the linear periodic problem

$$\begin{aligned}&x^{\Delta}(t)-x(t)=\alpha(t), \qquad \Delta-a.e.\ t\in[0,1]_{\mathbb{T}}\\ &x(0)=x(1)\end{aligned}$$

has a unique solution in the space $W^{1,H}(\mathbb{T},\mathbb{R}^n)$,

$$x(t)=e_1(t)\left(\frac{e_1(1)}{1-e_1(1)}(\mathrm{H})\int_{[0,1)_{\mathbb{T}}}\frac{\alpha(s)}{e_1(\sigma(s))}\Delta s+(\mathrm{H})\int_{[0,t)_{\mathbb{T}}}\frac{\alpha(s)}{e_1(\sigma(s))}\Delta s\right),$$

where

$$e_1(t)=e^{\int_{[0,t)_{\mathbb{T}}}\xi(\mu(s))\Delta s}$$

with

$$\xi(\tau)=\begin{cases}1, \text{ if } \tau=0;\\ \dfrac{\ln(1+\tau)}{\tau}, \text{ if } \tau\neq 0.\end{cases}$$

Proof. Using the equivalence between the integrability of a function on time scales and the integrability of its extension on a real interval (described in Section 2) we can apply the main result in [25] in order to find an increasing sequence X_n of closed sets whose union covers the unit interval such that on each of them α is Bochner integrable and

$$\lim_{n\to\infty}\int_{X_n\cap[0,t)_{\mathbb{T}}}\alpha(s)\Delta s=(\mathrm{H})\int_{[0,t)_{\mathbb{T}}}\alpha(s)\Delta s, \text{ uniformly on } [0,1]_{\mathbb{T}}.$$

Then $\alpha_n(t)=\alpha(t)\chi_{X_n}(t)$ is a sequence of Bochner-Δ-integrable functions converging to α. We shall prove that

$$\int_{[0,t)_{\mathbb{T}}}\frac{\alpha_n(s)}{e_1(\sigma(s))}\Delta s\to(\mathrm{H})\int_{[0,t)_{\mathbb{T}}}\frac{\alpha(s)}{e_1(\sigma(s))}\Delta s \text{ uniformly.}$$

To this aim, let's see that

$$\begin{aligned}\frac{1}{e_1(\sigma(t))}&=e^{-\int_{[0,\sigma(t))_{\mathbb{T}}}\xi(\mu(s))\Delta s}\\ &=e^{-\int_{[0,t)_{\mathbb{T}}}\xi(\mu(s))\Delta s}\cdot e^{-\int_{[t,\sigma(t))}\xi(\mu(s))\Delta s}\\ &=e^{-\int_{[0,t)_{\mathbb{T}}}\xi(\mu(s))\Delta s}\cdot e^{-\mu(t)\xi(\mu(t))}\\ &=e^{-\int_{[0,t)_{\mathbb{T}}}\xi(\mu(s))\Delta s}\cdot\frac{1}{1+\mu(t)}.\end{aligned}$$

Theorem 1.60.(ii) in [4] asserts that the function μ is rd-continuous, while ξ is continuous and so, by Theorem 1.60.(v) in [4], their composition is rd-continuous. It follows from Theorem 1.74 in [4] that its Δ-integral is continuous and Δ-differentiable and the Δ-derivative is $\xi(\mu(t))$. On the other hand, the real exponential function is continuously differentiable and so, by Lemma 1.4, $e^{-\int_{[0,t)_{\mathbb{T}}}\xi(\mu(s))\Delta s}$ is Δ-differentiable and there exists $c \in [t, \sigma(t)]$ such that

$$\left(e^{-\int_{[0,t)_{\mathbb{T}}}\xi(\mu(s))\Delta s}\right)^{\Delta} = e^{-\int_{[0,c)_{\mathbb{T}}}\xi(\mu(s))\Delta s} \cdot \xi(\mu(t)).$$

As this Δ-derivative is bounded because $\ln 2 \leq \xi(\mu(s)) \leq 1$, it follows that $e^{-\int_{[0,t)_{\mathbb{T}}}\xi(\mu(s))\Delta s}$ is of bounded variation.

Using now Lemma 1.2 gives that the map $t \mapsto \dfrac{1}{e_1(\sigma(t))}$ is of bounded variation, wherefrom

$$\begin{aligned} &\left\| \int_{[0,t)_{\mathbb{T}}} \frac{\alpha_n(s)}{e_1(\sigma(s))}\Delta s - (\mathrm{H})\int_{[0,t)_{\mathbb{T}}} \frac{\alpha(s)}{e_1(\sigma(s))}\Delta s \right\| \\ &\leq \left\| \frac{1}{e_1(\sigma(s))} \right\|_{BV} \cdot \sup_{s\in[0,t)_{\mathbb{T}}} \left\| (\mathrm{H})\int_0^s \alpha_n(u) - \alpha(u)\Delta u \right\| \end{aligned}$$

and so,

$$\int_{[0,t)_{\mathbb{T}}} \frac{\alpha_n(s)}{e_1(\sigma(s))}\Delta s \to (\mathrm{H})\int_{[0,t)_{\mathbb{T}}} \frac{\alpha(s)}{e_1(\sigma(s))}\Delta s \text{ uniformly.} \tag{3.1}$$

Since $\alpha_n \in L^1([0,1]_{\mathbb{T}}, \mathbb{R}^n)$, following Proposition 2.29 in [21], the problem

$$\begin{aligned} &x^{\Delta}(t) - x(t) = \alpha_n(t), \qquad \Delta - a.e.\ t \in [0,1]_{\mathbb{T}} \\ &x(0) = x(1) \end{aligned}$$

has a solution

$$x_n(t) = e_1(t)\left(\frac{e_1(1)}{1-e_1(1)}(\mathrm{H})\int_{[0,1)_{\mathbb{T}}} \frac{\alpha_n(s)}{e_1(\sigma(s))}\Delta s + (\mathrm{H})\int_{[0,t)_{\mathbb{T}}} \frac{\alpha_n(s)}{e_1(\sigma(s))}\Delta s\right).$$

When $n \to \infty$, the assertion (3.1) gives that the limit function defined by

$$x(t) = e_1(t)\left(\frac{e_1(1)}{1-e_1(1)}(\mathrm{H})\int_{[0,1)_{\mathbb{T}}} \frac{\alpha(s)}{e_1(\sigma(s))}\Delta s + (\mathrm{H})\int_{[0,t)_{\mathbb{T}}} \frac{\alpha(s)}{e_1(\sigma(s))}\Delta s\right)$$

is a solution for our problem.

Finally, the uniqueness comes immediately if one suppose that there are two different solutions x and y. Then their difference $z = x - y$ satisfies the linear periodic dynamic problem with Bochner integrable right hand side

$$\begin{aligned} &z^{\Delta}(t) - z(t) = 0, \qquad\qquad \Delta - a.e.\ t \in [0,1]_{\mathbb{T}} \\ &z(0) = z(1) \end{aligned}$$

that has the unique solution in the space $W^{1,1}(\mathbb{T}, \mathbb{R}^n)$

$$z(t) = 0.$$

We proceed now to give the main result of the paper. To this aim, we adapt to our framework the notion introduced in [17] which generalizes, to the multi-dimensional case, the concepts of lower and upper solutions. For the corresponding single-valued notion of solution tube in absolute integrability setting, see [21].

Definition 2.4 Let $v \in W^{1,H}([0,1]_{\mathbb{T}}, \mathbb{R}^n)$ and $r \in W^{1,1}([0,1]_{\mathbb{T}}, \mathbb{R}_+)$. We say that (v, r) is a solution tube for the inclusion (1.1) if:
i) for Δ-a.e. $t \in [0,1]_{\mathbb{T}}$ and every $x \in \mathbb{R}^n$ with $\|x - v(t)\| = r(t)$, there exists $y \in F(t,x)$ with

$$\langle x - v(t), y - v^{\Delta}(t)\rangle \leq r(t)r^{\Delta}(t);$$

ii) for Δ-a.e. $t \in [0,1]_{\mathbb{T}}$ for which $r(t) = 0$, $v^{\Delta}(t) \in F(t, v(t))$ and $r^{\Delta}(t) = 0$;
iii) $\|v(0) - v(1)\| \leq r(0) - r(1)$.

Denote by $T(v,r) = \{x \in C([0,1]_{\mathbb{T}}, \mathbb{R}^n); \|x(t) - v(t)\| \leq r(t), \forall t \in [0,1]_{\mathbb{T}}\}$.

Theorem 2.5 Let $F : [0,1]_{\mathbb{T}} \times \mathbb{R}^n \to \mathcal{P}_{kc}(\mathbb{R}^n)$ be a Carathéodory multifunction such that for every $R > 0$, there exists a Henstock-Δ-integrable multifunction $G_R : [0,1]_{\mathbb{T}} \to \mathcal{P}_{kc}(\mathbb{R}^n)$ such that

$$F(t,x) \subset G_R(t), \quad \forall t \in [0,1]_{\mathbb{T}},\ \forall x \in \mathbb{R}^n, \|x\| \leq R.$$

If the periodic dynamic inclusion (1.1) has a solution tube $(v,r) \in W^{1,H}([0,1]_{\mathbb{T}}, \mathbb{R}^n) \times W^{1,1}([0,1]_{\mathbb{T}}, \mathbb{R}_+)$, then it has at least one solution

$$x \in W^{1,H}([0,1]_{\mathbb{T}}, \mathbb{R}^n) \cap T(v,r).$$

Proof. We follow the method of proof given in [18], defining the truncated multifunction

$$\widetilde{F} : [0,1]_{\mathbb{T}} \times \mathbb{R}^n \to \mathcal{P}_{kc}(\mathbb{R}^n), \quad \widetilde{F}(t,x) = F(t, \widetilde{x}_t) \cap G(t,x)$$

where

$$\widetilde{x}_t = \begin{cases} x, & \text{if } \|x - v(t)\| \leq r(t) \\ v(t) + \dfrac{r(t)}{\|x - v(t)\|}(x - v(t)), & \text{otherwise} \end{cases}$$

and the multifunction G with closed values is given by

$$G(t,x) = \begin{cases} v^{\Delta}(t), \text{ if } r(t) = 0; \\ \mathbb{R}^n, \text{ if } \|x - v(t)\| \leq r(t) \text{ and } r(t) > 0; \\ \{z \in \mathbb{R}^n; \langle x - v(t), z - v^{\Delta}(t)\rangle \leq r^{\Delta}(t)\|x - v(t)\|\}, \text{otherwise.} \end{cases}$$

I. We shall prove that the modified problem

$$\begin{aligned} &x^{\Delta}(t) - x(t) \in \widetilde{F}(t, x(t)) - \widetilde{x(t)}_t, \quad \Delta - a.e.\ t \in [0,1]_{\mathbb{T}} \\ &x(0) = x(1) \end{aligned}$$

has at least one solution $x \in W^{1,H}([0,1]_{\mathbb{T}}, \mathbb{R}^n)$.
Consider the set-valued operator $\mathcal{F} : W^{1,H}([0,1]_{\mathbb{T}}, \mathbb{R}^n) \to W^{1,H}([0,1]_{\mathbb{T}}, \mathbb{R}^n)$,

$$\mathcal{F}(x) = \left\{ y \in W^{1,H}([0,1]_{\mathbb{T}}, \mathbb{R}^n), y^{\Delta}(t) - y(t) \in \widetilde{F}(t, x(t)) - \widetilde{x(t)}_t\ \Delta - a.e. \right\},$$

which has convex and nonempty values (by Theorem 2.3). Let us nos show that its values are weakly compact. For this purpose, take $x \in W^{1,H}([0,1]_{\mathbb{T}}, \mathbb{R}^n)$ and $(y_n)_n \subset \mathcal{F}(x)$. There exists a sequence z_n of Henstock-Δ-integrable selections of $\widetilde{F}(t, x(t))$ such that

$$y_n^{\Delta}(t) - y_n(t) = z_n(t) - \widetilde{x(t)}_t \quad \Delta - a.e.$$

whence, by Theorem 2.3,

$$y_n(t) = e_1(t)\left(\frac{e_1(1)}{1-e_1(1)}(\mathrm{H})\int_{[0,1)_{\mathbb{T}}}\frac{z_n(s)-\widetilde{x(s)}_s}{e_1(\sigma(s))}\Delta s + (\mathrm{H})\int_{[0,t)_{\mathbb{T}}}\frac{z_n(s)-\widetilde{x(s)}_s}{e_1(\sigma(s))}\Delta s\right). \tag{3.2}$$

Lemma 1.13 implies the existence of a sequence $\overline{z}_n \in conv\,\{z_m, m{\ge}n\}$ Δ-a.e. norm-convergent (pointwisely) to a measurable $\overline{z}$. As $\widetilde{F}$ has compact convex values, it follows that $\overline{z}(t) \in \widetilde{F}(t, x(t))$, $\Delta - a.e.\ t \in [0,1]_{\mathbb{T}}$. Using Theorem 1.12 we can apply the convergence Theorem 1.10 in order to obtain that $\overline{z} \in \mathcal{H}([0,1]_{\mathbb{T}}, \mathbb{R}^n)$ (so, $\overline{z}$ is a Henstock-Δ-integrable selection of $\widetilde{F}(t, x(t)))$ and that $\|\overline{z}_n - \overline{z}\|_A \to 0$. It follows, as in the proof of Theorem 2.3, that

$$\overline{y}_n \to \overline{y} \text{ uniformly},$$

where $\overline{y}_n$ is the corresponding convex combination of $\{y_m, m \ge n\}$ and

$$\overline{y}(t) = e_1(t)\left(\frac{e_1(1)}{1-e_1(1)}(\mathrm{H})\int_{[0,1)_{\mathbb{T}}}\frac{\overline{z}(s)-\widetilde{x(s)}_s}{e_1(\sigma(s))}\Delta s + (\mathrm{H})\int_{[0,t)_{\mathbb{T}}}\frac{\overline{z}(s)-\widetilde{x(s)}_s}{e_1(\sigma(s))}\Delta s\right).$$

From Corollary 2.2. in [14] it follows that $\mathcal{F}(x)$ is weakly compact.

Let us now prove that $\mathcal{F}$ is upper semi-continuous with respect to the weak topology. Let $M \subset W^{1,H}([0,1]_{\mathbb{T}}, \mathbb{R}^n)$ be weakly closed and $(x_n)_n \subset \{x \in W^{1,H}([0,1]_{\mathbb{T}}, \mathbb{R}^n); \mathcal{F}(x) \cap M \neq \emptyset\}$ converge to x_0. One can find $y_n \in \mathcal{F}(x_n) \cap M$, so

$$y_n^{\Delta}(t) - y_n(t) \in \widetilde{F}(t, x_n(t)) - \widetilde{x_n(t)}_t,\ \Delta - a.e.\ t \in [0,1]_{\mathbb{T}}.$$

By hypothesis, there exists a sequence $z_n \in conv\,\{y_m, m{\ge}n\}$ Δ-a.e. convergent to a measurable $\overline{y}$ and $\|\cdot\|_A$-convergent too (so, $\overline{y} \in M$). Since F is Carathéodory, for each neighborhood V of the origin, there exists $n_{t,V} \in \mathbb{N}$ such that, for every $n \ge n_{t,V}$, $F(t, x_n(t)) \subset F(t, x_0(t)) + V$. Then

$$\widetilde{F}(t, x_n(t)) - \widetilde{x_n(t)}_t \subset \widetilde{F}(t, x_0(t)) - \widetilde{x_0(t)}_t + V, \quad \forall n \ge n_{t,V}.$$

Consequently, for Δ-a.e. $t \in [0,1]_{\mathbb{T}}$,

$$\overline{y}(t) \in \widetilde{F}(t, x_0(t)) - \widetilde{x_0(t)}_t$$

and so, $\{x \in W^{1,H}([0,1]_{\mathbb{T}}, \mathbb{R}^n); \mathcal{F}(x) \cap M \neq \emptyset\}$ is closed.

Moreover, from (3.2) we obtain that for all $x \in W^{1,H}([0,1]_{\mathbb{T}}, \mathbb{R}^n)$,

$$\|\widetilde{x(t)}_t\| \le R = \max_{t\in[0,1]_{\mathbb{T}}} (\|v(t)\| + r(t)),$$

whence $\widetilde{F}(t, x(t)) \subset G_R(t)$ and from here $\bigcup\{\mathcal{F}(x); x \in W^{1,H}([0,1]_{\mathbb{T}}, \mathbb{R}^n)\}$ has for every fixed t values contained in a compact subset of $\mathbb{R}^n$.

We show now that $\bigcup\{\mathcal{F}(x); x \in W^{1,H}([0,1]_{\mathbb{T}}, \mathbb{R}^n)\}$ is equi-continuous (and this will imply that it is relatively weakly compact).

Indeed, let $y \in \mathcal{F}(x)$ for some $x \in W^{1,H}([0,1]_{\mathbb{T}}, \mathbb{R}^n)$ and let $t' < t" \in [0,1]_{\mathbb{T}}$. Then

$$\|y(t') - y(t")\| \leq |e_1(t') - e_1(t")| \frac{e_1(1)}{1-e_1(1)} \cdot$$
$$\cdot \left(\left\| (\mathrm{H}) \int_{[0,1)_{\mathbb{T}}} \frac{\overline{z}(s) - \widetilde{x(s)}_s}{e_1(\sigma(s))} \Delta s \right\| + \left\| (\mathrm{H}) \int_{[0,t')_{\mathbb{T}}} \frac{\overline{z}(s) - \widetilde{x(s)}_s}{e_1(\sigma(s))} \Delta s \right\| \right)$$
$$+ e_1(t") \frac{e_1(1)}{1-e_1(1)} \left\| (\mathrm{H}) \int_{[t',t")_{\mathbb{T}}} \frac{\overline{z}(s) - \widetilde{x(s)}_s}{e_1(\sigma(s))} \Delta s \right\|$$

and the equi-continuity is an immediate consequence of Theorem 1.12 and Lemma 3.3 in [30].

So, $\mathcal{F}$ is a compact upper semi-continuous operator with compact and convex values. By Kakutani fixed point theorem, it has fixed points.

II. In exactly the same way as in Theorem 4.3 in [21] (see also [18] for functions defined on a real interval), it follows that any solution found at the first step satisfies the condition $x \in T(v,r)$, whence $\widetilde{x(t)}_t = x(t)$ and so, x is in fact a solution for our initial problem (1.1).

We conclude by providing an example of first order dynamic inclusion for which our main theorem guarantees the existence of solutions, while classical results (like Theorem 3.7 in [20]) do not apply.

Example. Consider on an arbitrary time scale $\mathbb{T}$ the periodic problem

$$\begin{aligned} & x^{\Delta}(t) \in f(t) + F(t, x(t)), \qquad \Delta - a.e.\ t \in [0,1]_{\mathbb{T}}, \\ & x(0) = x(1) \end{aligned}$$

where

$$f(t) = \begin{cases} \left(2t \sin \frac{1}{t^2} - \frac{2}{t} \cos \frac{1}{t^2}\right) x_0, \text{ if } t \in (0,1]_{\mathbb{T}}, \\ 0, \text{ if } t = 0 \end{cases}$$

with $x_0 \in \mathbb{R}^n$ and $F : [0,1]_{\mathbb{T}} \times \mathbb{R}^n \to \mathcal{P}_{kc}(\mathbb{R}^n)$ is a Carathéodory multifunction satisfying that for every $R > 0$, one can find $g_R \in L^1([0,1]_{\mathbb{T}}, \mathbb{R}_+)$ such that

$$\max\{\|y\|, y \in F(t,x), \|x\| \leq R\} \leq g_R(t), \ \Delta - a.e.$$

and there exists a solution tube $(v,r) \in W^{1,H}([0,1]_{\mathbb{T}}, \mathbb{R}^n) \times W^{1,1}([0,1]_{\mathbb{T}}, \mathbb{R}_+)$.

Due to the fact that f is Henstock-Δ-integrable, the right-hand side of our inclusion satisfies the hypothesis of Theorem 2.5 and so, the considered problem has at least one solution in $W^{1,H}([0,1]_{\mathbb{T}}, \mathbb{R}^n) \cap T(v,r)$. But as $f \notin L^1([0,1]_{\mathbb{T}}, \mathbb{R}^n)$, none of the existence results given in absolute integrability setting can be applied.

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