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FIXED POINT THEOREMS FOR NONCONVEX VALUED CORRESPONDENCES AND APPLICATIONS IN GAME THEORY

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Abstract. In this paper, we introduce several types of correspondences: weakly naturally quasiconvex, *-weakly naturally quasiconvex, weakly biconvex and correspondences with *–weakly convex graph and we prove some fixed point theorems for these kinds of correspondences. As a consequence, using a version of W.K. Kim's quasi-point theorem, we obtain the existence of equilibria for a quasi-game.

Key Words and Phrases: Fixed point theorem, correspondences with *–weakly convex graph, weakly naturally quasiconvex correspondences, quasi game, quasi-quilibrium.
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1. Introduction

The aim of this paper is to prove some fixed points theorems for correspondences which are not continuous or convex valued and to give applications in game theory.

The significance of equilibrium theory stems from the fact that it develops important tools (as fixed point and selection theorems) to prove the existence of equilibrium for different types of games. In 1950, J. F. Nash [15] first proved a theorem of equilibrium existence for games where the player's preferences were representable by continuous quasi-concave utilities. G. Debreu's works on the existence of equilibrium in a generalized N-person game or on an abstract economy [6] were extended by several authors. In [16] W. Shafer and H. Sonnenschein proved the existence of equilibrium of an economy with finite dimensional commodity space and irreflexive preferences represented as correspondences with open graph. N. C. Yannelis and N. D. Prahbakar [19] developed new techniques based on selection theorems and fixed-point theorems. Their main result concerns the existence of equilibrium when the constraint and preference correspondences have open lower sections. They worked within different frameworks (countable infinite number of agents, infinite dimensional strategy spaces). K.J. Arrow and G. Debreu proved the existence of Walrasian equilibrium in [3]. In [20], X.Z. Yuan proposed a model of abstract economy more general than that introduced by Borglin and Keing in [4].

Within the last years, a lot of authors generalized the classical model of abstract economy. For example, K. Vind [18] defined the social system with coordination, X.Z. Yuan [20] proposed the model of the general abstract economy. Motivated by the fact that any preference of a real agent could be unstable by the fuzziness of consumers' behaviour or market situation, W.K. Kim and K.K. Tan [12] defined the generalized abstract economies. Also W.K. Kim [13] obtained a generalization of the quasi fixed-point theorem due to I. Lefebvre [14], and as an application, he proved an existence theorem of equilibrium for a generalized quasi-game with infinite number of agents. W. K. Kim's result concerns generalized quasi-games where the strategy sets are metrizable subsets in locally convex linear topological spaces.

Biconvexity was studied by R. Aumann, S. Hart in [2] and J. Gorski, F. Pfeuffer and K. Klamroth in [10].

An open problem of the fixed point theory is to prove the existence of fixed points for correspondences without continuity or convex values. X. Ding and He Yiran introduced in [7] the correspondences with weakly convex graph to prove a fixed point theorem. The result concerning the existence of the affine selection on a special type of sets (simplex) proves to be redundant, since a correspondence $T : X \to 2^Y$ has an affine selection if and only if it has a weakly convex graph. This result is stronger than it needs in order to obtain a fixed point theorem. We try to weaken these conditions by defining several types of correspondences which are not continuous or convex valued: weakly naturally quasiconvex, *-weakly naturally quasiconvex, correspondences with *–weakly convex graph and weakly biconvex correspondences. We prove fixed point theorems for these kinds of correspondences and using a version of W.K. Kim's quasi-point theorem, we prove the existence of equilibria for a quasi-game. We use the continuous selection technique introduced by N.C. Yannelis and N.D. Prahbakar in [19].

The paper is organized in the following way: Section 2 contains preliminaries and notation. The fixed point theorems are given in Section 3 and the equilibrium theorems are stated in Section 4.

2. Preliminaries and notations

Throughout this paper, we shall use the following notations and definitions:

Let A be a subset of a topological space X.

1. 2^A denotes the family of all subsets of A.

2. cl A denotes the closure of A in X.

3. If A is a subset of a vector space, co$A$ denotes the convex hull of A.

4. If F, $T : A \to 2^X$ are correspondences, then co$T$, cl T, $T \cap F : A \to 2^X$ are correspondences defined by $(\text{co}T)(x) = \text{co}T(x)$, $(\text{cl}T)(x) = \text{cl}T(x)$ and $(T \cap F)(x) = T(x) \cap F(x)$ for each $x \in A$, respectively.

5. The graph of $T : X \to 2^Y$ is the set $\text{Gr}(T) = \{(x, y) \in X \times Y \mid y \in T(x)\}$

6. The correspondence $\overline{T}$ is defined by $\overline{T}(x) = \{y \in Y : (x, y) \in \text{cl}_{X\times Y}\text{Gr}T\}$ (the set $\text{cl}_{X\times Y}\text{Gr}(T)$ is called the adherence of the graph of T).

It is easy to see that $\text{cl}T(x) \subset \overline{T}(x)$ for each $x \in X$.

Definition 2.1. Let X, Y be topological spaces and $T : X \to 2^Y$ be a correspondence. T is said to be *upper semicontinuous* if for each $x \in X$ and each open set V in Y with $T(x) \subset V$, there exists an open neighborhood U of x in X such that $T(y) \subset V$ for each $y \in U$.

Let $X \subset E_1$ and $Y \subset E_2$ be two nonempty convex sets, E_1, E_2 be topological vector spaces and let $B \subset X \times Y$.

Definition 2.2. [2] The set $B \subset X \times Y$ is called a *biconvex set* on $X \times Y$ if the section $B_x = \{y \in Y : (x, y) \in B\}$ is convex for every $x \in X$ and the section $B_y = \{x \in X : (x, y) \in B\}$ is convex for every $y \in Y$.

Definition 2.3. [2] Let $(x_i, y_i) \in X \times Y$ for $i = 1, 2, ...n$. A convex combination $(x, y) = \sum_{i=1}^{n} \lambda_i(x_i, y_i)$, (with $\sum_{i=1}^{n} \lambda_i = 1$, $\lambda_i \geq 0$ $i = 1, 2, ..., n$) is called *biconvex combination* if $x_1 = x_2 = ... = x_n = x$ or $y_1 = y_2 = ... = y_n = y$.

Theorem 2.4. *(Aumann and Hart [2]). A set $B \subseteq X \times Y$ is biconvex if and only if B contains all biconvex combinations of its elements.*

Definition 2.5. [2] Let $A \subseteq X \times Y$ be a given set. The set $H := \bigcap\{A_I : A \subseteq A_I$, A_I is biconvex$\}$ is called *biconvex hull of* A and is denoted biconv(A).

Theorem 2.6. *(Aumann and Hart [2]). The biconvex hull of a set A is biconvex. Furthermore, it is the smallest biconvex set (in the sense of set inclusion), which contains A.*

Lemma 2.7. *(Gorski, Pfeuffer and Klamroth [10]). Let $A \subseteq X \times Y$ be a given set. Then biconv(A) $\subseteq$conv(A).*

Notation. We denote the standard $(n-1)$- dimensional simplex by

$$\Delta_{n-1} = \{(\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbb{R}^n : \sum_{i=1}^{n} \lambda_i = 1, \lambda_i \geqslant 0, i = 1, 2, ..., n\}.$$

3. Selection theorems and fixed point theorems

An open problem of the fixed point theory is to prove the existence of fixed points for correspondences without continuity or convex values. In this section we introduce some types of correspondences which are not continuous or convex valued and prove selection theorems and fixed point theorems.

First, we introduce the concept of weakly naturally quasiconvex correspondence.

Definition 3.1. Let X, Y be nonempty convex subsets of topological vector spaces E, respectively F. The correspondence $T : X \longrightarrow 2^Y$ is said to be *weakly naturally quasiconvex (WNQ)* if for each $n \in \mathbb{N}$ and for each finite set $\{x_1, x_2, ..., x_n\} \subset X$, there exists $y_i \in T(x_i)$, $(i = 1, 2, ..., n)$ and $g = (g_1, g_2, ..., g_n) : \Delta_{n-1} \to \Delta_{n-1}$ a bijective function depending on $x_1, x_2, ..., x_n$ with g_i continuous, $g_i(1) = 1$, $g_i(0) = 0$ for each $i = 1, 2, ...n$, such that for every $(\lambda_1, \lambda_2, ..., \lambda_n) \in \Delta_{n-1}$, there exists $y \in T(\sum_{i=1}^{n} \lambda_i x_i)$ and $y = \sum_{i=1}^{n} g_i(\lambda_i) y_i$.

Remark 3.2. A weakly naturally quasiconvex correspondence may not be continuous or convex valued.

We give an economic interpretation of the weakly naturally quasiconvex correspondences.

We consider an abstract economy $\Gamma = (X_i, A_i, P_i)_{i\in I}$ with I - the set of agents. Each agent can choose a strategy from the set X_i and has a preferrence correspondence $P_i : X = \prod_{i\in I} X_i \to 2^{X_i}$ and a constraint correspondence $A_i : X = \prod_{i\in I} X_i \to 2^{X_i}$. The traditional approach considers that the preferrence of agent i is characterized by a binary relation $\succeq_i$ on the set X_i. A real valued function $u_i : X \to \mathbb{R}$ that satisfies $x \succeq_i y \Leftrightarrow u_i(x) \geq u_i(y)$ is called an utility function of the preferrence $\succeq_i$. The relation between the utility function u_i and the preferrence correspondence P_i, for each agent i is:

$P_i(x) = \{y_i \in X_i : u_i(x, y_i) > u_i(x, x_i)\}$, where, in this case, $u_i : X \times X_i \to \mathbb{R}$.

The aim of the equilibrium theory is to maximize each agent's utility on a strategy set.

For the case that, for each index i, P_i is a weakly naturally quasiconvex correspondence, the interpretation is the following: for all certain amounts $x^1, x^2, ...x^n \in X$, the agent i with the correspondence P_i will always prefer y_i, the weighted average of some quantities $y_i^k \in P_i(x^k)$, $k = 1, \cdots, n$. This implies that there exist $y_i^1 \in P_i(x^1), y_i^2 \in P_i(x^2), ..., y_i^n \in P_i(x^n)$ and $g = (g_1, g_2, ..., g_n) : \Delta_{n-1} \to \Delta_{n-1}$ a bijective function depending on $x^1, x^2, ..., x^n$ with g_i continuous for each $i = 1, 2, ..., n$, such that, for each $\lambda \in \Delta_{n-1}$, there exists $y_i = \sum_{k=1}^{n} g_i(\lambda_k) y_i^k$ and $y_i \in P_i(\sum_{k=1}^{n} \lambda_k x^k)$ (i.e., if there exist utility functions $u_i : X \times X_i \to \mathbb{R}$ such that, if $u_i(x^k, y_i^k) > u_i(x^k, x_i^k)$ for every $k \in \{1, 2, ...n\}$, we have that $y_i \in A_i(\sum_{k=1}^{n} \lambda_k x^k)$ and $u_i(\sum_{k=1}^{n} \lambda_k x^k, y_i) > u_i(\sum_{k=1}^{n} \lambda_k x^k, (\sum_{k=1}^{n} \lambda_k x^k)_i)$.

Theorem 3.3. *(selection theorem). Let K be a simplex in a topological vector space F and Y be a non-empty convex subset of a topological vector space E. Let $T : K \to 2^Y$ be a weakly naturally quasiconvex correspondence. Then, T has a continuous selection on K.*

Proof. Assume that K is a simplex, i.e., the convex hull of an affinely independent set $\{a_1, a_2, ..., a_n\}$. Since T is weakly naturally quasiconvex, there exist $b_i \in T(a_i)$, $(i = 1, 2, ..., n)$ and $g = (g_1, g_2, ..., g_n) : \Delta_{n-1} \to \Delta_{n-1}$ a bijective function with g_i continuous for each $i = 1, 2, ..., n$, such that for every $(\lambda_1, \lambda_2, ..., \lambda_n) \in \Delta_{n-1}$, there exists $y \in T(\sum_{i=1}^{n} \lambda_i a_i)$ with $y = \sum_{i=1}^{n} g_i(\lambda_i) y_i$.

Since K is a $(n-1)$-dimensional simplex with the vertices $a_1, ..., a_n$, there exists unique continuous functions $\lambda_i : K \to \mathbb{R}$, $i = 1, 2, ..., n$ such that for each $x \in K$, we have $(\lambda_1(x), \lambda_2(x), ..., \lambda_n(x)) \in \Delta_{n-1}$ and $x = \sum_{i=1}^{n} \lambda_i(x) a_i$.

Let's define $f : K \to Y$ by

$f(a_i) = b_i$ $(i = 1, ..., n)$ and
$f(\sum_{i=1}^{n} \lambda_i a_i) = \sum_{i=1}^{n} g_i(\lambda_i) b_i \in T(x)$.
We show that f is continuous.

Let $(x_m)_{m \in N}$ be a sequence which converges to $x_0 \in K$, where $x_m = \sum_{i=1}^{n} \lambda_i(x_m) a_i$ and $x_0 = \sum_{i=1}^{n} \lambda_i(x_0) a_i$. By the continuity of λ_i, it follows that for each $i = 1, 2, ..., n$, $\lambda_i(x_m) \to \lambda_i(x_0)$ as $m \to \infty$. Since $g_1, g_2, ..., g_n$ are continuous, we have $g_i(\lambda_i(x_m)) \to g_i(\lambda_i(x_0))$ as $m \to \infty$. Hence, $f(x_m) \to f(x_0)$ as $m \to \infty$, i.e. f is continuous.

We proved that T has a continuous selection on K.

By Brouwer's fixed point theorem, we obtain the following fixed point theorem for weakly naturally quasiconvex correspondences.

Theorem 3.4. *Let K be a simplex in a topological vector space F. Let $T : K \to 2^K$ be a weakly naturally quasiconvex correspondence. Then, T has a fixed point in K.*

Proof. By Theorem 2.5, T has a continuous selection on K, $f : K \to K$.
Since f has a fixed point $x^* \in K$, we have that $x^* = f(x^*) \in T(x^*)$.
Notation. For the correspondence $T : X \to 2^Y$ and for the set $V \in Y$, we denote T_V the correspondence $T_V : X \to 2^Y$, defined by $T_V(x) = (T(x) + V) \cap Y$ for each $x \in X$.
If $Y = K$, we obtain the following fixed point theorem:

Theorem 3.5. *Let K be a simplex in a topological vector space F and let $T : K \to 2^K$ be a correspondence. Assume that for each neighborhood V of the origin in F, there is $T^V : K \to 2^K$ a weakly naturally quasiconvex correspondence such that $GrT^V \subset clGrT_V$. Then there exists a point $x^* \in K$ such that $x^* \in \overline{T}(x^*)$.*

To prove Theorem 3.5, we need the following lemma from [20].

Lemma 3.6 (20)**.** *Let X be a topological space, Y be a non-empty subset of a topological vector space E, ß be a base of the neighborhoods of 0 in E and $T : X \to 2^Y$. If $x^* \in X$ and $\widehat{y} \in Y$ are such that $\widehat{y} \in \cap_{V \in ß} \overline{T_V}(x^*)$, then $\widehat{y} \in \overline{T}(x^*)$, where $\overline{T} : X \to 2^Y$ is defined by $\overline{T}(x) = \{y \in Y : (x, y) \in cl_{X \times Y} GrT\}$.*

Proof of Theorem 3.5. Let ß denote the family of all neighborhoods of zero in F. Let $V \in$ß. By the fixed point theorem 3.4, it follows that for each neighborhood V of the origin in Y, there exists $x_V^* \in T^V(x_V^*) \subset (T(x_V^*) + V) \cap K$.

For each $V \in$ ß, we define $Q_V = \{x \in K : x \in (T(x) + V) \cap K\}$.

Q_V is nonempty since $x_V^* \in Q_V$, then clQ_V is nonempty.

We prove that the family $\{\text{cl}Q_V : V \in ß\}$ has the finite intersection property.

Let $\{V^{(1)}, V^{(2)}, ..., V^{(n)}\}$ be any finite set of ß. Let $V = \bigcap_{k=1}^{n} V^{(k)}$, then $V \in$ ß. Clearly $Q_V \subset \bigcap_{k=1}^{n} Q_{V^{(k)}}$ so that $\bigcap_{k=1}^{n} Q_{V^{(k)}} \neq \emptyset$. Then $\bigcap_{k=1}^{n} \text{cl}Q_{V^{(k)}} \neq \emptyset$.

Since K is compact and the family $\{\text{cl}Q_V : V \in ß\}$ has the finite intersection property, we have that $\cap\{\text{cl}Q_V : V \in ß\} \neq \emptyset$. Take any $x^* \in \cap\{\text{cl}Q_V : V \in ß\}$, then for each $V \in$ ß, $x^* \in$cl$\{x^* \in K : x^* \in (T(x^*) + V) \cap K\}$. Hence $(x^*, x^*) \in$clGr$((T(x) + V) \cap K)$ for each $V \in$ß. By Lemma 3.6 we have that $x^* \in \overline{T}(x^*)$, i.e. x^* is a fixed point for $\overline{T}$. □

The weakly convex correspondences are defined in [7].

Definition 3.7. [7] Let X and Y be nonempty convex subsets of a topological vector space E. The correspondence $T : X \longrightarrow 2^Y$ is said to have *weakly convex graph* (in short it is a WCG correspondence), if for each finite set $\{x_1, x_2, ..., x_n\} \subset X$, there exists $y_i \in T(x_i)$, $(i = 1, 2, ..., n)$, such that

$$(1) \quad \text{co}(\{(x_1, y_1), (x_2, y_2), ..., (x_n, y_n)\}) \subset \text{Gr}(T)$$

The relation (1) is equivalent to

$$(2) \qquad \sum_{i=1}^{n} \lambda_i y_i \in T(\sum_{i=1}^{n} \lambda_i x_i) \qquad (\forall(\lambda_1, \lambda_2, ..., \lambda_n) \in \Delta_{n-1}).$$

It is clear that if either $\text{Gr}(T)$ is convex, or $\bigcap\{T(x) : x \in X\} \neq \emptyset$, then T has a weakly convex graph.

Remark 3.8. Let $T : X \to 2^Y$ be a WCG correspondence and X_0 be a non-empty convex subset of X. Then, the restriction of T on X_0, $T_{|X_0} : X_0 \to 2^Y$ is a WCG correspondence, too.

Now we introduce the following definition.

Definition 3.9. Let E, F be topological vector spaces, X and Y be nonempty convex subsets of E, respectively F and $T : X \to 2^Y$ be a correspondence. T is said to have *a *-weakly convex graph* if for each neighborhood V of the origin in F, the correspondence $T_V : X \to 2^Y$, defined by $T_V(x) = (T(x) + V) \cap Y$ for each $x \in X$ has an weakly convex graph.

The next theorem (its proof follows the same lines as that of Theorem 3.5) is a fixed point result for a correspondence with *-weakly convex graph.

Theorem 3.10. *Let K be a simplex in a topological vector space F. Let $T : K \to 2^K$ be a correspondence with *-weakly convex graph. Then, there exists a point $x^* \in K$ such that $x^* \in \overline{T}(x^*)$.*

We get the following corollary.

Corollary 3.11. *Let K be a simplex in a topological vector space F. Let $S, T : K \to 2^K$ be two correspondences with the following conditions:*

(i) *for each $x \in K$, $\overline{S}(x) \subset T(x)$ and $S(x) \neq \emptyset$,*
(ii) *S has *-weakly convex graph.*
Then, there exists a point $x^* \in K$ such that $x^* \in T(x^*)$.

Now, we introduce the concept of *-weakly naturally quasiconvex correspondence.

Definition 3.12. Let E, F be topological vector spaces, X and Y be nonempty convex subsets of E, respectively F and $T : X \to 2^Y$ be a correspondence. T is said to be **-weakly naturally quasiconvex* if for each neighborhood V of the origin in F, the corespondence $T_V : X \to 2^Y$, defined by $T_V(x) = (T(x) + V) \cap Y$ for each $x \in X$ is weakly naturally quasiconvex.

Theorem 3.13 is a fixed point theorem for *-weakly naturally quasiconvex correspondences.

Theorem 3.13. *Let K be a non-empty simplex in a topological vector space F. Let $T : K \to 2^K$ be a *-weakly naturally quasiconvex correspondence. Then there exists a point $x^* \in K$ such that $x^* \in \overline{T}(x^*)$.*

Proof. Let ß denote the family of all neighborhoods of zero in F and let $V \in$ß. The corespondence $T_V : K \to 2^K$, defined by $T_V(x) = (T(x) + V) \cap K$ for each $x \in K$ is *-weakly naturally quasiconvex. Then there exists a continuous selection $f_V : K \to K$ such that $f_V(x) \in T_V(x)$. The proof follows the same line as in Theorem 3.5. □

Now we introduce the following definition.

Definition 3.14. Let $B \subset X \times Y$ be a biconvex set, Z a nonempty convex subset of a topological vector space F and $T : B \to 2^Z$ a correspondence. T is called *weakly biconvex* if for each finite set $\{(x_1, y_1), (x_2, y_2), ..., (x_n, y_n)\} \subset B$, there exists $z_i \in T(x_i, y_i)$, $(i = 1, 2, ..., n)$ such that for every biconvex combination $(x, y) = \sum_{i=1}^{n} \lambda_i(x_i, y_i) \in B$ (with $\sum_{i=1}^{n} \lambda_i = 1$, $\lambda_i \geq 0$ $i = 1, 2, ..., n$), there exists $y' \in T(\sum_{i=1}^{n} \lambda_i(x_i, y_i))$ and $y' = \sum_{i=1}^{n} \lambda_i z_i$.

We state the following selection theorem for weakly biconvex correspondences.

Theorem 3.15. *(selection theorem). Let Y be a non-empty convex subset of a topological vector space F and $K \subset E_1 \times E_2$, where E_1, E_2 are topological vector spaces. Suppose that K is the biconvex hull of $\{(a_1, b_1), (a_2, b_2), ..., (a_n, b_n)\} \subset E_1 \times E_2$. Let $T : K \to 2^Y$ be a weakly biconvex correspondence. Then, T has a continuous selection on K.*

Proof. Since T is weakly biconvex, there exists $c_i \in T(a_i, b_i)$, $(i = 1, 2, ..., n)$, such that for every $(\lambda_1, \lambda_2, ..., \lambda_n) \in \Delta_{n-1}$, there exists $z \in T(\sum_{i=1}^{n} \lambda_i(a_i, b_i))$ with $z = \sum_{i=1}^{n} \lambda_i z_i$.

Since K is biconvex hull of $(a_1, b_1), ..., (a_n, b_n)$, there exist unique continuous functions $\lambda_i : K \to \mathbb{R}$, $i = 1, 2, ..., n$ such that for each $(x, y) \in K$, we have $(\lambda_1(x, y), \lambda_2(x, y), ..., \lambda_n(x, y)) \in \Delta_{n-1}$ and $(x, y) = \sum_{i=1}^{n} \lambda_i(x, y)(a_i, b_i)$.

Define $f : K \to 2^Y$ by
$f(a_i, b_i) = c_i$ $(i = 1, ..., n)$ and
$f(\sum_{i=1}^{n} \lambda_i(a_i, b_i)) = \sum_{i=1}^{n} \lambda_i c_i \in T(x, y)$.
We show that f is continuous.

Let $(x_m, y_m)_{m \in N}$ be a sequence which converges to $x_0 \in K$, where $(x_m, y_m) = \sum_{i=1}^{n} \lambda_i(x_m, y_m)(a_i, b_i)$ implies $a_1 = a_2 = ... = a_n = a$ or $b_1 = b_2 = ... = b_n = b$ and $(x_0, y_0) = \sum_{i=1}^{n} \lambda_i(x_0)(a_i, b_i)$ with $a_1 = a_2 = ... = a_n = a$ or $b_1 = b_2 = ... = b_n = b$. By the continuity of λ_i, it follows that for each $i = 1, 2, ..., n$, $\lambda_i(x_m, y_m) \to \lambda_i(x_0, y_0)$ as $m \to \infty$. Hence $f(x_m, y_m) \to f(x_0, y_0)$ as $m \to \infty$, i.e. f is continuous.

We proved that T has a continuous selection on K.

In order to prove the existence theorems of equilibria for a generalized quasi-game, we need the following version of Kim's quasi fixed-point theorem:

Theorem 3.16. *Let I and J be any (possible uncountable) index sets. For each $i \in I$ and $j \in J$, let X_i and Y_j be non-empty compact convex subsets of Hausdorff locally convex spaces E_i and respectively F_j.*

Let $X := \prod X_i$, $Y := \prod_{i\in I} Y_j$ and $Z := X \times Y$.

For each $i \in I$ let $\Phi_i : Z \to 2^{X_i}$ be a correspondence such that the set $W_i = \{(x, y) \in Z \mid \Phi_i(x, y) \neq \emptyset\}$ is open and Φ_i has a continuous selection f_i on W_i.

For each $j \in J$ let $\Psi_j : Z \to 2^{Y_j}$ be an upper semicontinuous correspondence with non-empty closed convex values.

Then there exists a point $(x^*, y^*) \in Z$ such that for each $i \in I$, either $\Phi_i(x^*, y^*) = \emptyset$ or $\bar{x}_i \in \Phi_i(x^*, y^*)$, and for each $j \in J$, $y^*_j \in \Psi_j(x^*, y^*)$.

Proof. We first endow $\prod_{i\in I} E_i$ and $\prod_{j\in J} F_j$ with the product topologies; and then $\prod_{i\in I} E_i \times \prod_{j\in J} F_j$ is also a locally convex Hausdorff topological vector space.

For each $i \in I$, we define a correspondence $\Phi'_i : Z \to 2^{X_i}$ by

$$\Phi'_i(x, y) := \begin{cases} \{f_i(x, y)\}, \text{ if } (x, y) \in W_i, \\ X_i, \quad \text{ if } (x, y) \notin W_i. \end{cases}$$

Then for each $(x, y) \in Z$, $\Phi'_i(x, y)$ is a non-empty closed convex subset of X_i. Also, Φ'_i is an upper semicontinuous correspondence on Z. In fact, for each proper open subset V of X_i, we have

$U := \{(x, y) \in Z \mid \Phi'_i(x, y) \subset V\}$

$=\{(x, y) \in W_i \mid \Phi'_i(x, y) \subset V \cup (x, y) \in Z \setminus W_i \mid \Phi'_i(x, y) \subset V\}$

$...=\{(x, y) \in W_i \mid f_i(x, y) \in V\} \cup \{(x, y) \in Z \setminus W_i \mid X_i \subset V\}$

$=\{(x, y) \in W_i \mid f_i(x, y) \in V\} = f_i^{-1}(V) \cap W_i.$

Since W_i is open and f_i is a continuous map on W_i, U is open, and hence Φ'_i is upper semicontinuous on Z.

Finally, we define a correspondence $\Phi : Z \to 2^Z$ by

$\Phi(x, y) := \prod_{i\in I} \Phi'_i(x, y) \times \prod_{j\in J} \Psi_j(x, y)$ for each $(x, y) \in Z$.

Then, by Lemma 3 in [8], Φ is an upper semicontinuous correspondence such that each $\Phi(x, y)$ is non-empty closed convex. Therefore, by Fan-Glicksebrg fixed point theorem [9] there exists a fixed point $(x^*,y^*) \in Z$ such that $(x^*, y^*) \in \Phi(x, y)$, i.e., for each $i \in I$, $x^*_i \in \Phi'_i(x, y)$, and for each $j \in J$, $y^*_j \in \Psi_j(x, y)$. If $(x^*, y^*) \in W_i$ for some $i \in I$, then $x^*_i = f_i(x^*, y^*) \in \Phi_i(x^*, y^*)$; and if $(x^*, y^*) \notin W_i$ for some $i \in I$, then $\Phi_i(x^*, y^*) = \emptyset$. Therefore, we have that for each $i \in I$, either $\Phi_i(x^*, y^*) = \emptyset$ or $x^*_i \in \Phi_i(x^*, y^*)$. Also, for each $j \in J$, we already have $y^*_j \in \Psi_j(x^*, y^*)$. This completes the proofs. □

We have the following corollary.

Corollary 3.17. *Let I and J be any (possible uncountable) index sets. For each $i \in I$ and $j \in J$, let X_i and Y_j be non-empty compact convex subsets of Hausdorff locally convex spaces E_i and respectivelly F_j.*

Let $X := \prod X_i$, $Y := \prod_{i\in I} Y_j$ *and* $Z := X \times Y$.

For each $i \in I$, *let* $S_i : Z \to 2^{X_i}$ *be a correspondence such that the set* $W_i = \{(x,y) \in Z \mid S_i(x,y) \neq \emptyset\}$ *is the interior of the biconvex hull of* $\{(a_1,b_1),(a_2,b_2),...,(a_n,b_n)\} \subset Z$ *and* S_i *is weakly biconvex on* W_i.

For each $j \in J$, *let* $T_j : Z \to 2^{Y_j}$ *be an upper semicontinuous correspondence with non-empty closed convex values.*

Then there exists a point $(x^*,y^*) \in Z$ *such that for each* $i \in I$, *either* $S_i(x^*,y^*) = \emptyset$ *or* $x_i^* \in S_i(x^*,y^*)$, *and for each* $j \in J$, $y_j^* \in T_j(x^*,y^*)$.

4. Applications in the equilibrium theory

In this paper, we study the following model of a generalized quasi-game.

Definition 4.1. Let I be a nonempty set (the set of agents). For each $i \in I$, let X_i be a non-empty topological vector space representing the set of actions and define $X := \prod_{i\in I} X_i$; let $A_i,\ B_i : X \times X \to 2^{X_i}$ be the constraint correspondences and $P_i : X \times X \to 2^{X_i}$ the preference correspondence. A *generalized quasi-game* $\Gamma = (X_i, A_i, B_i, P_i)_{i\in I}$ is defined as a family of ordered quadruples (X_i, A_i, B_i, P_i).

In particular, when I=$\{1,\ 2...n\}$, Γ is called n-person quasi-game.

Definition 4.2. An *equilibrium* for Γ is defined as a point $(x^*,y^*) \in X \times X$ such that for each $i \in I$, $y_i^* \in \mathrm{cl}B_i(x^*,y^*)$ and $A_i(x^*,y^*) \cap P_i(x^*,y^*) = \emptyset$.

If $A_i(x,y) = B_i(x,y)$ for each $(x,y) \in X \times X$ and $i \in I$, this model coincides with the one introduced by W. K. Kim [13].

If, in addition, for each $i \in I$, A_i, P_i are constant with respect to the first argument, this model coincides with the classical one of the abstract economy and the definition of equilibrium is the one given in [4].

In this work, Kim established an existence result for a generalized quasi-game with a possibly uncountable set of agents, in a locally convex Hausdorff topological vector space.

Here is his result:

Theorem 4.3 (11)**.** *Let* $\Gamma = (X_i, A_i, B_i, P_i)_{i\in I}$ *be a generalized quasi-game, where* I *is a (possibly uncountable) set of agents such that for each* $i \in I$:

(1) X_i *is a non-empty compact convex subset of a Hausdorff locally convex space* E_i *and denote* $X := \prod_{i\in I} X_i$ *and* $Z := X \times X$;

(2) The correspondence $A_i : X \times X \to 2^{X_i}$ *is upper semicontinuous such that* $A_i(x,y)$ *is a non-empty convex subset of* X_i *for each* $(x,y) \in Z$;

(3) $A_i^{-1}(x_i)$ *is (possibly empty) open for each* $x_i \in X_i$;

(4) the correspondence $P_i : Z \to 2^{X_i}$ *is such that* $(A_i \cap P_i)^{-1}(x_i)$ *is (possibly empty) open for each* $x_i \in X_i$;

(5) the set $W_i := \{(x,y) \in Z \mid (A_i \cap P_i)(x,y) \neq \emptyset\}$ *is perfectly normal;*

(6) for each $(x,y) \in W_i$, $x_i \notin coP_i(x,y)$.

Then there exists an equilibrium point $(x^*, y^*) \in X \times X$ *for* Γ*, i.e., for each* $i \in I$*,* $y_i^* \in \mathrm{cl} A_i(x^*, y^*)$ *and* $A_i(x^*, y^*) \cap P_i(x^*, y^*) = \emptyset$.

As application of the selection theorems from section 3, we state a theorem on the existence of the equilibrium for a generalized quasi-game.

Theorem 4.4. *Let* $\Gamma = (X_i, A_i, B_i, P_i)_{i \in I}$ *be a generalized quasi-game where* I *is a (possibly uncountable) set of agents such that for each* $i \in I$ *:*

(1) X_i *is a non-empty compact convex set in a Hausdorff locally convex space* E_i *and denote* $X := \prod_{i \in I} X_i$ *and* $Z := X \times X$*;*

(2) The correspondence $B_i : Z \to 2^{X_i}$ *is non-empty, convex valued such that for each* $(x, y) \in Z$*,* $A_i(x, y) \subset B_i(x, y)$ *and* $\mathrm{cl} B_i$ *is upper semicontinuous;*

(3) the correspondence $A_i \cap P_i : W_i \to 2^{X_i}$ *is weakly naturally quasiconvex;*

(4) the set $W_i := \{(x, y) \in Z \ / \ (A_i \cap P_i)(x, y) \neq \emptyset\}$ *is open and* $\mathrm{cl} W_i$ *is a* $(n-1)$ *dimensional simplex in* Z;

(5) for each $(x, y) \in W_i$*,* $x_i \notin P_i(x, y)$*.*

Then there exists an equilibrium point $(x^*, y^*) \in Z$ *for* Γ*, i.e., for each* $i \in I$*,* $y_i^* \in \mathrm{cl} B_i(x^*, y^*)$ *and* $A_i(x^*, y^*) \cap P_i(x^*, y^*) = \emptyset$.

Proof. For each $i \in I$, we define $\Phi_i : Z \to 2^{X_i}$ by

$$\Phi_i(x, y) = \begin{cases} (A_i \cap P_i)(x, y), & \text{if } (x, y) \in W_i, \\ \emptyset, & \text{if } (x, y) \notin W_i; \end{cases}$$

By applying Theorem 3.3 to the restrictions $A_i \cap P_i$ on W_i, we can obtain that there exists a continuous selection $f_i : W_i \to X_i$ such that $f_i(x, y) \in (A_i \cap P_i)(x, y)$ for each $(x, y) \in W_i$.

For each $j \in I$, we define $\Psi_j : Z \to 2^{X_i}$, by $\Psi_j(x, y) = \mathrm{cl} B_j(x, y)$ for each $(x, y) \in Z$.

Then Ψ_j is an upper semicontinuous correspondence and $\Psi_j(x, y)$ is a non-empty, convex, closed subset of X_j for each $(x, y) \in Z$.

By Theorem 3.16, it follows that there exists $(x^*, y^*) \in Z$ such that for each $i \in I$, either $\Phi_i(x^*, y^*) = \emptyset$ or $x_i^* \in \Phi_i(x^*, y^*)$ and for each $j \in J$, $y_j^* \in \Psi_j(x^*, y^*)$.

If $x_i^* \in \Phi_i(x^*, y^*)$ for some $i \in I$, then $x_i^* \in \Phi_i(x^*, y^*) = (A_i \cap P_i)(x^*, y^*) \subset P_i(x^*, y^*)$ which contradicts the assumption (5).

Therefore, for each $i \in I$, $\Phi_i(x, y) = \emptyset$ and then $(x^*, y^*) \notin W_i$. Hence, $(A_i \cap P_i)(x^*, y^*) = \emptyset$ and for each $i \in I$, $y^* \in \Psi_i(x^*, y^*) = \mathrm{cl} B_i(x^*, y^*)$. □

By using a similar type of proof and Theorem 3.15, we obtain Theorem 4.5.

Theorem 4.5. *Let* $\Gamma = (X_i, A_i, B_i, P_i)_{i \in I}$ *be a generalized quasi-game where* I *is a (possibly uncountable) set of agents such that for each* $i \in I$ *:*

(1) X_i *is a non-empty compact convex set in a Hausdorff locally convex space* E_i *and denote* $X := \prod_{i \in I} X_i$ *and* $Z := X \times X$*;*

(2) The correspondence $B_i : Z \to 2^{X_i}$ *is non-empty, convex valued such that for each* $(x, y) \in Z$*,* $A_i(x, y) \subset B_i(x, y)$ *and* $\mathrm{cl} B_i$ *is upper semicontinuous;*

(3) $A_i \cap P_i$ *is a weakly biconvex correspondence on* W_i*;*

(4) the set $W_i := \{(x, y) \in Z \ / \ (A_i \cap P_i)(x, y) \neq \emptyset\}$ *is the interior of the biconvex hull of* $\{(a_1, b_1), (a_2, b_2), ..., (a_n, b_n)\} \subset Z$*;*

(5) for each $(x, y) \in W_i$*,* $x_i \notin \mathrm{co} P_i(x, y)$*.*

Then there exists an equilibrium point $(x^*, y^*) \in Z$ *for* Γ, *i.e., for each* $i \in I$, $y_i^* \in \text{cl}B_i(x^*, y^*)$ *and* $A_i(x^*, y^*) \cap P_i(x^*, y^*) = \emptyset$.

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References

[1] R.P. Agarwal, O'Regan, *A Note on equilibria for abstract economies,* Math. Computer Modelling, **34**(2001), 331-343.

[2] R. Aumann, S. Hart, *Bi-convexity and bi-martingales,* Israel J. Math., **54**(1986), no. 2, 159-180.

[3] K.J. Arrow, G. Debreu, *Existence of an equilibrium for a competitive economy*, Econometrica, **22**(1954), 265-290.

[4] A. Borglin and H. Keiding, *Existence of equilibrium action and of equilibrium: A note on the "new" existence theorem.* J. Math. Econom., **3**(1976), 313-316.

[5] F.E. Browder, *A new generation of the Schauder fixed point theorems*, Math. Ann., **174**(1967), 285-290.

[6] G. Debreu, *A social equilibrium existence theorem*, Proc. Nat. Acad. Sci., **38**(1952), 886-893.

[7] X. Ding, He Yiran, *Best approximation theorem for set-valued mappings without convex values and continuity*, Appl. Math. Mech., **19**(9)(1998), 831-836.

[8] K. Fan, *Fixed-point and minimax theorems in locally convex topological linear spaces*, Proc. Nat. Acad. Sci. U.S.A., **38**(1952), 121-126.

[9] K. Fan, *A generalization of Tychonoff's fixed point theorem*, Math. Ann., **142**(1961), 305-310.

[10] J. Gorski, F. Pfeuffer, K. Klamroth, *Biconvex sets and optimization with biconvex functions: a survey and extension,* Math. Meth. Oper. Res., **66**(2007), 373-407.

[11] C.D. Horvath, *Point fixes et coincidences pour les applications multivoques sans convexite*, C.R. Acad. Sci. Paris, **296**(1983), 403-406.

[12] W.K. Kim, K.K. Tan, *New existence theorems of equilibria and applications*, Nonlinear Anal., **47**(2001), 531-542.

[13] W.K. Kim, *On a quasi fixed-point theorem*, Bull. Korean Math. Soc., **40**(2003), no. 2, 301-307.

[14] I. Lefebvre, *A quasi fixed-point theorem for a product of u.s.c. or l.s.c. correspondences with an economic application,* Set-Valued Anal., **9**(2001), 273-288.

[15] J.F. Nash, *Non-cooperative games*, Ann. Math., **54**(1951), 286-295.

[16] W. Shafer, H. Sonnenschein, *Equilibrium in abstract economies without ordered preferences*, J. Math. Econom., **2**(1975), 345-348.

[17] G. Tian, *Fixed points theorems for mappings with non-compact and non-convex domains,* J. Math. Anal. Appl., **158**(1991), 161-167.

[18] K. Vind, *Equilibrium with coordination,* J. Math. Econom., **12**(1983), 275-285.

[19] N.C. Yannelis and N.D. Prabhakar, *Existence of maximal elements and equilibrium in linear topological spaces*, J. Math. Econom., **12**(1983), 233-245.

[20] X.Z. Yuan, *The study of minimax inequalities and applications to economies and variational inequalities*, Memoirs of the American Math. Soc., **132**(625)(1988).

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