Fixed Point Theory, 14(2013), No. 2, 379-386
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

ON THE COMPLETENESS OF ORDERED SETS

M.A. KHAMSI*, D. MISANE** AND S.A. AL-MEZEL***

*Department of Mathematical Sciences, The University of Texas at El Paso
El Paso, TX 79968, U.S.A.
E-mail: mohamed@utep.edu

**Département de Mathématiques et d'Informatique, Université Mohammed V
Faculté de Sciences, Rabat, Morocco
E-mail: misane@fsr.ac.ma

***Department of Mathematics, King Abdulaziz University
P.O.Box 80203, Jeddah 21589, Saudi Arabia
E-mail: almezel@ut.edu.sa

Abstract. In this work, we introduce a density property in ordered sets that is weaker than the order density. Then, we prove a strong version of a result proved by Büber and Kirk, which is a special case of the Brouwer Reduction Theorem, in metric spaces relating completeness and density of ordered sets.

Key Words and Phrases: Compactness, complete lattice, convexity structure, fixed points, lattice, nonexpansive mappings, normal structure.
2010 Mathematics Subject Classification: 06F30, 46B20, 47E10, 47H10.

1. Introduction

In [2] Buber and Kirk proved that in separable metric spaces, countably compact convexity structures possess minimal elements. This is crucial to many existence fixed point theorems. A more general result may be obtained through the Brouwer Reduction Theorem [11] which states that if X is a topological space which has a countable base, then any family $\mathcal{F}$ of nonempty closed subsets of X has minimal element provided that the intersection of every descending sequence in $\mathcal{F}$ contains a member of $\mathcal{F}$. There is also the known set-theoretic fact (see [5, 17]) that if $\mathcal{F}$ is a countable family of nonempty subsets of a given set, and if the intersection of every descending sequence in $\mathcal{F}$ contains a member of $\mathcal{F}$; then $\mathcal{F}$ has a minimal element.

In this work, we follow the footsteps of the authors in [9] viewing ordered sets as metric spaces. Then we prove a strong version of Büber and Kirk's result in ordered sets. It is worth to mention that the proofs are given in a metric form although an ordered version could be found. Because it is our belief that this approach will support the idea that certain concepts of infinistic nature, like those which inspired metric spaces, can perfectly apply to the study of discrete sets.

2. Basic definitions and results

Consider a complete lattice $\mathcal{V}$, with a least element 0, greatest element 1, equipped with a semigroup operation $+$ and an involution satisfying the following properties:

1. $\mathcal{V}$ is an ordered semigroup, i.e.
   (i) 0 is its neutral element;
   (ii) if $p \preceq p'$ and $q \preceq q'$ then $p + q \preceq p' + q'$.
2. the involution (which satisfies $\overline{\overline{p}} = p$ for all $p \in \mathcal{V}$) is order-preserving and reverses the semigroup operation, i.e.
$$\overline{p+q} = \overline{q} + \overline{p} \ \text{ holds for all } \ p, q \in \mathcal{V}.$$

Let M be a set. A distance on M is a map $d : M \times M \longrightarrow \mathcal{V}$ satisfying:

(d_1) $d(x,y) = 0$ if and only if $x = y$;
(d_2) $d(x,y) \preceq d(x,z) + d(z,y)$, for all $x, y, z \in M$;
(d_3) $d(x,y) = \overline{d(y,x)}$, for all $x, y \in M$.

The pair (M, d) is said to be a generalized metric space over $\mathcal{V}$; if there is no confusion we will denote it M. We also denote $B_M(x,r)$ the ball with center $x \in M$ and radius $r \in \mathcal{V}$, i.e.
$$B_M(x,r) = \{y \in M; d(x,y) \preceq r\}.$$
If there is no confusion we will denote it $B(x,r)$ instead of $B_M(x,r)$.

Examples:

1. **Classical metric spaces:** We take $\mathcal{V} = \mathbb{R}_+ \cup \{+\infty\}$. Extend the addition to it in the obvious way. The spaces we get are just unions of disjoints copies of classical metric spaces . We assume that $\mathcal{V}$ is a complete ordered set only to have infinite products.
2. **Ordered sets:** Let $\mathcal{V}$ be the complete lattice defined by
$$\mathcal{V} = \{0, \alpha, \beta, 1\},$$
with α incomparable with β, $0 \preceq \alpha \preceq 1$ and $0 \preceq \beta \preceq 1$. The semigroup operation is $a + b = a \vee b$ and the involution is defined by
$$\overline{\alpha} = \beta, \ \overline{\beta} = \alpha, \ \overline{0} = 0 \ \text{ and } \ \overline{1} = 1.$$
If $(M, \preceq)$ is a partially ordered set, then the map $d : M \times M \longrightarrow \mathcal{V}$, defined by:
$$\begin{array}{lll} d(x,y) = 0 & \text{if} & x = y \\ d(x,y) = \alpha & \text{if} & x \preceq y \\ d(x,y) = \beta & \text{if} & y \preceq x \\ d(x,y) = 1 & \text{if} & x \text{ and } y \text{ are incomparable} \end{array}$$
is a generalized metric over $\mathcal{V}$. Conversely, if $(M, \preceq)$ is a generalized metric over $\mathcal{V}$, then the relation defined by
$$x \preceq y \ \text{ iff } \ d(x,y) \preceq \alpha$$
is a partial order on M. The balls of M over $\mathcal{V}$ are:
   (i) the set M;
   (ii) the singletons in M;

(iii) the principal initial segments $(\leftarrow, x] = \{m \in M;\ m \preceq x\}$ for all $x \in M$;
(iv) and the principal final segments $[x, \rightarrow) = \{m \in M;\ x \preceq m\}$ for all $x \in M$.

Definition 2.1. *Let $\mathcal{C}$ be a nonempty family of subsets of M. $\mathcal{C}$ defines a convexity structure on M if $\mathcal{C}$ is stable by intersection and contains the balls.*

In this work, we will consider the smallest convexity structure containing the balls, which we denote $\mathcal{A}(M)$ (or $\mathcal{A}$ if there is no confusion). Clearly, we have $C \in \mathcal{A}$ if and only if $C = \bigcap_{i \in I} B(x_i, r_i)$ where $x_i \in M$ and $r_i \in \mathcal{V}$. Such sets are called admissible. In this work, we will refer to them as convex. Notice that $A \in \mathcal{A}$ if and only if there exists M_1 and M_2 subsets of M such that:

$$A = \{a \in M; m_1 \preceq a \preceq m_2 \quad \text{for every} \quad m_i \in M_i, i = 1, 2\}.$$

Such sets are also called cuts (in ordered set theory [8]).

For the sake of simplicity, we will use the following:

$$A \preceq x \quad \text{iff} \quad a \preceq x, \ \text{ for all } \ a \in A,$$

where A is a subset of M and $x \in M$.

Proposition 2.1. *Let C be a nonempty convex subset of M.*

(i) *If c_1 and c_2 are two elements of C, then the segment $[c_1, c_2]$ is in C, i.e*

$$c_1 \preceq x \preceq c_2 \implies x \in C.$$

(ii) *Let $x \in C$. Then, we have*

$$dist(x, C) = \bigwedge_{c \in C} d(x, c) = 0 \implies x \in C.$$

(iii) *For any family $(c_i)_{i \in I}$ of elements of C, we have:*

$$\bigwedge_{i \in I} c_i \in C \quad \text{and} \quad \bigvee_{i \in I} c_i \in C$$

The proof of the above is obvious using the definition of a convex set.

Definition 2.2. *Let χ be an infinite cardinal.*

(1) *$\mathcal{C}$ is said to be χ-compact if any family $(C_i)_{i \in I}$, with $C_i \in \mathcal{A}$ and $card(I) \leq \chi$, such that $\bigcap_{i \in F} C_i \neq \emptyset$, for any finite subset F of I, has a nonempty intersection, i.e $\bigcap_{i \in I} C_i \neq \emptyset$.*

(2) *We will say that $\mathcal{C}$ is σ-compact (or countably compact) if $\chi = \chi_0$, and compact if $\mathcal{C}$ is χ-compact for any cardinal χ.*

Let us notice that, since for any infinite set I we have

$$card(I) = card\ \{F \subset I; F \text{ finite}\},$$

$\mathcal{C}$ is χ-compact if and only if any nonempty decreasing family $(C_i)_{i \in I}$ of convex sets, where I is downward directed, has a nonempty intersection provided that $card(I) \leq \chi$.

Since the main result of this work relates these properties to completeness, the following definition is needed.

Definition 2.3. *The ordered set M is said to be χ-complete if any subset $S \subset M$ such that $card(S) \leq \chi$, has a least upper bound and a greatest lower bound in M, that is*

$$\bigvee_{x \in S} x \quad \text{and} \quad \bigwedge_{x \in S} x \quad \text{exist in} \quad M.$$

Note that if M is a lattice, M is χ-complete if and only if for every chain $C \subset M$ such that $card(C) \leq \chi$, C has a least upper bound and a greatest lower bound in M. We will say that M is σ-complete if $\chi = \chi_0$ and complete if M is χ-complete for any cardinal χ.

Proposition 2.2. *Assume that M is a lattice. M is χ-complete if and only if $\mathcal{A}$ is χ-compact.*

Proof. Let $(C_i)_{i \in I}$ be a nonempty decreasing family of elements in $\mathcal{A}$. Let $x_i \in C_i$ for any $i \in I$. Since the family $(C_i)_{i \in I}$ is decreasing, we get $x_j \in C_i$ for $j \geq i$. Since C_i is convex and M is χ-complete, we deduce that $u_j = \bigvee\limits_{k \geq j} x_k \in C_i$ for any $j \geq i$. It is easy to see that $(u_j)_{j \in I}$ is decreasing. Therefore, we have

$$u = \bigwedge_{i \in I} u_i = \bigwedge_{k \geq j} u_k, \quad \text{for any} \;\; j \in I.$$

Using the facts $u_j \in C_i$ for any $j \geq i$ and the convexity of C_i, we get

$$u = \bigwedge_{j \geq i} u_j \in C_i, \quad \text{for any} \;\; i \in I.$$

Therefore, $\bigcap\limits_{i \in I} C_i$ is not empty. Conversely, let $C = (x_i)_{i \in I}$ be a chain in M. Take $A_i = [x_i, \rightarrow)$, then $(A_i)_{i \in I}$ is a nonempty decreasing family of elements in $\mathcal{A}$. By assumption, we have $A = \bigcap\limits_{i \in I} A_i \neq \emptyset$. Any $m \in A$ is an upper bound of C. Set $B_i = A_i \bigcap \bigcap\limits_{m \in A} (\leftarrow, m]$, for any $i \in I$. The family $(B_i)_{i \in I}$ is decreasing. Using the χ-compactness, we deduce that $B = \bigcap\limits_{i \in I} \neq \emptyset$. It is easy to check that B is a singleton, i.e. $B = \{s\}$ with s being the least upper bound of C. The same proof leads to the existence of the greatest lower bound. $\square$

Corollary 2.1. *Let M be a lattice.*

(i) *M is σ-complete if and only if $\mathcal{A}$ is σ-compact.*
(ii) *M is complete if and only if $\mathcal{A}$ is compact.*

3. Main result

In [2] Büber and Kirk proved that in a separable metric space, any convexity structure which is countably compact has minimal elements. In order to prove a similar or a stronger result in ordered sets, we will need to define separability and in general the notion of density in ordered sets. We should mention that our initial approach was to consider classical definitions of density in ordered sets. To our knowledge, almost nothing is known. This is why we start this section by some definitions and simple facts regarding density of subsets in ordered sets.

Definition 3.1. *(***Metric density***) Let D be a subset of M. We will say that D is metric dense in M if for every A and B two convex subsets of M such that $A \subset B$, $B \cap D = \emptyset$ and*

$$dist(d, A) = dist(d, B) \quad \textit{for all} \quad d \in D$$

then the equality $A = B$ holds.

Note that, although this definition uses the distance, it can be expressed using the order of M. Indeed, one can prove that D is metric dense in M if and only if for every convex subsets A and B such that $A \subset B$, $B \cap D = \emptyset$ and

(i) for every $d \in D$, if there exists $b \in B$ such that $d \preceq b$, then there exists $a \in A$ such that $d \preceq a$;
(ii) for every $d \in D$, if there exists $b \in B$ such that $b \preceq d$, then there exists $a \in A$ such that $a \preceq d$,

we have $A = B$.

Other natural densities that one could think of are:

(1) **Order density:** D is dense in M if for every $x \in M$, we have:

$$\begin{aligned} x &= \bigwedge\{d \in D; x \preceq d\} \\ &= \bigvee\{d \in D; d \preceq x\} \end{aligned}$$

It is worth to mention that almost any ordered set is order dense in its MacNeille completion [16].

(2) **Real density:** D is dense in M if
   (i) for every $x \in M$, there exist d_1 and d_2 in D such that $d_1 \preceq x \preceq d_2$;
   (ii) for every x and y in M such that $x \preceq y$, there exists $d \in D$ such that $x \preceq d \preceq y$.

These densities are related to the metric density through the following result.

Proposition 3.1. *Let M be a lattice. The real and order densities imply the metric density.*

Proof. (1) Real density implies metric density. Let B be a convex such that $B \cap D = \emptyset$; then B is a singleton. Indeed, let $x, y \in B$ with $x \neq y$. Put $b_1 = x \vee y$ and $b_2 = x \wedge y$, then $b_1, b_2 \in B$, since B is convex. We have $b_1 \preceq b_2$ and $b_1 \neq b_2$. The real density insure the existence of $d \in D$ such that $b_1 \preceq d \preceq b_2$; therefore $d \in B$. Contradiction. Hence, B is a singleton which clearly implies that D is metric dense in M.

(2) Order density implies metric density. Let A and B be convex subsets such that $B \cap D = \emptyset$ and $A \subset B$ satisfying $dist(d, A) = dist(d, B)$ for any $d \in D$. Since A is convex, then there exists A_1 et A_2 in M such that

$$A = \{a \in M;\ \ A_1 \preceq a \preceq A_2\}.$$

Let $x \in B$ and $d \in D$ such that $x \preceq d$. Then $dist(d, B) \leq \beta$. Since $d \notin B$, we have $dist(d, B) = \beta$. Hence $dist(d, A) = \beta$. Therefore, there exists $a \in A$ such that $a \preceq d$. Hence $A_1 \preceq d$, which implies that:

$$A_1 \preceq \bigwedge_{x \preceq d} d.$$

Since the order density implies that $x = \bigwedge\limits_{x \preceq d} d$, we get $A_1 \preceq x$. The same proof leads to $x \preceq A_2$. Therefore, x is in A which implies that $A = B$. □

The converse of Proposition 3.1 is false. Indeed, consider the ordered set M defined on $\{x_1, x_2, x_3, x_4, x_5, x_6\} \cup \mathbb{R}^+$ by: $x_1 \preceq x_2 \preceq x_3 \preceq x_4$; $x_1 \preceq x_6 \preceq x_3$; $x_2 \preceq x_5 \preceq x_4$; x_2 and x_6 are incomparable; x_3 and x_5 are incomparable; $x_4 \preceq x$ for every $x \in \mathbb{R}^+$. Let D be the set $\{x_1, x_4, x_5, x_6\} \cup \mathbb{Q}^+$. It is easy to see that D is metric dense in M; but it isn't neither real dense nor order dense in M. In this example, $\mathbb{R}$ and $\mathbb{Q}$ are respectively the sets of real and rational numbers.

The next theorem is the main result of this work.

Theorem 3.1. *Let M be a lattice. Assume that there exists D metric dense in M, with $card(D) \leq \chi$, and $\mathcal{A}$ is χ-compact. Then $\mathcal{A}$ is compact.*

Proof. Let $D = (x_t)_{t \in \Gamma}$ be metric dense in M, with $card(\Gamma) \leq \chi$. Let also $(C_i)_{i \in I}$ be a nonempty decreasing family of elements in $\mathcal{A}$. Set

$$\lambda(x) \ = \ \bigvee_{i \in I} dist(x, C_i), \quad \text{for} \quad x \in M.$$

For every $x \in M$, $\lambda(x)$ exists since $\mathcal{V}$ is complete. If there exists $x \in M$ such that $\lambda(x) = 0$, then $x \in C_i$ for any $i \in I$. Hence $\bigcap\limits_{i \in I} C_i$ is not empty. Let us assume that $\lambda(x) \neq 0$ for any $x \in D$. Since d is a distance over $\mathcal{V} = \{0, \alpha, \beta, 1\}$, it is easy to see that for any $x \in M$ there exists $i \in I$ such that $\lambda(x) = dist(x, C_i)$. Then for any $x_t \in D$ there exists $i(t) \in I$ such that $\lambda(x_t) = dist(x_t, C_{i(t)})$. Note $x_t \notin C_{i(t)}$, otherwise we will have $\lambda(x_t) = 0$. Set $C_\omega = \bigcap\limits_{t \in \Gamma} C_{i(t)}$. Then $C_\omega \neq \emptyset$. We claim that $\bigcap\limits_{i \in I} C_i = C_\omega$. Indeed, let $i \in I$.

Case 1. There exists $t \in \Gamma$ such that $i \leq i(t)$; then $C_\omega \subset C_{i(t)} \subset C_i$.

Case 2. $i \geq i(t)$ for all $t \in \Gamma$; hence, for any $t \in \Gamma$, we have

$$dist(x_t, C_i) \leq \lambda(x_t) = dist(x_t, C_{i(t)}) \leq dist(x_t, C_\omega) \leq dist(x_t, C_i)$$

Therefore $dist(x_t, C_i) = dist(x_t, C_\omega)$ for all $t \in \Gamma$. Since $C_\omega \cap D = \emptyset$ and $C_i \subset C_\omega$, we get from the metric density of D, that $C_i = C_\omega$. In both cases, we get $C_\omega \subset C_i$ for any $i \in I$. Therefore $C_\omega = \bigcap_{i \in I} C_i$. $\square$

An extension to Büber and Kirk's result will follow from the main theorem by taking $\chi = \chi_0$.

In general these results were motivated by the classical Kirk's fixed point theorem [14]. Recall that a convexity $\mathcal{C}$ on a metric space M is said to be normal if for every nonempty bounded element $C \in \mathcal{C}$, not reduced to one point, there exists $x \in C$ such that

$$r(x, C) = \sup\{d(x, y);\ y \in C\} < diam(C).$$

Kirk's classical fixed point theorem states that a metric space M which possesses a compact normal convexity structure, has the fixed point property, i.e. any map $T : M \to M$ which satisfies

$$d\Big(T(x), T(y)\Big) \leq d(x, y), \ \text{ for every } \quad x, y \in M,$$

has a fixed point $m \in M$, i.e. $T(m) = m$. Such mappings are called nonexpansive (see [1, 7] for more details). The original proof is based on the existence of minimal elements in the convexity structure via Zorn's lemma that holds due to the compactness property. Many authors [3, 12, 13, 15] proved that in many instances, the compactness property does not hold but a weaker version of it is satisfied. This appears to be enough for the existence part of Kirk's result [14]. The negative side of these results is the difficulty to get rid of the normal structure property. It is well known that the existence of minimal sets is crucial to the application of the well known Goebel-Karlovitz Lemma [6, 10] which exploits pathological behavior of minimal sets associated to nonexpansive maps that go beyond the normal structure property.

Let us mention, that among these results which do not use the compactness property and exhibit a kind of constructive behavior, the one proved by Kirk [15] which states that a metric space which possesses a countably compact convexity structure which is normal, has the fixed point property. Let us notice that this result can not be generalized to ordered sets.

Example 3.1. $M = \omega_1$. *Clearly, M is σ-complete and $\mathcal{A}$ has normal structure property, since for any $C \in \mathcal{A}$, we have $\delta(C) = 1$ and $R(C) = \alpha$. Indeed, any convex subset C, not empty, has a least element m. It is clear that $r(m, C) = \alpha$ and $r(x, C) = 1$ for any $x \in C, x \neq m$. But M fails the fixed point property. Indeed, according to Davis* [4]*, a lattice has the fixed point property if and only if it is complete. Obviously M is not complete.*

Acknowledgements. 1) The first and second author gratefully acknowledge the financial support provided by University of Tabuk through the project of international cooperation with the University of Texas at El Paso.
2) The authors thank the referee for pointing out some oversights and calling attention to some related literature.

References

[1] A.G. Aksoy, M.A. Khamsi, *Nonstandard Methods in Fixed Point Theory*, Springer-Verlag, Heidelberg, 1990.

[2] T. Büber, W.A. Kirk, *Constructive aspects of fixed point theory for nonexpansive mappings*, Proc. World Congress of Nonlinear Analysts, Vol. 2, Editor V. Lakshikantan, 2115-2125.

[3] S. Chen, M.A. Khamsi, W.M. Kozlowski, *Some geometrical properties and fixed point theorems in Orlicz spaces*, Jour. Math. Anal. Appl., **155**(2)(1991), 393-412.

[4] A.C. Davis, *A characterization of complete lattices*, Pacific J. Math., **5**(1955), 311-319.

[5] M.K. Fort, *A specialization of Zorn's Lemma*, Duke Math. J., **15**(1948), 763-765.

[6] K. Goebel, *On the structure of minimal invariant sets for nonexpansive mappings*, Annal. Univ. Marie Curie-Sklodowska, **29**(1975), 73-77.

[7] K. Goebel, W.A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge Univ. Press, Cambrigde, 1990.

[8] G. Gratzer, *General Lattice Theory*, Academic Press, New York, San Francisco, 1978.

[9] E. Jawhari, D. Misane, M. Pouzet, *Retracts: graphs and ordered sets from the metric point of view*, Contemporary Mathematics, **57**(1986), 175-226.

[10] L. Karlovitz, *Existence of fixed points for nonexpansive mappings in spaces without normal structure*, Pacific J. Math., **66**(1976), 153-156.

[11] J.L. Kelley, *General Topology*, Van Nostrand, Toronto, New York, 1955.

[12] M.A. Khamsi, *On metric spaces with uniform normal structure*, Proc. Amer. Math. Soc., **106**(3)(1989), 723-726.

[13] M.A. Khamsi, W.M. Kozlowski, S. Reich, *Fixed point theory in Modular function spaces*, Nonlinear Anal., **14**(1990), 935-953.

[14] W.A. Kirk, *A fixed point theorem for mappings which do not increase distances*, The Amer. Math. Monthly, **82**(1965), 1004-1006.

[15] W.A. Kirk, *Nonexpansive mappings in metric and Banach spaces*, Rend. Sem. Mat. Fis. Milano, **51**(1981), 640-642.

[16] H.M. MacNeille, *Partially ordered sets*, Trans. Amer. Math. Soc., **42**(1937), 416-460.

[17] A.N. Milgram, *Partially ordered sets, separating systems, and inductiveness*, Reports Math. Colloquium, Second Series, **1**(1939), 18-30.

Received: July 15, 2011; Accepted: November 10, 2011.