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FIXED POINTS, LIE $*$–HOMOMORPHISMS AND LIE $*$–DERIVATIONS ON LIE C^*–ALGEBRAS

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Abstract. In this paper, using fixed point methods we investigate Lie $*$–homomorphisms between Lie C^*–algebras, and Lie $*$–derivations on Lie C^*–algebras associated with the generalized Jensen–type functional equation

$$\mu f\left(\frac{\sum_{i=1}^{n} x_i}{n}\right) + \mu \sum_{j=2}^{n} f\left(\frac{\sum_{i=1, i\neq j}^{n} x_i - (n-1)x_j}{n}\right) - f(\mu x_1) = 0.$$

Key Words and Phrases: Approximate Lie $*$–homomorphism, approximate Lie $*$–derivation, Lie C $*$–algebra, alternative fixed point.
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1. Introduction

The theory of finite dimensional complex Lie algebras is an important part of Lie theory. It has several applications to physics and connections to other parts of mathematics. With an increasing amount of theory and applications concerning Lie algebras of various dimensions, it is becoming necessary to ascertain applicable tools for handling them. The miscellaneous characteristics of Lie algebras constitute such tools and have also found applications: Casimir operators [1], derived, lower central and upper central sequences, Lie algebra of derivations, radical, nilradical, ideals, subalgebras [36, 56] and recently megaideals [54]. These characteristics are particularly crucial when considering possible affinities among Lie algebras. Physically motivated relations between two Lie algebras, namely contractions and deformations, have been extensively studied, see e.g. [24, 43]. When investigating these kinds of relations in dimensions higher than five, one can encounter insurmountable difficulties. Firstly, aside the semisimple ones, Lie algebras are completely classified only up to dimension 5 and the nilpotent ones up to dimension 6. In higher dimensions, only special types, such as rigid Lie algebras [34] or Lie algebras with fixed structure of nilradical, are only classified [63] (for detailed survey of classification results in lower dimensions see e.g. [54] and references therein). Secondly, if all available characteristics of two results of contraction/deformation are the same then one cannot distinguish them at all.

This often occurs when the result of a contraction is oneparametric or moreparametric class of Lie algebras.

Consider the functional equation $\Im_1(f) = \Im_2(f)$ $(\Im)$ in a certain general setting. A function g is an approximate solution of $(\Im)$ if $\Im_1(g)$ and $\Im_2(g)$ are close in some sense. The Ulam stability problem asks whether or not there is a true solution of $(\Im)$ near g. A functional equation is superstable if every approximate solution of the equation is an exact solution of it.

The stability problem of functional equations originated from a question of Ulam [64] concerning the stability of group homomorphisms. Hyers [35] provided a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers theorem was generalized by T. Aoki [2] for additive mappings and by Th.M. Rassias [59] for linear mappings by considering an unbounded Cauchy difference. The paper of Th.M. Rassias [59] has provided a lot of influence in the development of what we now call generalized Hyers–Ulam stability or as Hyers–Ulam–Rassias stability of functional equations. In 1994, a generalization of the Rassias theorem was obtained by Găvruta [22] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias approach. For more details about various results concerning such problems the reader is referred to [6, 7, 11, 16, 15], [20, 21], [25]–[33], [37, 38, 42, 44, 49] and [57]–[62].

C. Park et al. proved the stability of homomorphisms and derivations in Banach algebras, Banach ternary algebras, C^*–algebras, Lie C^*–algebras and C^*–ternary algebras [44]–[53] (see also [3]–[7], [13]–[19]).

A unital C^*–algebra A, endowed with the Lie product $[x, y] = xy - yx$ on A, is called a Lie C^*–algebra. A $\mathbb{C}$–linear mapping D on a Lie C^*-algebra A is called a Lie derivation if $D([x, y]) = [D(x), y] + [x, D(y)]$ holds for all $x, y \in A$. A $\mathbb{C}$–linear mapping H of a Lie C^*–algebra A to a Lie C^*–algebra B is called a Lie homomorphism if $H([x, y]) = [H(x), H(y)]$ holds for all $x, y \in A$.

C. Park [45] investigated Lie $*$–homomorphisms in Lie C^*–algebras, and Lie $*$–derivations on Lie C^*–algebras associated with the additive functional equation. In this paper, using fixed point methods we investigate Lie $*$–homomorphisms between Lie C^*–algebras, and Lie $*$–derivations on Lie C^*–algebras associated with the generalized Jensen–type functional equation

$$\mu f(\frac{\sum_{i=1}^{n} x_i}{n}) + \mu \sum_{j=2}^{n} f(\frac{\sum_{i=1, i\neq j}^{n} x_i - (n-1)x_j}{n}) - f(\mu x_1) = 0 \qquad (1.1)$$

for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C}; |\lambda| = 1\}$, where $n \geq 2$. Before proceeding to the main results, we recall a fundamental result in fixed point theory.

Theorem 1.1. (*Cf.* [12, 55].) *Suppose that we are given a complete generalized metric space (Ω, d) and a strictly contractive function $T : \Omega \to \Omega$ with Lipschitz constant L. Then for each given $x \in \Omega$, either*

$$d(T^m x, T^{m+1} x) = \infty \quad \text{for all } m \geq 0,$$

or there exists a natural number m_0 such that

- $d(T^m x, T^{m+1} x) < \infty$ *for all* $m \geq m_0$;
- *the sequence $\{T^m x\}$ is convergent to a fixed point y^* of T;*

- *y^* is the unique fixed point of T in the set $\Lambda = \{y \in \Omega : d(T^{m_0}x, y) < \infty\}$;*
- *$d(y, y^*) \le \frac{1}{1-L} d(y, Ty)$ for all $y \in \Lambda$.*

Radu and Cadariu [8, 9, 55] applied the fixed point method to the investigation of functional equations (see also [10, 17, 23, 39, 50]).

Throughout this paper, let A be a Lie C^*–algebra with norm $\| \, . \, \|$ and unit e, and B a Lie C^*–algebra with norm $\| \, . \, \|$. Let $U(A) = \{u \in A | uu^* = u^*u = e\}$.

2. Approximation of Lie $*$-homomorphisms in Lie C^*-algebras

We start our work with the following theorem which establishes the stability of Lie $*$–homomorphisms in Lie C^*–algebras associated with the generalized Jensen–type functional equation (1.1), via fixed point method.

Theorem 2.1. *Let $\ell \in \{-1, 1\}$ be fixed and let $f : A \to B$ be a mapping with $f(0) = 0$ for which there exists a function $\phi : A^{n+2} \to [0, \infty)$ such that*

$$\begin{aligned}\|\mu f(\frac{\sum_{i=1}^n x_i + [z,w]}{n}) + \mu \sum_{j=2}^n f(\frac{\sum_{i=1, i\neq j}^n x_i - (n-1)x_j + [z,w]}{n}) \\ - f(\mu x_1) - \mu[f(z), f(w)]\| \\ \le \phi(x_1, x_2, \cdots, x_n, z, w) \qquad (2.1)\end{aligned}$$

$$\|f(n^m u^*) - f(n^m u)^*\| \le \phi(n^m u, n^m u, \cdots, n^m u, 0, 0) \qquad (2.2)$$

for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C}; |\lambda| = 1\}$, all $u \in U(A)$, $m = 0, 1, \cdots$, and all $x_1, \cdots, x_n, z, w \in A$. If there exists an $L < 1$ such that

$$\phi(x_1, x_2, \cdots, x_n, z, w) \le n^\ell L \phi(\frac{x_1}{n^\ell}, \frac{x_2}{n^\ell}, \cdots, \frac{x_n}{n^\ell}, \frac{z}{n^\ell}, \frac{w}{n^\ell}) \qquad (2.3)$$

for all $x_1, \cdots, x_n, z, w \in A$, then there exists a unique Lie $*$–homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\| \le \frac{L^{\frac{1+\ell}{2}}}{1-L} \phi(x, 0, 0, \cdots, 0) \qquad (2.4)$$

for all $x \in A$.

Proof. Consider the set $X := \{g \mid g : A \to B\}$ and introduce the generalized metric on X as follows:

$$d(g, h) := \inf\{C \in \mathbb{R}^+ : \|g(x) - h(x)\| \le C\phi(x, 0, \cdots, 0), \forall x \in A\}$$

It is easy to show that (X, d) is a generalized complete metric space [10]. Now we define the linear mapping $J : X \to X$ by

$$J(h)(x) = \frac{1}{n^\ell} h(n^\ell x)$$

for all $x \in A$. It is easy to see that

$$d(J(g), J(h)) \le L d(g, h)$$

for all $g, h \in X$. It follows from (2.3) that

$$\lim_{m\to\infty} \frac{1}{n^{m\ell}} \phi(n^{m\ell} x_1, n^{m\ell} x_2, \cdots, n^{m\ell} x_n, n^{m\ell} z, n^{m\ell} w) = 0 \qquad (2.5)$$

for all $x_1, \cdots, x_n, z, w \in A$. Putting $\mu = 1, x_1 = x$ and $z = w = x_j = 0$ for $j = 2, \cdots, n$ in (2.1), we obtain

$$\|nf(\frac{x}{n}) - f(x)\| \leq \phi(x, 0, \cdots, 0) \tag{2.6}$$

for all $x \in A$. Thus by using (2.3) with the case $\ell = 1$, we obtain that

$$\|\frac{1}{n}f(nx) - f(x)\| \leq \frac{1}{n}\phi(nx, 0, \cdots, 0) \leq L\phi(x, 0, \cdots, 0) \tag{2.7}$$

for all $x \in A$, that is,

$$d(f, J(f)) \leq L < \infty. \tag{2.8}$$

Also, it from (2.6) with the case $\ell = -1$, that

$$d(f, J(f)) \leq 1 < \infty. \tag{2.9}$$

By Theorem 1.1, in both case, J has a unique fixed point in the set $X_1 := \{h \in X : d(f, h) < \infty\}$. Let H be the fixed point of J. We note that H is the unique mapping with

$$H(nx) = nH(x)$$

for all $x \in A$, such that there exists $C \in (0, \infty)$ satisfying

$$\|f(x) - H(x)\| \leq C\phi(x, 0, \cdots, 0)$$

for all $x \in A$. On the other hand we have $\lim_{m\to\infty} d(J^m(f), H) = 0$, so

$$\lim_{m\to\infty} \frac{1}{n^{m\ell}} f(n^{m\ell}x) = H(x) \tag{2.10}$$

for all $x \in A$. Also by Theorem 1.1, we have

$$d(f, H) \leq \frac{1}{1-L} d(f, J(f)) \tag{2.11}$$

It follows from (2.8), (2.9) and (2.11), that

$$d(f, H) \leq \frac{L^{\frac{1+\ell}{2}}}{1-L}$$

This implies the inequality (2.4). It follows from (2.1), (2.5) and (2.10), we have

$$\begin{aligned}
&\|\mu H(\frac{\sum_{i=1}^n x_i}{n}) + \mu \sum_{j=2}^n H(\frac{\sum_{i=1, i\neq j}^n x_i - (n-1)x_j}{n}) - H(\mu x_1)\| \\
&= \lim_{m\to\infty} \frac{1}{n^{m\ell}} \|\mu f(n^{m\ell-1} \sum_{i=1}^n x_i) \\
&+ \mu \sum_{j=2}^n f(n^{m\ell-1}(\sum_{i=1, i\neq j}^n x_i - (n-1)x_j)) - f(\mu n^{m\ell} x_1)\| \\
&\leq \lim_{m\to\infty} \frac{1}{n^{m\ell}} \phi(n^{m\ell}x_1, n^{m\ell}x_2, \cdots, n^{m\ell}x_n, 0, 0) = 0
\end{aligned}$$

for all $x_1, \cdots, x_n \in A$. So

$$\mu H(\frac{\sum_{i=1}^n x_i}{n}) + \mu \sum_{j=2}^n H(\frac{\sum_{i=1, i\neq j}^n x_i - (n-1)x_j}{n}) = H(\mu x_1)$$

for all $\mu \in \mathbb{T}^1$ and all $x_1, \cdots, x_n \in A$. Put $\mu = 1$ in above equation then

$$H(\frac{\sum_{i=1}^n x_i}{n}) + \sum_{j=2}^n H(\frac{\sum_{i=1, i\neq j}^n x_i - (n-1)x_j}{n}) = H(x_1)$$

for all $x_1, \cdots, x_n \in A$. This means H satisfies (1.1). Putting $w_1 = \frac{\sum_{i=1}^n x_i}{n}$ and $w_j = \frac{\sum_{i=1, i\neq j}^n x_i - (n-1)x_j}{n}$ for $j = 2, 3, \cdots, n$ in above equation, we get

$$H(\sum_{j=1}^n w_j) = \sum_{j=1}^n H(w_j) \tag{2.12}$$

for all $w_1, \cdots, w_n \in A$. Setting $w_j = 0$ for $j = 3, 4, \cdots, n$ in (2.12), we get

$$H(w_1 + w_2) = H(w_1) + H(w_2)$$

Hence, H is cauchy additive. Letting $x_i = x$ and $z = w = 0$ for $i = 1, 2, \cdots, n$ in (2.1), we have

$$\|\mu f(x) - f(\mu x)\| \leq \phi(x, x, \cdots, x, 0, 0)$$

for all $x \in A$. It follows that

$$\begin{aligned}\|H(\mu x) - \mu H(x)\| &= \lim_{m\to\infty} \frac{1}{n^{m\ell}} \|f(\mu n^{m\ell} x) - \mu f(n^{m\ell} x)\| \\ &\leq \lim_{m\to\infty} \frac{1}{n^{m\ell}} \phi(n^{m\ell} x, n^{m\ell} x, \cdots, n^{m\ell} x, 0, 0) = 0\end{aligned}$$

for all $\mu \in \mathbb{T}^1$, and all $x \in A$. Thus

$$H(\mu x) = \mu H(x) \tag{2.13}$$

for all $\mu \in \mathbb{T}^1$, and all $x \in A$.

Now let $\lambda \in \mathbb{C}$ $(\lambda \neq 0)$ and M an integer greater than $(n+1)|\lambda|$. Then $|\frac{\lambda}{M}| < \frac{1}{n+1} < 1 - \frac{2}{n}$ $(n > 2)$. By ([41], Theorem 1), there exists n elements $\mu_1, \mu_2, \cdots, \mu_n \in \mathbb{T}^1$, such that $n\frac{\lambda}{M} = \mu_1 + \mu_2 + \cdots + \mu_n$. And $H(x) = H(n.\frac{1}{n}x) = nH(\frac{1}{n}x)$ for all $x \in A$. So $H(\frac{1}{n}x) = \frac{1}{n}H(x)$ for all $x \in A$. Thus by (2.13)

$$\begin{aligned}H(\lambda x) &= H(\frac{M}{n}.n\frac{\lambda}{M}x) = MH(\frac{1}{n}.n\frac{\lambda}{M}x) = \frac{M}{n}H(n\frac{\lambda}{M}x) \\ &= \frac{M}{n}H(\mu_1 x + \mu_2 x + \cdots + \mu_n x) = \frac{M}{n}(H(\mu_1 x) + H(\mu_2 x) + \cdots + H(\mu_n x)) \\ &\frac{M}{n}(\mu_1 + \mu_2 + \cdots + \mu_n)H(x) = \frac{M}{n}.n\frac{\lambda}{M}H(x) = \lambda H(x)\end{aligned}$$

for all $x \in A$. Hence

$$H(\xi x + \eta y) = H(\xi x) + H(\eta y) = \xi H(x) + \eta H(y)$$

for all $\xi, \eta \in \mathbb{C}$ $(\xi, \eta \neq 0)$ and all $x, y \in A$. And $H(0x) = 0 = 0H(x)$ for all $x \in A$. So the unique additive mapping $H : A \to B$ is a $\mathbb{C}$–linear mapping.

By (2.2) and (2.5), we get

$$\|H(u^*) - H(u)^*\| = \lim_{m\to\infty} \frac{1}{n^{m\ell}} \|f(n^{m\ell} u^*) - f(n^{m\ell} u)^*\|$$

$$\leq \lim_{m\to\infty} \frac{1}{n^{m\ell}}\phi(n^{m\ell}u, n^{m\ell}u, \cdots, n^{m\ell}u, 0, 0) = 0$$

for all $u \in U(A)$. Since $H : A \to B$ is $\mathbb{C}$–linear and each $x \in A$ is a finite linear combiniation of unitary elements ([40], Theorem 4.1.7), i.e., $x = \sum_{j=1}^{k} \lambda_j u_j$ ($\lambda_j \in \mathbb{C}, u_j \in U(A)$),

$$H(x^*) = H(\sum_{j=1}^{k} \overline{\lambda}_j u_j^*) = \sum_{j=1}^{k} \overline{\lambda}_j H(u_j^*) = \sum_{j=1}^{k} \overline{\lambda}_j H(u_j)^* = (\sum_{j=1}^{k} \lambda_j H(u_j))^*$$
$$= H((\sum_{j=1}^{k} \lambda_j u_j)^* = H(x)^*$$

for all $x \in A$. It follows from (2.10) that

$$H(x) = \lim_{m\to\infty} \frac{f(n^{2m\ell}x)}{n^{2m\ell}}$$

for all $x \in A$. Let $x_i = 0$ for $i = 1, 2, \cdots, n$ in (2.1), then we get

$$\|f([z, w]) - [f(z), f(w)]\| \leq \phi(0, 0, \cdots, 0, z, w)$$

for all $z, w \in A$. Since

$$\|H([z, w]) - [H(z), H(w)]\| = \lim_{m\to\infty} \frac{1}{n^{2m\ell}}\|f(n^{2m\ell}[z, w]) - [f(n^{m\ell}z), f(n^{m\ell}w)]\|$$
$$= \lim_{m\to\infty} \frac{1}{n^{2m\ell}}\|f([n^{m\ell}z, n^{m\ell}w]) - [f(n^{m\ell}z), f(n^{m\ell}w)]\|$$
$$\leq \lim_{m\to\infty} \frac{1}{n^{2m\ell}}\phi(0, 0, \cdots, 0, n^{m\ell}z, n^{m\ell}w) \leq \lim_{m\to\infty} \frac{1}{n^{m\ell}}\phi(0, 0, \cdots, 0, n^{m\ell}z, n^{m\ell}w) = 0$$

for all $z, w \in A$. So

$$H([z, w]) = [H(z), H(w)]$$

for all $z, w \in A$. Hence the $\mathbb{C}$–linear $H : A \to B$ is a Lie ∗–homomorphism satisfying the inequality (2.4), as desired. □

Example 2.2. *Let $\ell = 1$ and $L = \frac{1}{n}$ in above Theorem, and let A be a unital C^*–algebra, and let a mapping $f : A \to A$ be defined by*

$$f(x) = \begin{cases} x & \text{for} \quad \|x\| < 1, \\ 0 & \text{for} \quad \|x\| \geq 1, \end{cases}$$

for all $x \in A$. Let $\phi(x_1, x_2, \cdots, x_n, z, w) = n + 2$ for all $x_1, \cdots, x_n, z, w \in A$. Then

$$\|\mu f(\frac{\sum_{i=1}^{n} x_i + [z, w]}{n}) + \mu \sum_{j=2}^{n} f(\frac{\sum_{i=1, i\neq j}^{n} x_i - (n-1)x_j + [z, w]}{n})$$
$$- f(\mu x_1) - \mu[f(z), f(w)]\|$$
$$\leq \phi(x_1, x_2, \cdots, x_n, z, w) = n + 2,$$

$$\frac{L}{1-L}\phi(x, 0, 0, \cdots, 0) = \frac{\frac{1}{n}}{1-\frac{1}{n}}(n + 2) = \frac{n+2}{n-1} < \infty,$$

$$\|f(n^m u^*) - f(n^m u)^*\| = 0 \leq \phi(n^m u, n^m u, \cdots, n^m u, 0, 0) = n + 2$$

for all $\mu \in \mathbb{T}^1$, *all* $x_1, \cdots, x_n, z, w \in A$, $m = 0, 1, \cdots$, *and all* $u \in U(A)$. *But the mapping* $f : A \to A$ *is not a Lie* ∗*–homomorphism.*

(a) *For* $x = 0$,

$$H(0) = \lim_{m\to\infty} \frac{f(n^m 0)}{n^m} = \lim_{m\to\infty} \frac{f(0)}{n^m} = \lim_{m\to\infty} \frac{0}{n^m} = 0.$$

(b) *For each* $x \neq 0$, $\|n^m x\| = n^m \|x\| \geq 1$ *for all sufficiently large integer* m. *So*

$$H(x) = \lim_{m\to\infty} \frac{f(n^m x)}{n^m} = \lim_{m\to\infty} \frac{0}{n^m} = 0.$$

Therefore, the unique Lie ∗*–homomorphism* $H : A \to A$ *must be identically zero and satisfies*

$$\|f(x) - H(x)\| \leq \frac{n+2}{n-1}$$

for all $x \in A$.

Corollary 2.3. *Let* $\ell \in \{-1, 1\}$ *be fixed and* θ *and* p *be non–negative real numbers such that* $p\,\ell < \ell$. *Suppose that a function* $f : A \to B$ *with* $f(0) = 0$ *satisfies*

$$\begin{aligned}\|\mu f(\frac{\sum_{i=1}^n x_i + [z,w]}{n}) &+ \mu \sum_{j=2}^n f(\frac{\sum_{i=1, i\neq j}^n x_i - (n-1)x_j + [z,w]}{n}) \\ &- f(\mu x_1) - \mu[f(z), f(w)]\| \\ &\leq \theta(\sum_{i=1}^n \|x_i\|^p + \|z\|^p + \|w\|^p),\end{aligned}$$

$$\|f(n^m u^*) - f(n^m u)^*\| \leq n.n^{mp}\theta$$

for all $\mu \in \mathbb{T}^1$, *all* $u \in U(A)$, $m = 0, 1, \cdots$, *and all* $x_1, \cdots, x_n, z, w \in A$. *Then there exists a unique Lie* ∗*–homomorphism* $H : A \to B$ *such that*

$$\|f(x) - H(x)\| \leq \frac{2^p}{\ell(2 - 2^p)} \theta \|x\|^p$$

for all $x \in A$.

Proof. Define $\phi(x_1, x_2, \cdots, x_n, z, w) = \theta(\sum_{i=1}^n \|x_i\|^p + \|z\|^p + \|w\|^p)$ and apply Theorem 2.1. □

Theorem 2.4. *Let* $\ell \in \{-1, 1\}$ *be fixed and let* $f : A \to B$ *be a mapping with* $f(0) = 0$ *for which there exists a function* $\phi : A^{n+2} \to [0, \infty)$ *satisfying* (2.2) *and* (2.3), *such that*

$$\begin{aligned}\|\mu f(\frac{\sum_{i=1}^n x_i + [z,w]}{n}) &+ \mu \sum_{j=2}^n f(\frac{\sum_{i=1, i\neq j}^n x_i - (n-1)x_j + [z,w]}{n}) \\ &- f(\mu x_1) - \mu[f(z), f(w)]\| \\ &\leq \phi(x_1, x_2, \cdots, x_n, z, w) \qquad (2.14)\end{aligned}$$

for all $\mu = 1, i$, *and all* $x_1, \cdots, x_n, z, w \in A$. *If* $f(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in A$, *then there exists a unique Lie* ∗*–homomorphism* $H : A \to B$ *satisfying the inequality* (2.4).

Proof. Put $z = w = 0$ and $\mu = 1$ in (2.14). By the same reasoning as in the proof of Theorem 2.1, there exists a unique additive mapping $H : A \to B$ satisfying the inequality (2.4). The additive mapping $H : A \to B$ is given by

$$H(x) = \lim_{m\to\infty} \frac{1}{n^{m\ell}} f(n^{m\ell}x)$$

for all $x \in A$. By the same reasoning as in the proof of ([59], Theorem), the additive mapping $H : A \to B$ is $\mathbb{R}$–linear.

Putting $\mu = i$, $z = w = 0$ and $x_i = x$ for $i = 1, 2, \cdots, n$ in (2.14), we get

$$\|if(x) - f(ix)\| \leq \phi(x, x, \cdots, x, 0, 0)$$

for all $x \in A$. So

$$\|iH(x) - H(ix)\| = \lim_{m\to\infty} \frac{1}{n^{m\ell}} \|if(n^{m\ell}x) - f(n^{m\ell}ix)\|$$

$$\leq \lim_{m\to\infty} \frac{1}{n^{m\ell}} \phi(n^{m\ell}x, n^{m\ell}x, \cdots, n^{m\ell}x, 0, 0) = 0$$

for all $x \in A$. Hence

$$iH(x) = H(ix)$$

for all $x \in A$. For each element $\lambda \in \mathbb{C}$, $\lambda = s + it$, where $s, t \in \mathbb{R}$. So

$$H(\lambda x) = H(sx + itx) = sH(x) + tH(ix) = sH(x) + itH(x) = (s + it)H(x) = \lambda H(x)$$

for all $x \in A$. So

$$H(\xi x + \eta y) = H(\xi x) + H(\eta y) = \xi H(x) + \eta H(y)$$

for all $\xi, \eta \in \mathbb{C}$ ($\xi, \eta \neq 0$) and all $x, y \in A$. Hence the additive mapping $H : A \to B$ is $\mathbb{C}$–linear.

The rest of the proof is the same as in the proof of Theorem 2.1. □

3. Approximation of Lie $*$-derivations in Lie C^*-algebras

In this section, we prove the following stability problem for Lie $*$–derivations in Lie C^*–algebras associated with the generalized Jensen–type functional equation (1.1), via fixed point method.

Theorem 3.1. *Let $\ell \in \{-1, 1\}$ be fixed and let $f : A \to A$ be a mapping with $f(0) = 0$ for which there exists a function $\psi : A^{n+2} \to [0, \infty)$ such that*

$$\begin{aligned}\|\mu f\Big(\frac{\sum_{i=1}^n x_i + [z, w]}{n}\Big) + \mu \sum_{j=2}^{n} f\Big(\frac{\sum_{i=1, i\neq j}^n x_i - (n-1)x_j + [z, w]}{n}\Big) \\ - f(\mu x_1) - \mu[f(z), w] - \mu[z, f(w)]\| \\ \leq \psi(x_1, x_2, \cdots, x_n, z, w) \qquad (3.1)\end{aligned}$$

$$\|f(n^m u^*) - f(n^m u)^*\| \leq \psi(n^m u, n^m u, \cdots, n^m u, 0, 0) \qquad (3.2)$$

for all $\mu \in \mathbb{T}^1$, all $u \in U(A)$, $m = 0, 1, \cdots$, and all $x_1, \cdots, x_n, z, w \in A$. If there exists an $L < 1$ such that

$$\psi(x_1, x_2, \cdots, x_n, z, w) \leq \frac{L}{n^\ell} \psi(n^\ell x_1, n^\ell x_2, \cdots, n^\ell x_n, n^\ell z, n^\ell w) \qquad (3.3)$$

for all $x_1, \cdots, x_n, z, w \in A$, *then there exists a unique Lie* $*$*–derivation* $D : A \to A$ *such that*

$$\|f(x) - D(x)\| \leq \frac{L^{\frac{1+\ell}{2}}}{n(1-L)}\psi(nx, 0, 0, \cdots, 0) \tag{3.4}$$

for all $x \in A$.

Proof. Putting $\mu = 1,\ x_1 = x$ and $z = w = x_j = 0$ for $j = 2, \cdots, n$ in (3.1), we obtain

$$\|nf(\frac{x}{n}) - f(x)\| \leq \psi(x, 0, \cdots, 0) \tag{3.5}$$

for all $x \in A$. Replacing x by nx in (3.5), we get

$$\|\frac{1}{n}f(nx) - f(x)\| \leq \frac{1}{n}\psi(nx, 0, \cdots, 0) \tag{3.6}$$

for all $x \in A$. Consider the set $X' := \{g \mid g : A \to A\}$ and introduce the generalized metric on X' as follows:

$$d(g, h) := \inf\{C \in \mathbb{R}^+ : \|g(x) - h(x)\| \leq C\psi(nx, 0, \cdots, 0), \forall x \in A\}$$

It is easy to show that (X', d) is a generalized complete metric space.

Now we define the linear mapping $T : X' \to X'$ by

$$T(h)(x) = n^{\ell}h(\frac{x}{n^{\ell}})$$

for all $x \in A$. It is easy to see that

$$d(T(g), T(h)) \leq Ld(g, h)$$

for all $g, h \in X'$. It follows from (3.5) by using (3.3), with the case $\ell = 1$, that

$$\|nf(\frac{x}{n}) - f(x)\| \leq \psi(x, 0, \cdots, 0) \leq \frac{L}{n}\psi(nx, 0, \cdots, 0) \tag{3.7}$$

for all $x \in A$, that is,

$$d(f, T(f)) \leq \frac{L}{n} < \infty. \tag{3.8}$$

It follows from (3.6) with the case $\ell = -1$, that

$$d(f, T(f)) \leq \frac{1}{n} < \infty. \tag{3.9}$$

By Theorem 1.1, in both case, T has a unique fixed point in the set $X_2 := \{g \in X' : d(f, g) < \infty\}$. Let D be the fixed point of T. D is the unique mapping with $D(nx) = nD(x)$ for all $x \in A$, such that there exists $C \in (0, \infty)$ satisfying

$$\|f(x) - D(x)\| \leq C\psi(nx, 0, \cdots, 0)$$

for all $x \in A$. On the other hand we have $\lim_{m\to\infty} d(T^m(f), D) = 0$. It follows that

$$\lim_{m\to\infty} n^{m\ell}f(\frac{x}{n^{m\ell}}) = D(x) \tag{3.10}$$

for all $x \in A$. Also by Theorem 1.1, we have

$$d(f, D) \leq \frac{1}{1-L}d(f, T(f)) \tag{3.11}$$

It follows from (3.8), (3.9) and (3.11), that

$$d(f, D) \leq \frac{L^{\frac{1+\ell}{2}}}{n(1-L)}$$

This implies the inequality (3.4). It follows from (3.3) that

$$\lim_{m\to\infty} n^{m\ell}\psi(\frac{x_1}{n^{m\ell}}, \frac{x_2}{n^{m\ell}}, \cdots, \frac{x_n}{n^{m\ell}}, \frac{z}{n^{m\ell}}, \frac{w}{n^{m\ell}}) = 0 \tag{3.12}$$

for all $x_1, \cdots, x_n, z, w \in A$. By the same reasoning as the proof of Theorem 2.1, One can show that the mapping $D : A \to A$ is $\mathbb{C}$–linear. By (3.2) and (3.12), we get

$$\|D(u^*) - D(u)^*\| = \lim_{m\to\infty} n^{m\ell}\|f(\frac{u^*}{n^{m\ell}}) - f(\frac{u}{n^{m\ell}})^*\|$$
$$\leq \lim_{m\to\infty} n^{m\ell}\psi(\frac{u}{n^{m\ell}}, \frac{u}{n^{m\ell}}, \cdots, \frac{u}{n^{m\ell}}, 0, 0) = 0$$

for all $u \in U(A)$. By the same reasoning as the proof of Theorem 2.1, one can show that $D(x^*) = D(x)^*$ for all $x \in A$. It follows from (3.10) that

$$\lim_{m\to\infty} n^{2m\ell} f(\frac{x}{n^{2m\ell}}) = D(x)$$

for all $x \in A$. Let $x_i = 0$ for $i = 1, 2, \cdots, n$ in (3.1), then we get

$$\|f([z, w]) - [f(z), w] - [z, f(w)]\| \leq \psi(0, 0, \cdots, 0, z, w)$$

for all $z, w \in A$. Since

$$\begin{aligned}
&\|D([z, w]) - [D(z), w] - [z, D(w)]\| \\
&= \lim_{m\to\infty} n^{2m\ell}\|f(\frac{1}{n^{2m\ell}}[z, w]) - [f(\frac{z}{n^{m\ell}}), \frac{w}{n^{m\ell}}] - [\frac{z}{n^{m\ell}}, f(\frac{w}{n^{m\ell}})]\| \\
&= \lim_{m\to\infty} n^{2m\ell}\|f([\frac{z}{n^{m\ell}}, \frac{w}{n^{m\ell}}]) - [f(\frac{z}{n^{m\ell}}), \frac{w}{n^{m\ell}}] - [\frac{z}{n^{m\ell}}, f(\frac{w}{n^{m\ell}})]\| \\
&\leq \lim_{m\to\infty} n^{2m\ell}\psi(0, 0, \cdots, 0, \frac{z}{n^{m\ell}}, \frac{w}{n^{m\ell}}) = 0
\end{aligned}$$

for all $z, w \in A$. So

$$D([z, w]) = [D(z), w] + [z, D(w)]$$

for all $z, w \in A$. Hence the $\mathbb{C}$–linear $D : A \to A$ is a Lie $*$–homomorphism satisfying the inequality (3.4), as desired. □

Corollary 3.2. *Let $\ell \in \{-1, 1\}$ be fixed and ε and p be non–negative real numbers, such that $p\ \ell < \ell$. Suppose that a function $f : A \to A$ with $f(0) = 0$ such that*

$$\begin{aligned}
\|\mu f(\frac{\sum_{i=1}^n x_i + [z, w]}{n}) &+ \mu\sum_{j=2}^n f(\frac{\sum_{i=1, i\neq j}^n x_i - (n-1)x_j + [z, w]}{n}) \\
&- f(\mu x_1) - \mu[f(z), w] - \mu[z, f(w)]\| \\
&\leq \varepsilon(\sum_{i=1}^n \|x_i\|^p + \|z\|^p + \|w\|^p),
\end{aligned}$$

$$\|f(n^m u^*) - f(n^m u)^*\| \leq n.n^{mp}\varepsilon$$

for all $\mu \in \mathbb{T}^1$, *all* $u \in U(A)$, $m = 0, 1, \cdots$, *and all* $x_1, \cdots, x_n, z, w \in A$. *Then there exists a unique Lie* ∗*–derivation* $D : A \to A$ *such that*

$$\|f(x) - D(x)\| \le \frac{n^{p-1}2^p}{\ell(2-2^p)}\varepsilon\|x\|^p$$

for all $x \in A$.

Theorem 3.3. *Let* $\ell \in \{-1, 1\}$ *be fixed and let* $f : A \to A$ *be a mapping with* $f(0) = 0$ *for which there exists a function* $\psi : A^{n+2} \to [0, \infty)$ *satisfying* (3.2) *and* (3.3), *such that*

$$\begin{aligned}\|\mu f(\frac{\sum_{i=1}^n x_i + [z,w]}{n}) + \mu\sum_{j=2}^n f(\frac{\sum_{i=1,i\neq j}^n x_i - (n-1)x_j + [z,w]}{n}) \\ - f(\mu x_1) - \mu[f(z), w] - \mu[z, f(w)]\| \\ \le \psi(x_1, x_2, \cdots, x_n, z, w) \qquad (3.13)\end{aligned}$$

for all $\mu = 1, i$, *and all* $x_1, \cdots, x_n, z, w \in A$. *If* $f(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in A$, *then there exists a unique Lie* ∗*–derivation* $D : A \to A$ *satisfying the inequality* (3.4).

Proof. Put $z = w = 0$ and $\mu = 1$ in (3.13). By the same reasoning as in the proof of Theorem 2.1, there exists a unique additive mapping $D : A \to A$ satisfying the inequality (3.4). The additive mapping $D : A \to A$ is given by

$$D(x) = \lim_{m\to\infty} n^{m\ell} f(\frac{x}{n^{m\ell}})$$

for all $x \in A$. By the same reasoning as in the proof of ([59], Theorem), the additive mapping $D : A \to A$ is $\mathbb{R}$–linear.
Putting $\mu = i$, $z = w = 0$ and $x_i = x$ for $i = 1, 2, \cdots, n$ in (3.13), we get

$$\|if(x) - f(ix)\| \le \psi(x, x, \cdots, x, 0, 0)$$

for all $x \in A$. So

$$\begin{aligned}\|iD(x) - D(ix)\| &= \lim_{m\to\infty} n^{m\ell}\|if(\frac{x}{n^{m\ell}}) - f(\frac{ix}{n^{m\ell}})\| \\ &\le \lim_{m\to\infty} n^{m\ell}\psi(\frac{x}{n^{m\ell}}, \frac{x}{n^{m\ell}}, \cdots, \frac{x}{n^{m\ell}}, 0, 0) = 0\end{aligned}$$

for all $x \in A$. Hence

$$iD(x) = D(ix)$$

for all $x \in A$. For each element $\lambda \in \mathbb{C}$, $\lambda = s + it$, where $s, t \in \mathbb{R}$. So

$$D(\lambda x) = D(sx + itx) = sD(x) + tD(ix) = sD(x) + itD(x) = (s + it)D(x) = \lambda D(x)$$

for all $x \in A$. So

$$D(\xi x + \eta y) = D(\xi x) + D(\eta y) = \xi D(x) + \eta D(y)$$

for all $\xi, \eta \in \mathbb{C}$ $(\xi, \eta \neq 0)$ and all $x, y \in A$. Hence the additive mapping $D : A \to A$ is $\mathbb{C}$–linear.

The rest of the proof is the same as in the proof of Theorem 2.1. □

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