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RELAXED IMPLICIT EXTRAGRADIENT-LIKE METHODS FOR FINDING MINIMUM-NORM SOLUTIONS OF THE SPLIT FEASIBILITY PROBLEM

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Abstract. In this paper, we consider the split feasibility problem (SFP) in infinite-dimensional Hilbert spaces, and study the relaxed implicit extragradient-like methods for finding a common element of the solution set Γ of the SFP and the set $\mathrm{Fix}(S)$ of fixed points of a nonexpansive mapping S. Combining Mann's implicit iterative method and Korpelevich's extragradient method, we propose two implicit iterative algorithms for finding an element of $\mathrm{Fix}(S)\cap\Gamma$. On one hand, for $S = I$, the identity mapping, we derive the strong convergence of one implicit iterative algorithm to the minimum-norm solution of the SFP under appropriate conditions. On the other hand, we also derive the weak convergence of another implicit iterative algorithm to an element of $\mathrm{Fix}(S)\cap\Gamma$ under mild assumptions.

Key Words and Phrases: Relaxed implicit extragradient-like methods, split feasibility problems, fixed point problems, nonexpansive mappings, minimum-norm solutions, demiclosedness principle.
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1. Introduction

Let $\mathcal{H}$ be a real Hilbert space, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. We will denote by $x_n \to x$ (respectively, $x_n \rightharpoonup x$), the strong (respectively, weak) convergence of the sequence $\{x_n\}$ to x. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two infinite-dimensional real Hilbert spaces and let C and Q be the nonempty closed

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convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The split feasibility problem (SFP) is to find a point x^* with the property:

$$x^* \in C \quad \text{and} \quad Ax^* \in Q. \tag{1.1}$$

It was first considered by Censor and Elfving [7], in finite-dimensional Hilbert spaces, for modeling inverse problems which arise from phase retrievals and in medical image reconstruction [3]. It has been found that the SFP can also be used to model intensity-modulated radiation therapy (IMRT) [8] - [11]. In the recent past, a wide variety of iterative methods have been used in signal processing and image reconstruction and for solving the SFP; see, e.g., [12] - [10] and the references therein. A special case of the SFP is the following convex constrained linear inverse problem:

$$\text{find } x \in C \quad \text{such that } Ax = b. \tag{1.2}$$

It has been extensively investigated in the literature using the projected Landweber iterative method; see, e.g., [14], [18] and the references therein. Comparatively, the SFP has received much less attention so far, due to the complexity resulting from the set Q. Therefore, whether various versions of the projected Landweber iterative method [18] can be extended to solve the SFP remains an interesting open topics. The original algorithm given in [7] involves the computation of the inverse A^{-1} (assuming the existence of the inverse of A), and thus, did not become popular. A seemingly more popular algorithm that solves the SFP is the CQ algorithm of Byrne [4], [3] which is found to be a gradient-projection method (GPM) in convex minimization (it is also a special case of the proximal forward-backward splitting method [12]). The CQ algorithm only involves the computation of the projections P_C and P_Q onto the sets C and Q, respectively, and therefore it is implementable in the case where P_C and P_Q have closed-form expressions (for example, C and Q are closed balls or half-spaces). However, it remains a challenge how to implement the CQ algorithm in the case where the projections P_C and/or P_Q fail to have closed-form expressions, though theoretically we can prove the (weak) convergence of the algorithm. Very recently, Xu [30] gave a continuation of the study on the CQ algorithm and its convergence. He applied Mann's algorithm to the SFP and purposed an averaged CQ algorithm which is proved to be weakly convergent to a solution of the SFP. He also established the strong convergence result, which shows that the minimum-norm solution can be obtained.

On the other hand, Korpelevich [17] introduced the extragradient method for computing a solution of a variational inequality. She also proved that the sequences generated by this method converge to a solution of a variational inequality. Motivated by the idea of an extragradient method, Nadezhkina and Takahashi [19] introduced an iterative algorithm for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of a variational inequality. Meanwhile, Nadezhkina and Takahashi [20] provided a strong convergence theorem inspired by the extragradient method as well.

Furthermore, assume that the SFP is consistent, that is, the solution set $\mathit{\Gamma}$ of the SFP is nonempty. Let $\mathcal{H}$ be a real Hilbert space and $f : \mathcal{H} \to \mathbb{R}$ be a function. Then the minimization problem

$$\min_{x\in C} f(x) := \frac{1}{2}\|Ax - P_Q Ax\|^2 \tag{1.3}$$

is ill-posed. Xu [30] considered the following Tikhonov's regularization problem:

$$\min_{x\in C} f_\alpha(x) := \frac{1}{2}\|Ax - P_Q Ax\|^2 + \frac{1}{2}\alpha\|x\|^2, \tag{1.4}$$

where $\alpha > 0$ is the regularization parameter. The regularized minimization (1.4) has a unique solution which is denoted by x_α. The following result is not hard to prove.

Proposition 1.1. (see [30], [5]) There hold the following statements:

(i) the gradient

$$\nabla f_\alpha(x) = \nabla f(x) + \alpha I = A^*(I - P_Q)A + \alpha I$$

is $(\alpha + \|A\|^2)$-Lipschitzian and α-strongly monotone;

(ii) the mapping $P_C(I - \lambda\nabla f_\alpha)$ is a contraction with coefficient

$$\sqrt{1 - \lambda(2\alpha - \lambda(\|A\|^2 + \alpha)^2)}\ (\le \sqrt{1-\alpha\lambda} \le 1 - \frac{1}{2}\alpha\lambda),$$

where $0 < \lambda \le \frac{\alpha}{(\|A\|^2+\alpha)^2}$;

(iii) if the SFP is consistent, then the strong $\lim_{\alpha\to 0} x_\alpha$ exists and is the minimum-norm solution of the SFP.

It is worth noting that x_α is a fixed point of the mapping $P_C(I - \lambda\nabla f_\alpha)$ for any $\lambda > 0$ satisfying $0 < \lambda \le \frac{\alpha}{(\|A\|^2+\alpha)^2}$, and can be obtained through the limit as $n \to \infty$ of the sequence of Picard iterates

$$x_{n+1}^\alpha = P_C(I - \lambda\nabla f_\alpha)x_n^\alpha.$$

Secondly, letting $\alpha \to 0$ yields $x_\alpha \to x_{\min}$ in norm. It is a very subtle work that Ceng, Ansari and Yao [5] very recently combined these two steps to get $x_{\min}$ in a relaxed extragradient algorithm. The following result shows that for suitable choices of λ and α, the minimum-norm solution $x_{\min}$ can be obtained by the relaxed extragradient algorithm.

Theorem 1.1. (see [5], Theorem 3.1) *Define a sequence $\{x_n\}$ through the following Mann's type extragradient algorithm:*

$$\begin{cases} x_0 = x \in \mathcal{H}_1 \quad \text{chosen arbitrarily,} \\ y_n = P_C(x_n - \lambda_n\nabla f_{\alpha_n}(x_n)), \\ x_{n+1} = \beta_n x_n + \gamma_n y_n + \delta_n P_C(x_n - \lambda_n\nabla f_{\alpha_n}(y_n)), \quad \forall n \ge 0, \end{cases} \tag{1.5}$$

where $\nabla f_{\alpha_n} = \alpha_n I + A^*(I - P_Q)A$, and the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ and $\{\lambda_n\}$ satisfy the following conditions:

(i) $0 < \lambda_n \le \frac{\alpha_n}{(\|A\|^2+\alpha_n)^2}$ for all (large enough) n;

(ii) $\alpha_n \to 0$ and $\lambda_n \to 0$;

(iii) $\sum_{n=0}^\infty \alpha_n^2\lambda_n\delta_n = \infty$;

(iv) $\frac{|\lambda_{n+1}-\lambda_n| + \lambda_n|\alpha_{n+1}-\alpha_n|}{\alpha_{n+1}^3\lambda_{n+1}^2\delta_{n+1}} \to 0$;

(v) $\{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset [0, 1]$ and $\beta_n + \gamma_n + \delta_n = 1$ for all $n \ge 0$;

(vi) $\frac{2\delta_n}{\alpha_n+\|A\|^2} \leq \gamma_n\lambda_n$ *for all (large enough)* n.

Then, both the sequences $\{x_n\}$ *and* $\{y_n\}$ *converge in norm to the minimum-norm solution of the SFP.*

At the same time, they also proposed a Mann's type extragradient-like algorithm if the SFP is consistent (i.e., the solution set Γ of the SFP is nonempty) and $\mathrm{Fix}(S)\cap\Gamma \neq \emptyset$, where $S : C \to C$ is a nonexpansive mapping.

Theorem 1.2. (see [5], Theorem 3.2]) *Let* $S : C \to C$ *be a nonexpansive mapping such that* $\mathrm{Fix}(S) \cap \Gamma \neq \emptyset$. *Assume that* $0 < \lambda < \frac{2}{\|A\|^2}$, *and let* $\{x_n\}$ *and* $\{y_n\}$ *be the sequences in* C *generated by the following Mann type extragradient-like algorithm:*

$$\begin{cases} x_0 = x \in C \quad \text{chosen arbitrarily}, \\ y_n = (1-\beta_n)x_n + \beta_n P_C(x_n - \lambda\nabla f_{\alpha_n}(x_n)), \\ x_{n+1} = \gamma_n x_n + (1-\gamma_n)SP_C(y_n - \lambda\nabla f_{\alpha_n}(y_n)), \quad \forall n \geq 0, \end{cases} \tag{1.6}$$

where the sequences of parameters $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *satisfy the following conditions:*

(i) $\sum_{n=0}^{\infty} \alpha_n < \infty$;

(ii) $\{\beta_n\} \subset [0,1]$ *and* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iii) $\{\gamma_n\} \subset [0,1]$ *and* $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$.

Then, both the sequences $\{x_n\}$ *and* $\{y_n\}$ *converge weakly to an element* $z \in \mathrm{Fix}(S)\cap\Gamma$.

In this paper, our purpose is to study the relaxed implicit extragradient-like methods for finding a common element of the solution set Γ of the SFP and the set $\mathrm{Fix}(S)$ of fixed points of a nonexpansive mapping S in the setting of infinite -dimensional real Hilbert spaces. Combining Mann's implicit iterative method and Korpelevich's extragradient method, we propose two implicit iterative algorithms for finding an element of $\mathrm{Fix}(S) \cap \Gamma$. On one hand, we propose an implicit iterative algorithm in the case when $S = I$ the identity mapping and the SFP is consistent (i.e., the solution set Γ of the SFP is nonempty). That is, define a sequence $\{x_n\}$ through the following Mann's type implicit extragradient-like algorithm:

$$\begin{cases} x_0 &= x \in \mathcal{H}_1 \quad \text{chosen arbitrarily}, \\ y_n &= P_C(x_n - \lambda_n\nabla f_{\alpha_n}(x_n)), \\ x_{n+1} &= \beta_n x_n + \gamma_n P_C(x_{n+1} - \lambda_n\nabla f_{\alpha_n}(x_{n+1})) \\ &\quad +\delta_n P_C(x_{n+1} - \lambda_n\nabla f_{\alpha_n}(y_n)), \quad \forall n \geq 0, \end{cases} \tag{1.7}$$

where $\nabla f_{\alpha_n} = \alpha_n I + A^*(I - P_Q)A$, and the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ and $\{\lambda_n\}$ satisfy the following conditions:

(i) $0 < \lambda_n \leq \frac{\alpha_n}{(\|A\|^2+\alpha_n)^2}$ for all (large enough) n;

(ii) $\alpha_n \to 0$ and $\lambda_n \to 0$;

(iii) $\sum_{n=0}^{\infty} \alpha_n^2\lambda_n\delta_n = \infty$;

(iv) $\frac{|\lambda_{n+1}-\lambda_n|+\lambda_n|\alpha_{n+1}-\alpha_n|}{\alpha_{n+1}^3\lambda_{n+1}^2\delta_{n+1}} \to 0$;

(v) $\{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset [0,1]$ and $\beta_n + \gamma_n + \delta_n = 1$ for all $n \geq 0$;

(vi) $\frac{2\delta_n}{\alpha_n+\|A\|^2} \leq \gamma_n\lambda_n$ for all (large enough) n.

It is proven that the sequence $\{x_n\}$ generated by (1.7) converges in norm to the minimum-norm solution of the SFP. On the other hand, we propose another Mann's type implicit extragradient-like algorithm in the case when $S : C \to C$ is a nonexpansive mapping such that $\mathrm{Fix}(S) \cap \Gamma \neq \emptyset$. For $0 < \lambda < \frac{2}{\|A\|^2}$, define the sequences $\{x_n\}$ and $\{y_n\}$ in C through the following Mann type implicit extragradient-like algorithm:

$$\begin{cases} x_0 = x \in C \quad \text{chosen arbitrarily,} \\ y_n = (1-\beta_n)x_n + \beta_n P_C(y_n - \lambda \nabla f_{\alpha_n}(y_n)), \\ x_{n+1} = \gamma_n x_n + (1-\gamma_n) S P_C(x_{n+1} - \lambda \nabla f_{\alpha_n}(y_n)), \quad \forall n \geq 0, \end{cases} \tag{1.8}$$

where the sequences of parameters $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:

(i) $\sum_{n=0}^{\infty} \alpha_n < \infty$;

(ii) $\{\beta_n\} \subset [0,1]$ and $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iii) $\{\gamma_n\} \subset [0,1]$ and $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$.

It is also shown that both the sequences $\{x_n\}$ and $\{y_n\}$ generated by (1.8) converge weakly to an element $z \in \mathrm{Fix}(S) \cap \Gamma$. It is worth emphasizing that our results are novel in the Hilbert spaces setting. Our results represent the supplementation, improvement, extension and development of the corresponding results in [30], [5] to a great extent.

Compared with the above Theorems 1.1 and 1.2, our results improve, extend, supply and develop them in the following aspects:

(i) The relaxed implicit extragradient-like methods of this paper are superior to the relaxed extragradient methods in Theorems 1.1 and 1.2 because the relaxed implicit extragradient-like methods under consideration are essentially the predictor-corrector methods, which comprise one predictor step and another corrector step.

(ii) The iterative algorithm (1.5) is extended to develop the Mann's type implicit extragradient-like algorithm (1.7) with $S \equiv I$ the identity mapping.

(iii) The iterative algorithm (1.6) is extended to develop the Mann's type implicit extragradient-like algorithm (1.8).

(iv) The relaxed implicit extragradient-like methods of this paper combine Mann's implicit iterative method with Korpelevich's extragradient method to be designed.

(v) Under the same conditions imposed on the sequences of parameters as in Theorem 1.1, the sequence $\{x_n\}$ generated by (1.7) converges in norm to the minimum-norm solution of the SFP.

(vi) Under the same conditions imposed on the sequences of parameters as in Theorem 1.2, both the sequences $\{x_n\}$ and $\{y_n\}$ generated by (1.8) converge weakly to an element $z \in \mathrm{Fix}(S) \cap \Gamma$.

2. Preliminaries

Let $\mathcal{H}$ be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. Throughout the paper, unless otherwise specified, we denote by $x_n \to x$ (respectively, $x_n \rightharpoonup x$), the strong (respectively, weak) convergence of the sequence $\{x_n\}$ to x. In addition, we use $\omega_w(x_n)$ to denote the weak ω-limit set of the sequence $\{x_n\}$; namely,

$$\omega_w(x_n) := \{x : x_{n_i} \rightharpoonup x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\}.$$

Let C be a nonempty, closed and convex subset of $\mathcal{H}$ and $V : C \to \mathcal{H}$ be a (possibly nonself) ρ-contraction mapping with coefficient $\rho \in [0, 1)$, that is, there exists a constant $\rho \in [0, 1)$ such that $\|Vx - Vy\| \leq \rho\|x - y\|$, $\forall x, y \in C$. Now we present some known results and definitions which will be used in the sequel.

The metric (or nearest point) projection from $\mathcal{H}$ onto C is the mapping $P_C : \mathcal{H} \to C$ which assigns to each point $x \in \mathcal{H}$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y \in C} \|x - y\| =: d(x, C).$$

The following properties of projections are useful and pertinent to our purpose.

Proposition 2.1. (see [15]) Given any $x \in \mathcal{H}$ and $z \in C$. We have:

(i) $z = P_C x \;\Leftrightarrow\; \langle x - z, y - z\rangle \leq 0,\ \forall y \in C$;

(ii) $z = P_C x \;\Leftrightarrow\; \|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2,\ \forall y \in C$;

(iii) $\langle P_C x - P_C y, x - y\rangle \geq \|P_C x - P_C y\|^2,\ \forall x, y \in \mathcal{H}$, which hence implies that P_C is nonexpansive and monotone.

Definition 2.1. A mapping $T : \mathcal{H} \to \mathcal{H}$ is said to be:

(a) nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in \mathcal{H};$$

(b) firmly nonexpansive if $2T - I$ is nonexpansive, or equivalently,

$$\langle x - y, Tx - Ty\rangle \geq \|Tx - Ty\|^2, \quad \forall x, y \in \mathcal{H};$$

alternatively, T is firmly nonexpansive if and only if T can be expressed as

$$T = \frac{1}{2}(I + S),$$

where $S : \mathcal{H} \to \mathcal{H}$ is nonexpansive; projections are firmly nonexpansive.

Definition 2.2. Let T be a nonlinear operator whose domain is $D(T) \subseteq \mathcal{H}$ and whose range is $R(T) \subseteq \mathcal{H}$.

(a) T is said to be monotone if

$$\langle x - y, Tx - Ty\rangle \geq 0, \quad \forall x, y \in D(T).$$

(b) Given a number $\beta > 0$, T is said to be β-strongly monotone if

$$\langle x - y, Tx - Ty\rangle \geq \beta\|x - y\|^2, \quad \forall x, y \in D(T).$$

(c) Given a number $\nu > 0$, T is said to be ν-inverse strongly monotone (ν-ism) if

$$\langle x - y, Tx - Ty\rangle \geq \nu\|Tx - Ty\|^2, \quad \forall x, y \in D(T).$$

Example 2.1. (see [28]) Let $V : C \to \mathcal{H}$ be a ρ-contraction with $\rho \in [0, 1)$ and $T : C \to C$ be a nonexpansive mapping. Then

(i) $I - V$ is $(1 - \rho)$-strongly monotone:

$$\langle (I - V)x - (I - V)y, x - y\rangle \geq (1 - \rho)\|x - y\|^2, \quad \forall x, y \in C;$$

(ii) $I - T$ is monotone:

$$\langle (I - T)x - (I - T)y, x - y\rangle \geq 0, \quad \forall x, y \in C.$$

It can be easily seen that the projection P_C is 1-ism. Inverse strongly monotone (also referred to as co-coercive) operators have been applied widely in solving practical problems in various fields, for instance, in traffic assignment problems; see [1], [16].

Definition 2.3. A mapping $T : \mathcal{H} \to \mathcal{H}$ is said to be an averaged mapping if it can be written as the average of the identity mapping I and a nonexpansive mapping, that is,

$$T \equiv (1 - \alpha)I + \alpha S$$

where $\alpha \in (0, 1)$ and $S : \mathcal{H} \to \mathcal{H}$ is nonexpansive. More precisely, when the last equality holds, we say that T is α-averaged. Thus firmly nonexpansive mappings (in particular, projections) are $\frac{1}{2}$-averaged maps.

Proposition 2.2. (see [4]) Let $T : \mathcal{H} \to \mathcal{H}$ be a given mapping.

(i) T is nonexpansive if and only if the complement $I - T$ is $\frac{1}{2}$-ism.

(ii) If T is ν-ism, then for $\gamma > 0$, γT is $\frac{\nu}{\gamma}$-ism.

(iii) T is averaged if and only if the complement $I - T$ is ν-ism for some $\nu > 1/2$. Indeed, for $\alpha \in (0, 1)$, T is α-averaged if and only if $I - T$ is $\frac{1}{2\alpha}$-ism.

Proposition 2.3. (see [4], [13]) Let $S, T, V : \mathcal{H} \to \mathcal{H}$ be given operators.

(i) If $T = (1 - \alpha)S + \alpha V$ for some $\alpha \in (0, 1)$ and if S is averaged and V is nonexpansive, then T is averaged.

(ii) T is firmly nonexpansive if and only if the complement $I - T$ is firmly nonexpansive.

(iii) If $T = (1 - \alpha)S + \alpha V$ for some $\alpha \in (0, 1)$ and if S is firmly nonexpansive and V is nonexpansive, then T is averaged.

(iv) The composite of finitely many averaged mappings is averaged. That is, if each of the mappings $\{T_i\}_{i=1}^N$ is averaged, then so is the composite $T_1 \circ ... \circ T_N$. In particular, if T_1 is α_1-averaged and T_2 is α_2-averaged, where $\alpha_1, \alpha_2 \in (0, 1)$, then the composite $T_1 \circ T_2$ is α-averaged, where $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$.

The following result is useful when we prove the weak convergence of a sequence.

Proposition 2.4. (see [21]) Let K be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Let $\{x_n\}$ be a bounded sequence which satisfies the following properties:

(i) every weak limit point of $\{x_n\}$ lies in K;

(ii) $\lim_{n\to\infty} \|x_n - x\|$ exists for every $x \in K$.

Then $\{x_n\}$ converges weakly to a point in K.

The following so-called demiclosedness principle for nonexpansive mappings will often be used.

Lemma 2.1. (see [2], Demiclosedness principle) *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$ and let $T : C \to C$ be a nonexpansive mapping with* $\mathrm{Fix}(T) \neq \emptyset$. *If $\{x_n\}$ is a sequence in C converging weakly to x and if $\{(I - T)x_n\}$ converges strongly to y, then $(I - T)x = y$; in particular, if $y = 0$, then $x \in \mathrm{Fix}(T)$.*

The following lemma plays a key role in proving strong convergence of the sequences generated by our algorithms.

Lemma 2.2. (see [26], Lemma 2.1) *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the condition*

$$a_{n+1} \leq (1 - s_n)a_n + s_n t_n, \quad \forall n \geq 0,$$

where $\{s_n\}$, $\{t_n\}$ *are sequences of real numbers such that*

(i) $\{s_n\} \subset [0,1]$ *and* $\sum_{n=0}^{\infty} s_n = \infty$*, or equivalently*

$$\prod_{n=0}^{\infty}(1-s_n) := \lim_{n\to\infty}\prod_{k=0}^{n}(1-s_k) = 0;$$

(ii) $\limsup_{n\to\infty} t_n \le 0$*, or*

(ii)′ $\sum_{n=0}^{\infty} s_n t_n$ *is convergent.*

Then, $\lim_{n\to\infty} a_n = 0$.

It is easy to see that the following lemma holds.

Lemma 2.3. (see [15]) *Let* $\mathcal{H}$ *be a real Hilbert space. Then*

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2, \quad \forall x, y \in \mathcal{H},\ \forall \lambda \in [0,1].$$

The following elementary fact on real sequences is well-known but useful.

Lemma 2.4. (see [22] p. 80) *Let* $\{a_n\}$, $\{b_n\}$ *and* $\{\delta_n\}$ *be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \le (1+\delta_n)a_n + b_n, \quad \forall n \ge 0.$$

If $\sum_{n=0}^{\infty} \delta_n < \infty$ *and* $\sum_{n=0}^{\infty} b_n < \infty$*, then* $\lim_{n\to\infty} a_n$ *exists. If in addition* $\{a_n\}$ *has a subsequence which converges to zero, then* $\lim_{n\to\infty} a_n = 0$.

Corollary 2.1. (see [25], p. 303) *Let* $\{a_n\}$ *and* $\{b_n\}$ *be two sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \le a_n + b_n, \quad \forall n \ge 0.$$

If $\sum_{n=0}^{\infty} b_n$ *converges, then* $\lim_{n\to\infty} a_n$ *exists.*

3. Relaxed Implicit Extragradient-Like Methods

Throughout this paper, we assume that the SFP is consistent, that is, the solution set Γ of the SFP is nonempty. Let $\mathcal{H}$ be a real Hilbert space and $f : \mathcal{H} \to \mathbb{R}$ be a function. Then the minimization problem

$$\min_{x\in C} f(x) := \frac{1}{2}\|Ax - P_Q Ax\|^2$$

is ill-posed. Xu [30] considered the following Tikhonov's regularization problem:

$$\min_{x\in C} f_\alpha(x) := \frac{1}{2}\|Ax - P_Q Ax\|^2 + \frac{1}{2}\alpha\|x\|^2, \tag{3.1}$$

where $\alpha > 0$ is the regularization parameter. The regularized minimization (3.1) has a unique solution which is denoted by x_α. The following result is easy to prove.

Proposition 3.1. (see [30]) If the SFP is consistent, then the strong $\lim_{\alpha\to 0} x_\alpha$ exists and is the minimum-norm solution of the SFP.

Let $x_{\min}$ be a minimum-norm solution of the SFP; namely, $x_{\min} \in \Gamma$ has the property

$$\|x_{\min}\| = \min\{\|x^*\| : x^* \in \Gamma\}.$$

$x_{\min}$ can be obtained by two steps. First, observing that the gradient

$$\nabla f_\alpha(x) = \nabla f(x) + \alpha I = A^*(I - P_Q)A + \alpha I$$

is $(\alpha + \|A\|^2)$-Lipschitzian and α-strongly monotone, we know that the mapping $P_C(I - \lambda \nabla f_\alpha)$ is a contraction with coefficient

$$\sqrt{1 - \lambda(2\alpha - \lambda(\|A\|^2 + \alpha)^2)} \ (\leq \sqrt{1 - \alpha\lambda} \leq 1 - \frac{1}{2}\alpha\lambda),$$

where

$$0 < \lambda \leq \frac{\alpha}{(\|A\|^2 + \alpha)^2}. \tag{3.2}$$

Indeed, observe that

$$\begin{aligned} &\|P_C(I - \lambda\nabla f_\alpha)(x) - P_C(I - \lambda\nabla f_\alpha)(y)\|^2 \\ &\leq \|(I - \lambda\nabla f_\alpha)(x) - (I - \lambda\nabla f_\alpha)(y)\|^2 \\ &= \|x - y\|^2 - 2\lambda\langle\nabla f_\alpha(x) - \nabla f_\alpha(y), x - y\rangle + \lambda^2\|\nabla f_\alpha(x) - \nabla f_\alpha(y)\|^2 \\ &\leq (1 - 2\lambda\alpha + \lambda^2(\|A\|^2 + \alpha)^2)\|x - y\|^2 \\ &\leq (1 - 2\lambda\alpha + \lambda\tfrac{\alpha}{(\|A\|^2+\alpha)^2} \cdot (\|A\|^2 + \alpha)^2)\|x - y\|^2 \\ &= (1 - \lambda\alpha)\|x - y\|^2. \end{aligned} \tag{3.3}$$

It is worth noting that x_α is a fixed point of the mapping $P_C(I - \lambda\nabla f_\alpha)$ for any $\lambda > 0$ satisfying (3.2), and can be obtained through the limit as $n \to \infty$ of the sequence of Picard iterates

$$x^\alpha_{n+1} = P_C(I - \lambda\nabla f_\alpha)x^\alpha_n.$$

Secondly, letting $\alpha \to 0$ yields $x_\alpha \to x_{\min}$ in norm. It is interesting to know whether these two steps can be combined to get $x_{\min}$ in a relaxed implicit extragradient-like algorithm. The following result shows that for suitable choices of λ and α, the minimum-norm solution $x_{\min}$ can be obtained by the relaxed implicit extragradient-like algorithm.

Theorem 3.1. *Define a sequence $\{x_n\}$ through the following Mann's type implicit extragradient-like algorithm:*

$$\begin{cases} x_0 &= x \in \mathcal{H}_1 \quad \text{chosen arbitrarily}, \\ y_n &= P_C(x_n - \lambda_n\nabla f_{\alpha_n}(x_n)), \\ x_{n+1} &= \beta_n x_n + \gamma_n P_C(x_{n+1} - \lambda_n\nabla f_{\alpha_n}(x_{n+1})) \\ &\quad +\delta_n P_C(x_{n+1} - \lambda_n\nabla f_{\alpha_n}(y_n)), \quad \forall n \geq 0, \end{cases} \tag{3.4}$$

where $\nabla f_{\alpha_n} = \alpha_n I + A^*(I - P_Q)A$, and the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$ and $\{\lambda_n\}$ satisfy the following conditions:

(i) $0 < \lambda_n \leq \frac{\alpha_n}{(\|A\|^2+\alpha_n)^2}$ for all (large enough) n;

(ii) $\alpha_n \to 0$ and $\lambda_n \to 0$;

(iii) $\sum_{n=0}^\infty \alpha_n^2\lambda_n\delta_n = \infty$;

(iv) $\frac{|\lambda_{n+1}-\lambda_n|+\lambda_n|\alpha_{n+1}-\alpha_n|}{\alpha^3_{n+1}\lambda^2_{n+1}\delta_{n+1}} \to 0$;

(v) $\{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset [0, 1]$ and $\beta_n + \gamma_n + \delta_n = 1$ for all $n \geq 0$;

(vi) $\frac{2\delta_n}{\alpha_n+\|A\|^2} \leq \gamma_n\lambda_n$ for all (large enough) n.

Then, both the sequences $\{x_n\}$ and $\{y_n\}$ converge in norm to the minimum-norm solution of the SFP.

Note that $\alpha_n = n^{-\delta}$, $\lambda_n = n^{-\sigma}$ and $\delta_n = n^{-\epsilon}$ with $0 < \delta < \sigma \leq \epsilon < 1$ and $3\delta + 2\sigma + \epsilon < 1$ satisfy conditions (i)-(iv).

Proof. For any λ satisfying (3.2), x_α is a fixed point of the mapping $P_C(I-\lambda\nabla f_\alpha)$. For each $n \geq 0$, let z_n be a unique fixed point of the contraction $T_n := P_C(I - \lambda_n \nabla f_{\alpha_n})$. Then, $z_n := x_{\alpha_n}$, and so $z_n \to x_{\min}$ in norm. So, it is sufficient to prove that

$$\|x_{n+1} - z_n\| \to 0.$$

Noting that T_n has a contraction coefficient $(1 - \frac{1}{2}\alpha_n\lambda_n)$, we have

$$\|y_n - z_n\| = \|T_n x_n - T_n z_n\| \leq (1 - \frac{1}{2}\alpha_n\lambda_n)\|x_n - z_n\|. \tag{3.5}$$

Analogously, we have

$$\|P_C(I - \lambda_n\nabla f_{\alpha_n})x_{n+1} - z_n\| = \|T_n x_{n+1} - T_n z_n\| \leq (1 - \frac{1}{2}\alpha_n\lambda_n)\|x_{n+1} - z_n\|. \tag{3.6}$$

We now estimate

$$\begin{aligned} \|z_n - z_{n-1}\| &= \|T_n z_n - T_{n-1} z_{n-1}\| \\ &\leq \|T_n z_n - T_n z_{n-1}\| + \|T_n z_{n-1} - T_{n-1} z_{n-1}\| \\ &\leq (1 - \tfrac{1}{2}\alpha_n\lambda_n)\|z_n - z_{n-1}\| + \|T_n z_{n-1} - T_{n-1} z_{n-1}\|, \end{aligned}$$

which hence implies that

$$\|z_n - z_{n-1}\| \leq \frac{2}{\alpha_n\lambda_n}\|T_n z_{n-1} - T_{n-1} z_{n-1}\|. \tag{3.7}$$

However, since ∇f is Lipschitzian and $\{z_n\}$ is bounded, we have

$$\begin{aligned} \|T_n z_{n-1} &- T_{n-1} z_{n-1}\| = \|P_C(I - \lambda_n\nabla f_{\alpha_n})z_{n-1} - P_C(I - \lambda_{n-1}\nabla f_{\alpha_{n-1}})z_{n-1}\| \\ &\leq \|(I - \lambda_n\nabla f_{\alpha_n})z_{n-1} - (I - \lambda_{n-1}\nabla f_{\alpha_{n-1}})z_{n-1}\| \\ &= \|\lambda_n\nabla f_{\alpha_n}(z_{n-1}) - \lambda_{n-1}\nabla f_{\alpha_{n-1}}(z_{n-1})\| \\ &= \|(\lambda_n - \lambda_{n-1})\nabla f_{\alpha_n}(z_{n-1}) + \lambda_{n-1}(\nabla f_{\alpha_n}(z_{n-1}) - \nabla f_{\alpha_{n-1}}(z_{n-1}))\| \\ &\leq |\lambda_n - \lambda_{n-1}|\|\nabla f(z_{n-1}) + \alpha_n z_{n-1}\| + \lambda_{n-1}|\alpha_n - \alpha_{n-1}|\|z_{n-1}\| \\ &\leq (|\lambda_n - \lambda_{n-1}| + \lambda_{n-1}|\alpha_n - \alpha_{n-1}|)M, \end{aligned} \tag{3.8}$$

where $M = \sup_{n\geq 1} \max\{\|\nabla f(z_{n-1}) + \alpha_n z_{n-1}\|, \|z_{n-1}\|\} < \infty$. Utilizing conditions (i), (vi), and inequalities (3.5)-(3.8), we obtain

$$\begin{aligned} \|x_{n+1} - z_n\| &= \|\beta_n(x_n - z_n) + \gamma_n(P_C(I - \lambda_n\nabla f_{\alpha_n})x_{n+1} - z_n) \\ &\quad +\delta_n(P_C(x_{n+1} - \lambda_n\nabla f_{\alpha_n}(y_n)) - z_n)\| \\ &\leq \beta_n\|x_n - z_n\| + \gamma_n\|P_C(I - \lambda_n\nabla f_{\alpha_n})x_{n+1} - z_n\| \\ &\quad +\delta_n\|P_C(x_{n+1} - \lambda_n\nabla f_{\alpha_n}(y_n)) - P_C(z_n - \lambda_n\nabla f_{\alpha_n}(z_n))\| \\ &\leq \beta_n\|x_n - z_n\| + \gamma_n(1 - \tfrac{1}{2}\alpha_n\lambda_n)\|x_{n+1} - z_n\| \\ &\quad +\delta_n\|(x_{n+1} - \lambda_n\nabla f_{\alpha_n}(y_n)) - (z_n - \lambda_n\nabla f_{\alpha_n}(z_n))\| \\ &\leq \beta_n\|x_n - z_n\| + \gamma_n(1 - \tfrac{1}{2}\alpha_n\lambda_n)\|x_{n+1} - z_n\| \\ &\quad +\delta_n[\|x_{n+1} - z_n\| + \lambda_n\|\nabla f_{\alpha_n}(y_n) - \nabla f_{\alpha_n}(z_n)\|] \\ &\leq \beta_n\|x_n - z_n\| + \gamma_n(1 - \tfrac{1}{2}\alpha_n\lambda_n)\|x_{n+1} - z_n\| \\ &\quad +\delta_n[\|x_{n+1} - z_n\| + \lambda_n(\alpha_n + \|A\|^2)\|y_n - z_n\|] \\ &\leq \beta_n\|x_n - z_n\| + \gamma_n(1 - \tfrac{1}{2}\alpha_n\lambda_n)\|x_{n+1} - z_n\| \\ &\quad +\delta_n[\|x_{n+1} - z_n\| + \lambda_n(\alpha_n + \|A\|^2)(1 - \tfrac{1}{2}\alpha_n\lambda_n)\|x_n - z_n\|] \\ &= [\beta_n + \delta_n\lambda_n(\alpha_n + \|A\|^2)(1 - \tfrac{1}{2}\alpha_n\lambda_n)]\|x_n - z_n\| \\ &\quad +(1 - \beta_n - \tfrac{1}{2}\alpha_n\gamma_n\lambda_n)\|x_{n+1} - z_n\|, \end{aligned}$$

and hence

$$\begin{aligned}
&\|x_{n+1}-z_n\| \le \tfrac{\beta_n+\delta_n\lambda_n(\alpha_n+\|A\|^2)(1-\frac{1}{2}\alpha_n\lambda_n)}{\beta_n+\frac{1}{2}\alpha_n\gamma_n\lambda_n}\|x_n-z_n\| \\
&= [\tfrac{\beta_n}{\beta_n+\frac{1}{2}\alpha_n\gamma_n\lambda_n} + \tfrac{\delta_n\lambda_n(\alpha_n+\|A\|^2)(1-\frac{1}{2}\alpha_n\lambda_n)}{\beta_n+\frac{1}{2}\alpha_n\gamma_n\lambda_n}]\|x_n-z_n\| \\
&\le [1-\tfrac{\alpha_n\gamma_n\lambda_n}{2(\beta_n+\frac{1}{2}\alpha_n\gamma_n\lambda_n)} + \tfrac{\delta_n\alpha_n(1-\frac{1}{2}\alpha_n\lambda_n)}{(\beta_n+\frac{1}{2}\alpha_n\gamma_n\lambda_n)(\alpha_n+\|A\|^2)}]\|x_n-z_n\| \\
&= [1-\tfrac{\alpha_n\gamma_n\lambda_n}{2(\beta_n+\frac{1}{2}\alpha_n\gamma_n\lambda_n)} + \tfrac{\delta_n\alpha_n}{(\beta_n+\frac{1}{2}\alpha_n\gamma_n\lambda_n)(\alpha_n+\|A\|^2)} - \tfrac{\alpha_n^2\lambda_n\delta_n}{2(\beta_n+\frac{1}{2}\alpha_n\gamma_n\lambda_n)(\alpha_n+\|A\|^2)}]\|x_n-z_n\| \\
&\le [1-\tfrac{\alpha_n^2\lambda_n\delta_n}{2(\beta_n+\frac{1}{2}\alpha_n\gamma_n\lambda_n)(\alpha_n+\|A\|^2)}]\|x_n-z_n\| \quad \text{(using condition (vi))} \\
&\le [1-\tfrac{\alpha_n^2\lambda_n\delta_n}{2(\beta_n+\frac{1}{2}\alpha_n\gamma_n\lambda_n)(\alpha_n+\|A\|^2)}]\|x_n-z_{n-1}\| + \|z_n-z_{n-1}\| \\
&\le [1-\tfrac{\alpha_n^2\lambda_n\delta_n}{2(\beta_n+\frac{1}{2}\alpha_n\gamma_n\lambda_n)(\alpha_n+\|A\|^2)}]\|x_n-z_{n-1}\| + \tfrac{2}{\alpha_n\lambda_n}\|T_nz_{n-1}-T_{n-1}z_{n-1}\| \\
&\le [1-\tfrac{\alpha_n^2\lambda_n\delta_n}{2(\beta_n+\frac{1}{2}\alpha_n\gamma_n\lambda_n)(\alpha_n+\|A\|^2)}]\|x_n-z_{n-1}\| + \tfrac{2M(|\lambda_n-\lambda_{n-1}|+\lambda_{n-1}|\alpha_n-\alpha_{n-1}|)}{\alpha_n\lambda_n} \\
&= [1-\tfrac{\alpha_n^2\lambda_n\delta_n}{2(\beta_n+\frac{1}{2}\alpha_n\gamma_n\lambda_n)(\alpha_n+\|A\|^2)}]\|x_n-z_{n-1}\| \\
&\quad + \tfrac{\alpha_n^2\lambda_n\delta_n}{2(\beta_n+\frac{1}{2}\alpha_n\gamma_n\lambda_n)(\alpha_n+\|A\|^2)} \cdot \tfrac{4M(\beta_n+\frac{1}{2}\alpha_n\gamma_n\lambda_n)(\alpha_n+\|A\|^2)(|\lambda_n-\lambda_{n-1}|+\lambda_{n-1}|\alpha_n-\alpha_{n-1}|)}{\alpha_n^3\lambda_n^2\delta_n} \\
&= (1-s_n)\|x_n-z_{n-1}\| + s_nt_n,
\end{aligned} \tag{3.9}$$

where $s_n := \frac{\alpha_n^2\lambda_n\delta_n}{2(\beta_n+\frac{1}{2}\alpha_n\gamma_n\lambda_n)(\alpha_n+\|A\|^2)}$ and

$$t_n := \frac{4M(\beta_n+\frac{1}{2}\alpha_n\gamma_n\lambda_n)(\alpha_n+\|A\|^2)(|\lambda_n-\lambda_{n-1}|+\lambda_{n-1}|\alpha_n-\alpha_{n-1}|)}{\alpha_n^3\lambda_n^2\delta_n} \to 0$$

(due to conditions (ii) and (iv)). Taking into account conditions (ii), (v) and (vi), we have for all (large enough) n,

$$\begin{aligned}
s_n &= \tfrac{\alpha_n^2\lambda_n\delta_n}{2(\beta_n+\frac{1}{2}\alpha_n\gamma_n\lambda_n)(\alpha_n+\|A\|^2)} \\
&\le \tfrac{\alpha_n^2\lambda_n\delta_n}{\alpha_n\gamma_n\lambda_n(\alpha_n+\|A\|^2)} = \tfrac{\delta_n}{\gamma_n}\cdot\tfrac{\alpha_n}{\alpha_n+\|A\|^2} \\
&\le \tfrac{\alpha_n}{\alpha_n+\|A\|^2}\cdot\tfrac{1}{2}\lambda_n(\alpha_n+\|A\|^2) = \tfrac{1}{2}\alpha_n\lambda_n \\
&\le \tfrac{1}{2}(\tfrac{\alpha_n}{\alpha_n+\|A\|^2})^2.
\end{aligned}$$

So, it follows that $s_n \in [0,1]$ for all (large enough) n. In the meantime, we also derive from (v)

$$s_n = \frac{\alpha_n^2\lambda_n\delta_n}{2(\beta_n+\frac{1}{2}\alpha_n\gamma_n\lambda_n)(\alpha_n+\|A\|^2)} \ge \frac{\alpha_n^2\lambda_n\delta_n}{(2+\alpha_n\lambda_n)(\alpha_n+\|A\|^2)}.$$

Thus, it follows from conditions (ii), (iii) that $\sum_{n=0}^{\infty} s_n = \infty$. By applying Lemma 2.2 to (3.9) we conclude that $\|x_{n+1}-z_n\| \to 0$; hence, $x_n \to x_{\min}$ in norm. Taking into account the strong convergence of both $\{x_n\}$ and $\{z_n\}$ to $x_{\min}$, we deduce from (3.5) that

$$\|y_n - z_n\| \le \|x_n - z_n\| \to 0.$$

Therefore, $y_n \to x_{\min}$ in norm. This completes the proof. □

Remark 3.1. (see [5], Remark 3.1) In Theorem 3.1, put $\alpha_n = n^{-\delta}$, $\lambda_n = n^{-\sigma}$ and $\delta_n = n^{-\epsilon}$ where $\delta = \frac{1}{10}$, $\sigma = \frac{1}{5}$ and $\epsilon = \frac{1}{4}$. Then it is easy to see that $0 < \delta < \sigma \le \epsilon < 1$ and $3\delta + 2\sigma + \epsilon = \frac{19}{20} < 1$. Thus, conditions (i)-(iv) in Theorem 3.1 are satisfied.

In particular, if $\liminf_{n\to\infty} \gamma_n > 0$ additionally, then it is clear that condition (vi) is also satisfied.

Remark 3.2 It is worth pointing out that the Mann type implicit extragradient-like algorithm in Theorem 3.1 is essentially the predictor-corrector algorithm. Indeed, the first iterative step $y_n = P_C(x_n - \lambda_n \nabla f_{\alpha_n}(x_n))$ is the predictor one, and the second implicit iterative step $x_{n+1} = \beta_n x_n + \gamma_n P_C(I - \lambda_n \nabla f_{\alpha_n})x_{n+1} + \delta_n P_C(x_{n+1} - \lambda_n \nabla f_{\alpha_n}(y_n))$ is actually the corrector one. Obviously, both the iterative algorithms in [30] (Theorem 5.5) and [5] (Theorem 3.1) are extended to develop Mann's type implicit extragradient-like algorithm in Theorem 3.1. Therefore, although those algorithms in [30] (Theorem 5.5) and [5] (Theorem 3.1) are explicit, Mann's type implicit extragradient-like algorithm in Theorem 3.1 is superior to them to a certain extent.

Under the assumptions of Theorem 3.1, the sequence $\{\lambda_n\}$ is forced to tend to zero. If we keep it as a constant, then we have weak convergence as shown below.

Theorem 3.2. *Let $S : C \to C$ be a nonexpansive mapping such that* $\mathrm{Fix}(S) \cap \Gamma \neq \emptyset$. *Assume that $0 < \lambda < \frac{2}{\|A\|^2}$, and let $\{x_n\}$ and $\{y_n\}$ be the sequences in C generated by the following Mann type implicit extragradient-like algorithm:*

$$\begin{cases} x_0 = x \in C \quad \text{chosen arbitrarily}, \\ y_n = (1-\beta_n)x_n + \beta_n P_C(y_n - \lambda \nabla f_{\alpha_n}(y_n)), \\ x_{n+1} = \gamma_n x_n + (1-\gamma_n) S P_C(x_{n+1} - \lambda \nabla f_{\alpha_n}(y_n)), \quad \forall n \geq 0, \end{cases} \tag{3.10}$$

where the sequences of parameters $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:

(i) $\sum_{n=0}^{\infty} \alpha_n < \infty$;

(ii) $\{\beta_n\} \subset [0,1]$ and $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iii) $\{\gamma_n\} \subset [0,1]$ and $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$.

Then, both the sequences $\{x_n\}$ and $\{y_n\}$ converge weakly to an element $z \in$ $\mathrm{Fix}(S) \cap \Gamma$.

Proof. First of all, in terms of conditions (ii) and (iii), without loss of generality we may assume that $\{\beta_n\} \subset [a,b]$ for some $a, b \in (0,1)$ and $\{\beta_n\} \subset [c,d]$ for some $c, d \in (0,1)$. Now we assert that $P_C(I - \lambda \nabla f_\alpha)$ is ζ-averaged for each $\lambda \in (0, \frac{2}{\alpha + \|A\|^2})$, where

$$\zeta = \frac{2 + \lambda(\alpha + \|A\|^2)}{4}.$$

Indeed, it is easy to see that $\nabla f = A^*(I - P_Q)A$ is $\frac{1}{\|A\|^2}$-ism, that is,

$$\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \frac{1}{\|A\|^2} \|\nabla f(x) - \nabla f(y)\|^2.$$

Observe that

$$\begin{aligned} &(\alpha + \|A\|^2)\langle \nabla f_\alpha(x) - \nabla f_\alpha(y), x - y \rangle \\ &= (\alpha + \|A\|^2)[\alpha \|x - y\|^2 + \langle \nabla f(x) - \nabla f(y), x - y \rangle] \\ &= \alpha^2 \|x - y\|^2 + \alpha \langle \nabla f(x) - \nabla f(y), x - y \rangle \\ &+ \alpha \|A\|^2 \|x - y\|^2 + \|A\|^2 \langle \nabla f(x) - \nabla f(y), x - y \rangle \\ &\geq \alpha^2 \|x - y\|^2 + 2\alpha \langle \nabla f(x) - \nabla f(y), x - y \rangle + \|\nabla f(x) - \nabla f(y)\|^2 \\ &= \|\alpha(x - y) + \nabla f(x) - \nabla f(y)\|^2 \\ &= \|\nabla f_\alpha(x) - \nabla f_\alpha(y)\|^2. \end{aligned}$$

Hence, it follows that $\nabla f_\alpha = \alpha I + A^*(I-P_Q)A$ is $\frac{1}{\alpha+\|A\|^2}$-ism. Thus, $\lambda\nabla f_\alpha$ is $\frac{1}{\lambda(\alpha+\|A\|^2)}$-ism according to Proposition 2.2 (ii). By Proposition 2.2 (iii) the complement $I-\lambda\nabla f_\alpha$ is $\frac{\lambda(\alpha+\|A\|^2)}{2}$-averaged. Therefore, noting that P_C is $\frac{1}{2}$-averaged and utilizing Proposition 2.3 (iv), we know that for each $\lambda\in(0,\frac{2}{\alpha+\|A\|^2})$, $P_C(I-\lambda\nabla f_\alpha)$ is ζ-averaged with

$$\zeta = \frac{1}{2} + \frac{\lambda(\alpha+\|A\|^2)}{2} - \frac{1}{2}\cdot\frac{\lambda(\alpha+\|A\|^2)}{2} = \frac{2+\lambda(\alpha+\|A\|^2)}{4} \in (0,1).$$

This shows that $P_C(I-\lambda\nabla f_\alpha)$ is nonexpansive. Furthermore, for $\lambda\in(0,\frac{2}{\|A\|^2})$, utilizing the fact that $\lim_{n\to\infty}\frac{2}{\alpha_n+\|A\|^2} = \frac{2}{\|A\|^2}$ we may assume that

$$0 < \lambda < \frac{2}{\alpha_n+\|A\|^2}, \quad \forall n\geq 0.$$

Consequently, it follows that for each integer $n\geq 0$, $P_C(I-\lambda\nabla f_{\alpha_n})$ is ζ_n-averaged with

$$\zeta_n = \frac{1}{2} + \frac{\lambda(\alpha_n+\|A\|^2)}{2} - \frac{1}{2}\cdot\frac{\lambda(\alpha_n+\|A\|^2)}{2} = \frac{2+\lambda(\alpha_n+\|A\|^2)}{4} \in (0,1).$$

This immediately implies that $P_C(I-\lambda\nabla f_{\alpha_n})$ is nonexpansive for all $n\geq 0$.

Next we divide the remainder of the proof into several steps.

Step 1. $\{x_n\}$ is bounded.

Indeed, take a fixed $p\in \mathrm{Fix}(S)\cap\Gamma$ arbitrarily. Then, we get $Sp=p$ and $P_C(I-\lambda\nabla f)p=p$ for $\lambda\in(0,\frac{2}{\|A\|^2})$. Hence, we have

$$\begin{aligned}
\|y_n-p\| &= \|(1-\beta_n)(x_n-p)+\beta_n[P_C(I-\lambda\nabla f_{\alpha_n})y_n-p]\| \\
&\leq (1-\beta_n)\|x_n-p\|+\beta_n\|P_C(I-\lambda\nabla f_{\alpha_n})y_n-p\| \\
&= (1-\beta_n)\|x_n-p\|+\beta_n\|P_C(I-\lambda\nabla f_{\alpha_n})y_n-P_C(I-\lambda\nabla f)p\| \\
&\leq (1-\beta_n)\|x_n-p\|+\beta_n[\|P_C(I-\lambda\nabla f_{\alpha_n})y_n-P_C(I-\lambda\nabla f_{\alpha_n})p\| \\
&\quad +\|P_C(I-\lambda\nabla f_{\alpha_n})p-P_C(I-\lambda\nabla f)p\|] \\
&\leq (1-\beta_n)\|x_n-p\|+\beta_n[\|y_n-p\|+\|(I-\lambda\nabla f_{\alpha_n})p-(I-\lambda\nabla f)p\|] \\
&= (1-\beta_n)\|x_n-p\|+\beta_n[\|y_n-p\|+\lambda\alpha_n\|p\|],
\end{aligned}$$

which implies that

$$\|y_n-p\| \leq \|x_n-p\| + \frac{\beta_n}{1-\beta_n}\lambda\alpha_n\|p\| \leq \|x_n-p\| + \frac{b}{1-b}\lambda\alpha_n\|p\|. \qquad (3.11)$$

Thus, we obtain that

$$\begin{aligned}
\|x_{n+1}-p\| &= \|\gamma_n(y_n-p)+(1-\gamma_n)[SP_C(I-\lambda\nabla f_{\alpha_n})x_{n+1}-p]\| \\
&\leq \gamma_n\|y_n-p\|+(1-\gamma_n)\|SP_C(I-\lambda\nabla f_{\alpha_n})x_{n+1}-p\| \\
&\leq \gamma_n\|y_n-p\|+(1-\gamma_n)\|P_C(I-\lambda\nabla f_{\alpha_n})x_{n+1}-p\| \\
&= \gamma_n\|y_n-p\|+(1-\gamma_n)\|P_C(I-\lambda\nabla f_{\alpha_n})x_{n+1}-P_C(I-\lambda\nabla f)p\| \\
&\leq \gamma_n\|y_n-p\|+(1-\gamma_n)[\|P_C(I-\lambda\nabla f_{\alpha_n})x_{n+1}-P_C(I-\lambda\nabla f_{\alpha_n})p\| \\
&\quad +\|P_C(I-\lambda\nabla f_{\alpha_n})p-P_C(I-\lambda\nabla f)p\|] \\
&\leq \gamma_n\|y_n-p\|+(1-\gamma_n)[\|x_{n+1}-p\|+\|(I-\lambda\nabla f_{\alpha_n})p-(I-\lambda\nabla f)p\|] \\
&= \gamma_n\|y_n-p\|+(1-\gamma_n)[\|x_{n+1}-p\|+\lambda\alpha_n\|p\|],
\end{aligned}$$

which together with (3.11) implies that

$$\begin{aligned}\|x_{n+1}-p\| &\le \|y_n-p\|+\tfrac{1-\gamma_n}{\gamma_n}\lambda\alpha_n\|p\| \\ &\le \|x_n-p\|+\tfrac{b}{1-b}\lambda\alpha_n\|p\|+\tfrac{1-c}{c}\lambda\alpha_n\|p\| \\ &\le \|x_n-p\|+2\max\{\tfrac{b}{1-b},\tfrac{1-c}{c}\}\lambda\alpha_n\|p\|.\end{aligned}$$

Since $\sum_{n=0}^{\infty}\alpha_n<\infty$, according to Corollary 2.1 we conclude that

$$\lim_{n\to\infty}\|x_n-p\| \quad \text{exists for each } p\in \mathrm{Fix}(S)\cap\Gamma. \tag{3.12}$$

Therefore, $\{x_n\}$ is bounded and so are $\{y_n\}$, $\{\nabla f(x_n)\}$ and $\{\nabla f(y_n)\}$.

Step 2. $\lim_{n\to\infty}\|u_n-x_n\|=\lim_{n\to\infty}\|x_n-y_n\|=\lim_{n\to\infty}\|v_n-Sv_n\|=\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$, where $u_n=P_C(I-\lambda\nabla f_{\alpha_n})x_{n+1}$ and $v_n=P_C(I-\lambda\nabla f_{\alpha_n})y_n$.

Indeed, observe that

$$\begin{aligned}\|y_n-p\|^2 &= \|(1-\beta_n)(x_n-p)+\beta_n(v_n-p)\|^2 \\ &= (1-\beta_n)\|x_n-p\|^2+\beta_n\|v_n-p\|^2-\beta_n(1-\beta_n)\|x_n-v_n\|^2 \\ &\le (1-\beta_n)\|x_n-p\|^2+\beta_n[\|y_n-p\|+\lambda\alpha_n\|p\|]^2-\beta_n(1-\beta_n)\|x_n-v_n\|^2 \\ &= (1-\beta_n)\|x_n-p\|^2+\beta_n\|y_n-p\|^2+\alpha_n\beta_n(2\lambda\|p\|\|y_n-p\|+\alpha_n\lambda^2\|p\|^2) \\ &\quad -\beta_n(1-\beta_n)\|x_n-v_n\|^2,\end{aligned}$$

and hence

$$\begin{aligned}\|y_n-p\|^2 &\le \|x_n-p\|^2+\alpha_n\tfrac{\beta_n}{1-\beta_n}(2\lambda\|p\|\|y_n-p\|+\alpha_n\lambda^2\|p\|^2)-\beta_n\|x_n-v_n\|^2 \\ &\le \|x_n-p\|^2+\alpha_nM_1-\beta_n\|x_n-v_n\|^2,\end{aligned} \tag{3.13}$$

where $M_1=\sup_{n\ge 0}\{\frac{\beta_n}{1-\beta_n}(2\lambda\|p\|\|y_n-p\|+\alpha_n\lambda^2\|p\|^2)\}<\infty$.

Also, observe that

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|\gamma_n(y_n-p)+(1-\gamma_n)(Su_n-p)\|^2 \\ &= \gamma_n\|y_n-p\|^2+(1-\gamma_n)\|Su_n-p\|^2-\gamma_n(1-\gamma_n)\|y_n-Su_n\|^2 \\ &\le \gamma_n\|y_n-p\|^2+(1-\gamma_n)\|u_n-p\|^2-\gamma_n(1-\gamma_n)\|y_n-Su_n\|^2 \\ &\le \gamma_n\|y_n-p\|^2+(1-\gamma_n)[\|x_{n+1}-p\| \\ &\quad +\lambda\alpha_n\|p\|]^2-\gamma_n(1-\gamma_n)\|y_n-Su_n\|^2 \\ &= \gamma_n\|y_n-p\|^2+(1-\gamma_n)[\|x_{n+1}-p\|^2 \\ &\quad +\alpha_n(2\lambda\|p\|\|x_{n+1}-p\|+\alpha_n\lambda^2\|p\|^2)] \\ &\quad -\gamma_n(1-\gamma_n)\|y_n-Su_n\|^2,\end{aligned}$$

which together with (3.13) yields that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \|y_n-p\|^2+\alpha_n\tfrac{1-\gamma_n}{\gamma_n}(2\lambda\|p\|\|x_{n+1}-p\|+\alpha_n\lambda^2\|p\|^2) \\ &\quad -(1-\gamma_n)\|y_n-Su_n\|^2 \\ &\le \|y_n-p\|^2+\alpha_nM_2-(1-\gamma_n)\|y_n-Su_n\|^2 \\ &\le \|x_n-p\|^2+\alpha_nM_1-\beta_n\|x_n-v_n\|^2+\alpha_nM_2-(1-\gamma_n)\|y_n-Su_n\|^2 \\ &= \|x_n-p\|^2+\alpha_n(M_1+M_2)-\beta_n\|x_n-v_n\|^2-(1-\gamma_n)\|y_n-Su_n\|^2,\end{aligned}$$

where $M_2=\sup_{n\ge 0}\{\frac{1-\gamma_n}{\gamma_n}(2\lambda\|p\|\|x_{n+1}-p\|+\alpha_n\lambda^2\|p\|^2)\}<\infty$. Hence, it follows that

$$\beta_n\|x_n-v_n\|^2+(1-\gamma_n)\|y_n-Su_n\|^2\le\|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n(M_1+M_2). \tag{3.14}$$

Since $\lim_{n\to\infty}\alpha_n = 0$, $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$ and $0 < \liminf_{n\to\infty}\gamma_n \le \limsup_{n\to\infty}\gamma_n < 1$, we deduce from the existence of $\lim_{n\to\infty}\|x_n - p\|$ that

$$\lim_{n\to\infty}\|x_n - v_n\| = \lim_{n\to\infty}\|y_n - Su_n\| = 0. \tag{3.15}$$

Thus, utilizing (3.10) we get

$$\lim_{n\to\infty}\|y_n - x_n\| = \lim_{n\to\infty}\beta_n\|v_n - x_n\| = 0,$$

$$\lim_{n\to\infty}\|x_{n+1} - y_n\| = \lim_{n\to\infty}(1-\gamma_n)\|Su_n - y_n\| = 0.$$

and so

$$\|x_{n+1} - x_n\| \le \|x_{n+1} - y_n\| + \|y_n - x_n\| \to 0 \quad \text{as } n \to \infty.$$

Furthermore, note that

$$\begin{aligned}\|Sv_n - v_n\| &\le \|Sv_n - Su_n\| + \|Su_n - y_n\| + \|y_n - v_n\| \\ &\le \|v_n - u_n\| + \|Su_n - y_n\| + \|y_n - v_n\| \\ &= \|P_C(I-\lambda\nabla f_{\alpha_n})y_n - P_C(I-\lambda\nabla f_{\alpha_n})x_{n+1}\| \\ &\quad + \|Su_n - y_n\| + (1-\beta_n)\|x_n - v_n\| \\ &\le \|y_n - x_{n+1}\| + \|Su_n - y_n\| + (1-\beta_n)\|x_n - v_n\|.\end{aligned}$$

This implies that

$$\lim_{n\to\infty}\|Sv_n - v_n\| = \lim_{n\to\infty}\|u_n - v_n\| = 0, \tag{3.16}$$

and so

$$\|u_n - x_n\| \le \|u_n - v_n\| + \|v_n - x_n\| \to 0 \quad \text{as } n \to \infty.$$

Step 3. $\omega_w(x_n) \subset \mathrm{Fix}(S) \cap \Gamma$.

Indeed, suppose that $\hat{x} \in \omega_w(x_n)$ and $\{x_{n_j}\}$ is a subsequence of $\{x_n\}$ such that $x_{n_j} \rightharpoonup \hat{x}$. Set $T = P_C(I - \lambda\nabla f)$. Then for each $\lambda \in (0, \frac{2}{\|A\|^2})$, T is nonexpansive. As a matter of fact, we have seen that $\nabla f = A^*(I - P_Q)A$ is $\frac{1}{\|A\|^2}$-ism and $\lambda\nabla f = \lambda A^*(I - P_Q)A$ is $\frac{1}{\lambda\|A\|^2}$-ism. Hence, by Proposition 2.2 (iii) the complement $I - \lambda\nabla f$ is $\frac{\lambda\|A\|^2}{2}$-averaged. Therefore, noting that P_C is $\frac{1}{2}$-averaged and applying Proposition 2.3 (iv), we know that for each $\lambda \in (0, \frac{2}{\|A\|^2})$, $T = P_C(I - \lambda\nabla f)$ is α-averaged, with

$$\alpha = \frac{1}{2} + \frac{\lambda\|A\|^2}{2} - \frac{1}{2}\cdot\frac{\lambda\|A\|^2}{2} = \frac{2+\lambda\|A\|^2}{4} \in (0,1).$$

Consequently, it is clear that T is nonexpansive.

Now observe that

$$\begin{aligned}\|x_n - Tx_n\| &\le \|x_n - u_n\| + \|u_n - Tx_n\| \\ &= \|x_n - u_n\| + \|P_C(I-\lambda\nabla f_{\alpha_n})x_{n+1} - P_C(I-\lambda\nabla f)x_n\| \\ &\le \|x_n - u_n\| + \|P_C(I-\lambda\nabla f_{\alpha_n})x_{n+1} - P_C(I-\lambda\nabla f_{\alpha_n})x_n\| \\ &\quad + \|P_C(I-\lambda\nabla f_{\alpha_n})x_n - P_C(I-\lambda\nabla f)x_n\| \\ &\le \|x_n - u_n\| + \|x_{n+1} - x_n\| + \lambda\alpha_n\|x_n\|.\end{aligned}$$

So, from $\|x_n - u_n\| \to 0$, $\|x_{n+1} - x_n\| \to 0$, $\alpha_n \to 0$ and the boundedness of $\{x_n\}$ it follows that

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0. \tag{3.17}$$

Taking into account $x_{n_j} \rightharpoonup \hat{x}$ and utilizing Lemma 2.1, we obtain $\hat{x} \in \mathrm{Fix}(T)$. But $\mathrm{Fix}(T) = \Gamma$; we therefore have $\hat{x} \in \Gamma$. Furthermore, since $x_{n_j} \rightharpoonup \hat{x}$ and $\lim_{n\to\infty} \|u_n - x_n\| = \lim_{n\to\infty} \|u_n - v_n\| = \lim_{n\to\infty} \|Sv_n - v_n\| = 0$, it is known that $v_{n_j} \rightharpoonup \hat{x}$ and $\lim_{n\to\infty} \|Sv_{n_j} - v_{n_j}\| = 0$. Thus, from Lemma 2.1 we get $\hat{x} \in \mathrm{Fix}(S)$. Therefore, we have $\hat{x} \in \mathrm{Fix}(S) \cap \Gamma$. This shows that there holds the relation

$$\omega_w(x_n) \subset \mathrm{Fix}(S) \cap \Gamma. \tag{3.18}$$

Step 4. Both the sequences $\{x_n\}$ and $\{y_n\}$ converge weakly to an element $z \in \mathrm{Fix}(S) \cap \Gamma$.

Indeed, according to (3.12) and (3.18) we apply Proposition 2.4 to $\mathrm{Fix}(S) \cap \Gamma$ to show that $\{x_n\}$ converges weakly to a point $z \in \mathrm{Fix}(S) \cap \Gamma$. Moreover, from $\|x_n - y_n\| \to 0$ it follows that $y_n \rightharpoonup z$. This completes the proof. □

Remark 3.3. Theorem 3.2 improves, extends and develops [30] (Theorem 5.7) in the following aspects:

(a) The iterative algorithm in [30] (Theorem 5.7) is extended to develop the Mann's type implicit extragradient-like algorithm in Theorem 3.2.

(b) The technique of proving weak convergence in Theorem 3.2 is very different from that in [30] (Theorem 5.7) because our technique depends on the demiclosedness principle for nonexpansive mappings in Hilbert spaces.

(c) The problem of finding an element of $\mathrm{Fix}(S) \cap \Gamma$ is more general than the one of finding a solution of the SFP in [30] (Theorem 5.7).

4. Concluding remarks

In this paper, we considered the split feasibility problem (SFP) in infinite-dimensional Hilbert spaces, and studied the relaxed implicit extragradient-like methods for finding a common element of the solution set Γ of the SFP and the set $\mathrm{Fix}(S)$ of fixed points of a nonexpansive mapping S. Two implicit iterative algorithms for finding an element of $\mathrm{Fix}(S) \cap \Gamma$ which are combinations of Mann's implicit iterative method and Korpelevich's extragradient method are presented. Whenever $S \equiv I$ (the identity mapping), strong convergence of one algorithm to the minimum-norm solution of the SFP is obtained under very appropriate conditions. Whenever $S \neq I$, weak convergence of the other algorithm is also obtained under quite mild conditions. As mentioned, in this paper, $\Gamma = \mathrm{Fix}(T)$, where $T = P_C(I - \lambda\nabla f)$ is a nonexpansive mapping. Thus finding a point in $\mathrm{Fix}(S) \cap \Gamma$ is equivalent to finding a point in $\mathrm{Fix}(S) \cap \mathrm{Fix}(T)$ for two nonexpansive mappings S and T. On the other hand, Xu and Ori [27] introduced an implicit iteration process for finding a common fixed point of a finite family of nonexpansive mappings in infinite-dimensional Hilbert spaces. Subsequently, Zeng and Yao [32] proposed another implicit iteration scheme with perturbed mapping for the approximation of common fixed points of a finite family of nonexpansive mappings in infinite-dimensional Hilbert spaces. Obviously, the relaxed implicit extragradient-like methods of this paper are superior to the two implicit iterative algorithms in [27], [32] because the relaxed implicit extragradient-like methods under consideration comprises one predictor step and another corrector step and thus are quite reasonable in the practical implementation.

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